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EXISTENCE RESULTS, INEQUALITIES AND A PRIORI BOUNDS TO FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. This paper presents new results to fractional boundary value problems in relation to the existence of solutions and providing *a priori* bounds on all possible solutions subject to a differential inequality. This provides the scope of these results to be applicable to a vast range of problems that may have unrestricted growth to prove the existence of solutions. This contributes to the practically of these results through their application in modelling real world phenomena.

1. Introduction

The aim of this paper and its merits are to present novel existence results and provide *a priori* bounds for fractional nonlinear boundary value problems with unrestricted growth by using fractional differential inequalities and topological methods. The results generalise and contribute to the current existence theory allowing the existence of solutions for a larger scope of problems and useful qualitative information about their bounds. The paper can be broken down into five sections, an introduction to theory; key results for producing an equivalent integral representation including fractional derivative definitions and interesting results relevant to the problems; fractional inequalities and proving *a priori* bounds on all possible solutions; novel existence results; and finally examples illustrating the type of problems now applicable which is motivating the purpose of the theory.

Let $p\in(0,1]$ and $a,b\in\mathbb{R}, a<b$. We consider the following fractional boundary value problem:

$$(D_a^{p+1}y)(t)=f(t,y(t)),\qquad t\in(a,b), \tag{1}$$

$$(D_a^{p}y)(a)=c_1,\quad y(b)=c_2, \tag{2}$$

where $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. The fractional derivative in the BVP (1), (2) is the fractional Riemann-Liouville derivative of order $\alpha\in\mathbb{R}^+$ is defined as

$$(D_a^{\alpha}y)(t)=\left(\frac{d}{dt}\right)^k(J_a^{k-\alpha}y)(t)$$

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where

$$(J_a^{k-\alpha}y)(t) := \frac{1}{\Gamma(k-\alpha)}\int_a^t (t-\tau)^{k-\alpha-1}y(\tau)\,d\tau,$$

and $k = \lceil \alpha \rceil$. However, there are alternative definitions of a fractional derivative and the results herein still hold with minor adaptations. For example, the fractional derivative can be interchanged with the Caputo derivative with the following boundary conditions

$$y'(a) = c_1, \quad y(b) = c_2$$

where the Caputo derivative is defined as

$$({}^C D_a^{\alpha} y)(t) := \frac{1}{\Gamma(k-\alpha)}\int_a^t (t-\tau)^{k-\alpha-1}D^{\lceil \alpha \rceil}(y)(\tau)\,d\tau.$$

Furthermore, there are natural right fractional derivative boundary value problems with accompanying right-focal conditions which naturally follow from the same techniques and results herein too. In this paper, we will focus on the results using a Riemann-Liouville derivative, the similar results with a Caputo derivative will be found the following paper [11] and the right fractional focal problems can be deduced from the results within this paper. In the following, we begin by examining a particular case when $c_1 = 0$ and it turns out that this condition has some natural and interesting properties which allows these results to be well defined and presented elegantly. A particular consequence is if $(D_a^p y)(a) = 0$, y is continuous and integrable then $y(a) = 0$.

The study of fractional calculus dates back to the late 1600's from Leibniz and more recently has seen a surge of research found in surveys and the works of [8], [14], [15], [16], [19], [20] and their references therein. In particular, fractional boundary value problems has been developing over the last few decades due to being a very effective and appropriate approach to modelling various real world phenomena. The reason is the fractional order assists in more effectively capturing the past history of complex dynamical systems from past to present [6], [7], [18] and [26]. This allows us to understand the memory effects of problems which naturally appear in science compared to the classical setting where we are limited to integer-order rates of change. Some real world applications include chronic disorders, epidemic models, biological processes, groundwater models, electrical engineering, chemistry, control theory and physics with examples found in Podlubny [19], Kilbas et al. [14], Xuan et al. [24], Atangana [3], Ibe [13] and Zheng & Zhang [25].

The first main result of this paper is to provide an equivalent integral representation to the fractional nonlinear boundary value problem. Section 3 of the paper provides a series of results relating to fractional differential inequalities which prove to be very versatile and useful, and provides some of the backbone to the main results. The next main result is to present novel *a priori* bounds to all possible solutions to the fractional BVP (1), (2) under the key differential inequality,

$$\|\mathbf{f}(t,\mathbf{u})\| \leqslant 2V\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u})\rangle + W, \quad \text{for all} \quad (t,\mathbf{u}) \in (a,b)\times\mathbb{R}^n \tag{3}$$

where V,W are non-negative constants. Also, the norm is defined via $\|\mathbf{u}\|^2 = \langle \mathbf{u},\mathbf{u}\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual Euclidean dot product of two vectors in $\mathbb{R}^n$. This inequality (3) is satisfied for a variety of different functions $\mathbf{f}$, including functions that are unbounded and allows the right hand side of the fractional BVP to be unrestricted for a continuous system of fractional equations. The results in the paper will be presented for the scalar case, however, the results are naturally true for a system if this inequality is satisfied. This inequality first appeared in the work of Hartman [12] for second order boundary value problems and more useful recents results for BVPs have been proved by Fewster-Young and Tisdell [9]. In the literature, currently results are known when f is Lipschitz, satisfies certain Lypunaov conditions or monotonicity, or Osgood condition or if f is a bounded function [4], [5], [8], [19], [23], [22], [21] and references therein. If we combine these results with topological methods such as Schauder's fixed point theorem then this proves a novel existence result of solutions to the fractional BVP (1), (2) under these conditions. Finally, the paper concludes with applications to examples showcasing the new results.

2. The equivalent integral representation

This section presents equivalent integral representations and *a priori* bounds for all solutions to the BVP (1), (2). This allows the problem to be reformulated in a versatile way to apply topological methods to determine the existence of solutions. In addition, this section will present definitions and outline key points to present some lemmas in the general theory of fractional calculus which play an important role in proofs of the upcoming main results. Also, the particular behaviour of a fractional derivative and the consequences of the solution at $t = a$ will play an important role in some of the results.

The following lemmas provide alternative definitions of a Riemann-Liouville fractional derivative which can be found in [8] and [19].

LEMMA 1. *Let $p > 0$, $p \notin \mathbb{N}$ and $m = \lceil p \rceil$. Suppose $y \in C^m([a,b];\mathbb{R})$ then*

$$(D_a^p y)(t) = \frac{1}{\Gamma(-p)} \int_a^t (t-\tau)^{-p-1} y(\tau)\, d\tau.$$

LEMMA 2. *Let $p \in \mathbb{R}^+$ and $0 < p < 1$. If y is a continuous function and its derivative is integrable in $[a,b]$ then $D_a^p y$ exists almost everywhere in $[a,b]$, $D_a^p y \in L_q[a,b]$ for $1 \leqslant q \leqslant 1/p$ and*

$$(D_a^p y)(t) = \frac{1}{\Gamma(1-p)} \left[\frac{y(a)}{(t-a)^p} + \int_a^t (t-\tau)^{-p} y'(\tau)\, d\tau \right]. \tag{4}$$

LEMMA 3. *Let $p \in \mathbb{R}^+$ and $0 < p < 1$. If y is a continuous function, its derivative is integrable in $[a,b]$ and*

$$\lim_{t \to a^+} \left[(D_a^p y)(t) \right] = 0$$

then

$$y(a) = 0.$$

Proof. By multiplying both sides of (4) by $(t-a)^p$ then Lemma 2 and

$$\lim_{t\to a^+}\left[(D_a^p y)(t)\right]=0$$

implies $y(a)=0$. □

REMARK 1. Moreover, the conditions are equivalent to each other in Lemma 3.

This naturally results in some consequences in operations of fractional calculus [19] presented below:

LEMMA 4. *If* $0<p<1$ *and Lemma* 3 *holds then*

$$D_a^p(D_a^1 y)(t)=(D_a^{p+1}y)(t)=\frac{d}{dt}(D_a^p y)(t).$$

Naturally, the following is always true by the definition of a Riemann-Liouville fractional derivative $(D_a^{p+1}y)(t)=\frac{d}{dt}(D_a^p y)(t)$. Furthermore, we notice that roughly speaking, fractional integration improves the smoothness properties of functions. This provides some interesting properties relating to the behaviour of solutions at the lower terminal. If we suppose that y is continuous and has at least one continuous derivative in the closed interval $[a,t]$ then by considering the fractional power series of order $q\geqslant 0$

$$(J_a^q y)(t)=\sum_{k=0}^{\infty}\frac{y^{(k)}(a)}{\Gamma(k+q+1)}(t-a)^{k+q}$$

and as $t\longrightarrow a^+$ then

$$\lim_{t\to a^+}(J_a^q y)(t)=0,\quad \text{for}\quad q>0$$

and

$$\lim_{t\to a^+}(J_a^q y)(t)=y(a),\quad \text{for}\quad q=0.$$

We can naturally take this one step further to widen the range of functions by supposing a solution $y(t)$ could have an integrable singularity up to order $r=1-p$ where $0<p\leqslant 1$. Moreover $y(a)$ may not exist and this results in considering a more appropriate solution space. Therefore, it can be written as $y(t)=(t-a)^{-r}y_*(t)$ where $y_*(a)\neq 0$ and $y_*(t)$ is a continuous function with a continuous derivative as $t\longrightarrow a^+$. By considering the fractional integral power series of order q on $[a,t]$,

$$(J_a^q y)(t)=\sum_{k=0}^{\infty}\frac{y_*^{(k)}(a)}{k!}\frac{\Gamma(k+1-r)}{\Gamma(k+1-r+q)}(t-a)^{k-r+q}.$$

Thus,

$$\lim_{t\to a^+}(J_a^q y)(t)=0,\quad \text{for}\quad q>r;\qquad \lim_{t\to a^+}(J_a^q y)(t)=y_*(a)\Gamma(1-r),\quad \text{for}\quad q=r$$

and moreover if $r = 1-p$ then

$$\lim_{t\to a^+}(J_a^q y)(t) = 0, \quad \text{for} \quad p+q>1; \quad \lim_{t\to a^+}(J_a^q y)(t) = y_*(a)\Gamma(p), \quad \text{for} \quad p+q=1.$$

Another result [17, Property 2.10] which plays an important role in fractional calculus is a formula for integration by parts. The formula is similar to the classical version, however, it changes the type of differentiation of left Riemann-Liouville fractional derivative to a right Caputo fractional derivative where the definition of the derivative can be found in [17, Definition 2.5].

LEMMA 5. *Suppose* $0<p<1$, $f \in AC([a,b];\mathbb{R})$ *and* $g \in L^r((a,b);\mathbb{R})$ $(1 \leqslant r \leqslant \infty)$. *Then, the following integration by parts formula holds:*

$$\int_a^b f(t)(D_a^p g)(t)\,dt = \int_a^b g(t)({}_t^C D_b^p f)(t)\,dt + \left[f(t)(J_a^{1-p}g)(t)\right]_a^b \tag{5}$$

where

$$({}_t^C D_b^p f)(t) := -\left({}_t J_b^{1-p}\frac{df}{dt}\right)(t)$$

and

$$({}_t J_b^{1-p} f)(t) := \frac{1}{\Gamma(1-p)}\int_t^b (\tau-t)^{-p} f(\tau)\,d\tau.$$

If $f(t)=1$ for $t\in[a,b]$ then $({}_t^C D_b^p 1)(t) = 0$ and this yields the following useful corollary.

COROLLARY 1. *If* $0<p<1$ *and* $g \in L^r((a,b);\mathbb{R})$ *(*$1\leqslant r\leqslant\infty$*), then*

$$\int_a^b (D_a^p g)(t)\,dt = \left[(J_a^{1-p}g)(t)\right]_a^b \tag{6}$$

We now establish two equivalent integral representations for the fractional BVP, (1), (2). The first equivalent representation relates to when $c_1 = 0$.

THEOREM 1. *Suppose* y *is a continuous function and its derivative is integrable in* $[a,b]$. *A function* y *is a solution to the fractional BVP* (1), (2) *with* $c_1 = 0$ *if and only if it is a solution to the equivalent integral representation given by*

$$y(t) = c_2 - \frac{1}{\Gamma(p)}\int_t^b\int_a^s (s-\tau)^{p-1} f(\tau,y(\tau))\,d\tau ds, \quad \textit{for} \quad t\in[a,b]. \tag{7}$$

Proof. Suppose y is a solution to the fractional BVP (1), (2) and consider

$$(D_a^{p+1}y)(\tau) = f(\tau,y(\tau)), \qquad \tau\in(a,b).$$

The conditions of Lemma 4 are satisfied, this implies

$$\left(D_a^p\left[\frac{dy}{d\tau}\right]\right)(\tau) = f(\tau,y(\tau)), \qquad \tau\in(a,b).$$

By taking J_a^p of both sides yields

$$\left(J_a^p\left(D_a^p\left[\frac{dy}{ds}\right]\right)\right)(s)=\frac{dy(s)}{dt}-\lim_{s\to a^+}\left[(D_a^p y)(s)\right]\frac{(s-a)^{p-1}}{\Gamma(p)}=(J_a^p f(\cdot,y(\cdot)))(s).$$

Therefore, it follows

$$y'(s)=(J_a^p f(\cdot,y(\cdot)))(s),\qquad s\in(a,b].$$

By Fundamental Theorem of Calculus, this gives

$$y(t)=y(b)-\frac{1}{\Gamma(p)}\int_t^b\int_a^s(s-\tau)^{p-1}f(\tau,y(\tau))\,d\tau ds,\quad \text{for}\quad t\in[a,b].$$

Imposing the boundary condition $y(b)=c_2$ produces (7). In addition, the left boundary condition and Lemma 3 imply $y(a)=0$. It still suffices to prove the equivalence, if we differentiate then

$$y'(t)=\frac{1}{\Gamma(p)}\int_a^t(t-\tau)^{p-1}f(\tau,y(\tau))\,d\tau=(J_a^p f(\cdot,y(\cdot)))(t),$$

and by using Lemma 2, this implies

$$(D_a^p y)(t)=\frac{1}{\Gamma(1-p)}\left[\frac{y(a)}{(t-a)^p}+\int_a^t(t-s)^{-p}(J_a^p f(\cdot,y(\cdot)))(s)\,ds\right].$$

Notice the above integral is actually $(J_a^{1-p})(J_a^p)(\cdot)=(J_a^1)(\cdot)$, therefore

$$(D_a^p y)(t)=\frac{1}{\Gamma(1-p)}\frac{y(a)}{(t-a)^p}+\int_a^t f(s,y(s))\,ds.$$

By the remark after Lemma 3 and $t\to a^+$, the boundary condition is satisfied as $t\to a^+$ since $y(a)=0$. Finally, differentiating both sides yields

$$(D_a^{p+1}y)(t)=f(t,y(t)).\quad\square$$

An alternative integral representation can be established for the general boundary condition as follows:

THEOREM 2. *Suppose y is a continuous function and its derivative is integrable in $(a,b]$. A function y is a solution to the fractional BVP* (1), (2) *if and only if it is a solution to the equivalent integral representation given by*

$$y(t)=c_1\frac{(t-a)^p}{\Gamma(p+1)}+\frac{(t-a)^{p-1}}{(b-a)^{p-1}}\left(c_2-\left[c_1\frac{(b-a)^p}{\Gamma(p+1)}+(J_a^{p+1}f(\cdot,y(\cdot)))(b)\right]\right)+(J_a^{p+1}f(\cdot,y(\cdot)))(t)\qquad(8)$$

for $t\in(a,b]$.

Proof. Suppose y is a solution to the fractional BVP (1), (2) and consider

$$(D_a^{p+1}y)(t) = f(t,y(t)), \qquad t \in (a,b),$$

By taking J_a^{p+1} of both sides and using the identity [19, Equation 2.115]

$$J_a^{p+1}(D_a^{p+1}y)(t) = y(t) - \lim_{t\to a^+}\left[(D_a^p y)(t)\right]\frac{(t-a)^p}{\Gamma(p+1)} - \lim_{t\to a^+}\left[(D_a^{p-1}y)(t)\right]\frac{(t-a)^{p-1}}{\Gamma(p)}$$

then this yields

$$y(t) = \lim_{t\to a^+}\left[(D_a^p y)(t)\right]\frac{(t-a)^p}{\Gamma(p+1)} + \lim_{t\to a^+}\left[(D_a^{p-1}y)(t)\right]\frac{(t-a)^{p-1}}{\Gamma(p)} + (J_a^{p+1}f(\cdot,y(\cdot)))(t),$$

for $t \in (a,b]$. By using the initial condition, and letting

$$A := \lim_{t\to a^+}\left[(D_a^{p-1}y)(t)\right] = \lim_{t\to a^+}\left[(J_a^{1-p}y)(t)\right]$$

then we can simplify this to

$$y(t) = c_1\frac{(t-a)^p}{\Gamma(p+1)} + A\frac{(t-a)^{p-1}}{\Gamma(p)} + (J_a^{p+1}f(\cdot,y(\cdot)))(t), \quad \text{for} \quad t \in (a,b].$$

If we substitute the other boundary condition and rearrange then

$$A = \frac{\Gamma(p)}{(b-a)^{p-1}}\left(c_2 - \left[c_1\frac{(b-a)^p}{\Gamma(p+1)} + (J_a^{p+1}f(\cdot,y(\cdot)))(b)\right]\right)$$

and in turn we have the following

$$y(t) = c_1\frac{(t-a)^p}{\Gamma(p+1)} + \frac{(t-a)^{p-1}}{(b-a)^{p-1}}\left(c_2 - \left[c_1\frac{(b-a)^p}{\Gamma(p+1)} + (J_a^{p+1}f(\cdot,y(\cdot)))(b)\right]\right) + J_a^{p+1}f(t,y(t)),$$

for $t \in (a,b]$. Note that the following identity holds $J_a^{p+1}f(t,y(t)) = J_a(J_a^p f(t,y(t)))$. It suffices to show the equivalence, if we consider the integral representation and apply D_a^p to both sides then

$$(D_a^p y)(t) = c_1 + (J_a^1 f(\cdot,y(\cdot)))(t).$$

This can be deduced by using the following [8][Example 2.4] for $\beta > -1$

$$(D_a^p[\cdot - a]^\beta)(t) = \frac{\Gamma(\beta+1)}{\Gamma(2-p+\beta)}(D_a^1[(\cdot-a)^{1-p+\beta}])(t)$$

which when applied with $\beta = p-1$, $\beta = p$ yields

$$(D_a^p[\cdot - a]^{p-1})(t) = \frac{\Gamma(p)}{\Gamma(1)}(D_a^1[1])(t) = 0,$$

$$(D_a^p[\cdot - a]^p)(t) = \frac{\Gamma(p+1)}{\Gamma(2)}(D_a^1[(\cdot-a)^1])(t) = \Gamma(p+1).$$

Thus, we see that $\lim_{t\to a^+}(D_a^p y)(t) = c_1$ and by differentiating once more then we obtain

$$(D_a^{p+1}y)(t) = f(t,y(t)). \quad \square$$

3. Fractional differential inequalities & a priori bounds

A key fractional differential inequality that has plenty of applications and variations is the following and we provide a new proof to our knowledge of this result which is joint work with Professor Martin Bohner in discrete fractional calculus [10]. Earlier proofs can be attributed to Alsaedi & Ahmad & Kirane [2], however, we provide an elegant proof below tailored to our problem for the readers.

LEMMA 6. *If* $u:[a,b]\to\mathbb{R}$, $u\in C^1((a,b);\mathbb{R})$ *and* $p\in(0,1)$ *then*

$$(D_a^p[u]^2)(t)\leqslant 2u(t)(D_a^p u)(t)$$

where $t\in(a,b)$.

Proof. We start with the left hand side of the inequality and recall that

$$[u(t)]^2=(u(t)-u(\tau))^2+2u(t)u(\tau)-[u(\tau)]^2.$$

By using the fractional derivation definition, then

$$\begin{aligned}
D_a^p([u(t)]^2) &= \frac{1}{\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}[u(\tau)]^2\,d\tau\\
&= \frac{1}{\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}[u(t)-u(\tau)]^2\,d\tau\\
&\quad + 2u(t)\frac{1}{\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}[u(\tau)]\,d\tau\\
&\quad - [u(t)]^2(D_a^p\,1)(t)\\
&= \frac{1}{\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}[u(t)-u(\tau)]^2\,d\tau\\
&\quad + 2u(t)D_a^p([u(t)]) - [u(t)]^2\left[\frac{1}{\Gamma(1-p)}(t-a)^{-p}\right]\\
&\leqslant 2u(t)D_a^p([u(t)])
\end{aligned}$$

since $\Gamma(1-p)\geqslant 0$ and $\Gamma(-p)\leqslant 0$. □

Notice that a tighter inequality is possible here from the proof,

$$D_a^p([u(t)]^2)\leqslant 2u(t)D_a^p([u(t)])-[u(t)]^2\left[\frac{1}{\Gamma(1-p)}(t-a)^{-p}\right]$$

for $t\in(a,b)$.

Furthermore, another inequality that stems from this result is a fractional integral inequality, that can be deduced from this Lemma by letting $u(t):=(D_a^p y)(t)$.

LEMMA 7. *If* $y:[a,b]\to\mathbb{R}$, $y\in C^1((a,b);\mathbb{R})$ *and* $D_a^p y\in C^1((a,b);\mathbb{R})$ *with* $p\in(0,1)$ *then*

$$J_a^{p+1}((Dy)(\cdot)(D_a^p y)(\cdot))(t)\geqslant 0 \tag{9}$$

for $t\in[a,b]$.

Proof. The proof of this result can be deduced from Lemma 6 by letting $u(t) = (D_a^p y)(t)$, and replacing p with $1-p$ to obtain the following inequality,

$$(D_a^{1-p}[(D_a^p y)(\cdot)]^2)(t) \leqslant 2(D_a^p y)(t)(D_a^{1-p}(D_a^p y)(\cdot))(t)$$

for $t \in (a,b)$. The law for the composition of fractional derivatives is given below

$$(D_a^{1-p}(D_a^p y))(t) = (Dy)(t) - \lim_{t \to a^+}(J_a^{1-p} y)(t)\frac{(t-a)^{p-2}}{\Gamma(p-1)}$$

for $t \in (a,b)$. Since either our solutions admit $\lim_{t\to a^+}(D_a^p y)(t) = 0$ implying $y(a) = 0$, or since ultimately in general they are continuous and have at one continuous derivative then $\lim_{t\to a^+}(J_a^{1-p} y)(t) = 0$, thus

$$(D_a^{1-p}(D_a^p y)(\cdot))(t) = (Dy)(t)$$

for $t \in (a,b)$. By applying this inequality, we see that

$$\begin{aligned} J_a^{p+1}((Dy)(\cdot)(D_a^p y)(\cdot))(t) &\geqslant \frac{1}{2} J_a^{p+1}(D_a^{1-p}[(D_a^p y)(\cdot)]^2)(t) \\ &= \frac{1}{2} J_a^{2p}([(D_a^p y)(\cdot)]^2)(t) \\ &\geqslant 0 \end{aligned}$$

for $t \in [a,b]$. □

We now present one of the main novel results proving that a solution to the fractional BVP (1), (2) satisfies the following *a priori* bounds.

THEOREM 3. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. If there exists non-negative constants V, W such that*

$$\|f(t,u)\| \leqslant 2V\langle u, f(t,u)\rangle + W \tag{10}$$

then all possible solutions to (1), (2) *satisfy*

$$\|y(t)\| \leqslant \|c_2\| + V\|c_2\|^2 + \frac{W}{\Gamma(p+2)}(b-a)^{p+1}, \quad \textit{for all} \quad t \in [a,b].$$

Proof. By Theorem 1, we have the equivalent integral representation of all solutions to the BVP (1), (2) and is given by

$$y(t) = c_2 - \frac{1}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} f(\tau, y(\tau))\, d\tau ds, \qquad t \in [a,b].$$

We now estimate $y(t)$ and by applying condition (10) yields

$$\begin{aligned} \|y(t)\| &\leqslant \|c_2\| + \frac{1}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} \|f(\tau, y(\tau))\|\, d\tau ds \\ &\leqslant \|c_2\| + \frac{1}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} \big[2Vy(\tau)f(\tau,y(\tau)) + W\big]\, d\tau ds \end{aligned}$$

for $t \in [a,b]$. Recall that y is a solution to the differential equation and Lemma 4 implies

$$\|y(t)\| \leqslant \|c_2\| + \frac{2V}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} y(\tau)(D_a^p y')(\tau)\, d\tau ds \\ + \frac{W}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1}\, d\tau ds,$$

for $t \in [a.b]$. We now focus on finding an estimate for the first integral when $t = a$ and let

$$H(t) := \frac{2V}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} y(\tau)(D_a^p y')(\tau)\, d\tau ds.$$

See that $H(t)$ can be written as follows

$$H(t) = \frac{2V}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} \left[y(\tau)(D_a^p y')(\tau) + y'(\tau)(D_a^p y)(\tau)\right] d\tau ds \\ - \frac{2V}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} \left[y'(\tau)(D_a^p y)(\tau)\right] d\tau ds$$

Now Lemma 4 and Lemma 7 implies

$$H(a) \leqslant \frac{2V}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} \left[y(\tau)(D_a^p[y])(\tau)\right]' d\tau ds$$

since

$$J_a^{p+1}((Dy)(\cdot)(D_a^p y)(\cdot))(b) = \frac{1}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} \left[y'(\tau)(D_a^p y)(\tau)\right] d\tau\, ds \geqslant 0.$$

Furthermore, we can simplify the integral with the use of Lemma 2 and the boundary condition to produce

$$H(a) \leqslant 2V \int_a^b (D_a^{1-p} y(\cdot)(D_a^p[y])(\cdot))(s)\, ds.$$

We apply the definition of a fractional derivative given by Lemma 1 to produce

$$H(a) \leqslant \frac{2V}{\Gamma(p-1)} \int_a^b \int_a^s (s-\tau)^{p-2} [y(\tau)(D_a^p[y])(\tau)]\, d\tau\, ds.$$

By employing the Lemma 6 and since $\Gamma(p-1) < 0$ then

$$H(a) \leqslant \frac{V}{\Gamma(p-1)} \int_a^b \int_a^s (s-\tau)^{p-2} (D_a^p[y]^2)(\tau)\, d\tau\, ds.$$

However, this is the composition of two fractional derivatives and since $y(a) = 0$ then

$$H(a) \leqslant V \int_a^b \frac{d[y(s)]^2}{ds}\, ds = V([y(b)]^2 - [y(a)]^2) = V[c_2]^2.$$

Therefore, by putting everything together, we have

$$\begin{aligned}\|y(t)\| &\leqslant \|c_2\| + V\|c_2\|^2 + \frac{W}{\Gamma(p)}\int_a^b \int_a^s (s-\tau)^{p-1}\, d\tau ds \\ &= \|c_2\| + V\|c_2\|^2 + \frac{W}{\Gamma(p+2)}(b-a)^{p+1}, \quad \text{for all} \quad t \in [a,b]. \quad \square\end{aligned}$$

We now consider the fractional boundary value problem where the derivative on the lower terminal is not necessarily equal to zero. However, we must emphasis the class of functions for solutions to be smooth enough such that the following holds

$$\lim_{t\to a^+} (J_a^{1-p} y)(t) = 0.$$

This means that the solution must be integrable near the lower terminal and not have a singularity of order greater than $1-p$. We are mainly interested in showing the existence of continuous solutions, therefore the space of continuous functions does this job nicely. We define the space of continuous functions with the following norm,

$$X^* := \{y \in C((a,b];\mathbb{R}) \mid \|y\|_* := \sup_{t\in(a,b]} |(t-a)^{1-p} y(t)|\}.$$

We present the next novel result proving that if $y \in X^*$ is a continuous solution to the fractional BVP (1), (2) then it satisfies the following *a priori* bounds.

THEOREM 4. *Let $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. If there exists non-negative constants V,W such that* (10) *is satisfied then all possible solutions to* (1), (2) *satisfy*

$$\|y\|_* \leqslant \frac{|c_1|(b-a)}{\Gamma(p+1)} + \frac{1}{(b-a)^{p-1}}\left(|c_2| + 2V\left[[c_2]^2 + \left[\frac{c_1(b-a)^p}{\Gamma(p+1)}\right]^2\right] + \frac{2W(b-a)^{p+1}}{\Gamma(p+2)}\right),$$

for all $t \in (a,b]$.

Proof. By using Theorem 2, we have the following equivalent integral representation for solutions to (1), (2) given by

$$\begin{aligned} y(t) = c_1\frac{(t-a)^p}{\Gamma(p+1)} + \frac{(t-a)^{p-1}}{(b-a)^{p-1}}\left(c_2 - \left[c_1\frac{(b-a)^p}{\Gamma(p+1)} + (J_a^{p+1} f(\cdot,y(\cdot)))(b)\right]\right) \\ + (J_a^{p+1} f(t,y(t)))(t), \end{aligned}$$

for $t \in (a,b]$. We estimate $y(t)$ with the norm in X^* and by applying condition (10) this yields

$$|(t-a)^{1-p} y(t)| \leqslant \frac{|c_1|(b-a)}{\Gamma(p+1)} + \frac{\left(|c_2| + |(J_a^{p+1} f(\cdot,y(\cdot)))(b)| + |J_a^{p+1} f(t,y(t))|\right)}{(b-a)^{p-1}}$$

and moreover

$$|(J_a^{p+1} f(\cdot,y(\cdot)))(t)| \leqslant \frac{1}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} \left[2Vy(\tau)f(\tau,y(\tau)) + W\right] d\tau ds,$$

for $t \in (a,b]$. Recall that y is a solution to the differential equation, therefore

$$|(J_a^{p+1} f(\cdot,y(\cdot))(t)| \leqslant \frac{2V}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} y(\tau)(D_a^{p+1}y)(\tau)\, d\tau ds + \frac{W(b-a)^{p+1}}{\Gamma(p+2)}.$$

If we just consider the integral part as in the proof of Theorem 3 and apply Lemma 4 then

$$\begin{aligned}\int_a^b \int_a^s (s-\tau)^{p-1} y(\tau)(D_a^{p+1}y)(\tau)\, d\tau ds \leqslant & \int_a^b \int_a^s (s-\tau)^{p-1} (D[y(\cdot)(D_a^p y)(\cdot)]))(\tau) d\tau ds \\ & - \int_a^b \int_a^s (s-\tau)^{p-1} [(Dy)(\cdot)(D_a^p y)(\cdot)])(\tau)\, d\tau ds.\end{aligned}$$

By Lemma 7, we know

$$\int_a^b (J_a^p[(Dy)(\cdot)(D_a^p y)(\cdot)])(s)\, ds \geqslant 0$$

and furthermore we can simplify the expression with the composition rule to obtain

$$\begin{aligned}\frac{2V}{\Gamma(p)} \int_a^b \int_a^s (s-\tau)^{p-1} y(\tau)(D_a^{p+1}y)(\tau)\, d\tau ds = 2V \Bigg[& \int_a^b (D_a^{1-p}([y(\cdot)(D_a^p y)(\cdot)]))(s) \\ & - \frac{y(a)(D_a^p y)(a)}{\Gamma(p)(s-a)^{1-p}}\, ds \Bigg]\end{aligned}$$

Thus, this yields

$$\begin{aligned}|(J_a^{p+1} f(\cdot,y(\cdot))(t)| \leqslant 2V \left[\int_a^b (D_a^{1-p}([y(\cdot)(D_a^p y)(\cdot)]))(s)\, ds - \frac{y(a)c_1(b-a)^p}{\Gamma(p+1)} \right] \\ + \frac{W(b-a)^{p+1}}{\Gamma(p+2)}\end{aligned}$$

for $t \in (a,b]$. In the same approach as Theorem 3 and using the fractional differential inequality result of Lemma 6, we have

$$|(J_a^{p+1} f(\cdot,y(\cdot))(t)| \leqslant V \int_a^b (D_a^{1-p}(D^p[y(\cdot)]^2))(s)\, ds - \frac{2Vy(a)c_1(b-a)^p}{\Gamma(p+1)} + \frac{W(b-a)^{p+1}}{\Gamma(p+2)}$$

for $t \in (a,b]$. By law for the composition of fractional derivatives,

$$\int_a^b (D_a^{1-p}(D_a^p f))(s)\, ds = \int_a^b (Df)(s)\, ds - \int_a^b \lim_{t \to a^+} (J_a^{1-p} f)(t)(D_a^{2-p} 1)(s)\, ds.$$

Furthermore, the function $f(s) := \|y(s)\|^2$ is a continuous non-negative function, so employing the fundamental theorem of calculus and moreover, the second integral is a Hadamard finite-part integral yields

$$\begin{aligned}\int_a^b (D_a^{1-p}(D_a^p f))(s)\, ds &= \int_a^b (Df)(s)\, ds - \lim_{t\to a^+} (J_a^{1-p} f)(t)(D_a^{1-p}1)(b) \\ &\leqslant f(b) - f(a) = [y(b)]^2 - [y(a)]^2\end{aligned}$$

since

$$\lim_{t\to a^+} (J_a^{1-p} f)(t) = \lim_{t\to a^+} (J_a^{1-p}[y(\cdot)]^2)(t) \geqslant 0 \quad \text{and} \quad (D_a^{1-p}1)(b) = \frac{(b-a)^{p-1}}{\Gamma(p)} > 0.$$

Thus this implies

$$|(J_a^{p+1} f(\cdot,y(\cdot))(t)| \leqslant V\left[[y(b)]^2 - [y(a)]^2 - \frac{2y(a)c_1(b-a)^p}{\Gamma(p+1)}\right] + \frac{W(b-a)^{p+1}}{\Gamma(p+2)}.$$

Furthermore, by applying Young's inequality this implies

$$\frac{2y(a)c_1(b-a)^p}{\Gamma(p+1)} \leqslant [y(a)]^2 + \left[\frac{c_1(b-a)^p}{\Gamma(p+1)}\right]^2.$$

Therefore,

$$|(J_a^{p+1} f(\cdot,y(\cdot))(t)| \leqslant V\left[[c_2]^2 + \left[\frac{c_1(b-a)^p}{\Gamma(p+1)}\right]^2\right] + \frac{W(b-a)^{p+1}}{\Gamma(p+2)}$$

for $t \in (a,b]$. Thus, if we put all our estimates together then

$$\begin{aligned}\|y\|_* &= \sup_{t\in(a,b]} |(t-a)^{1-p} y(t)| \\ &\leqslant \frac{|c_1|(b-a)}{\Gamma(p+1)} + \frac{1}{(b-a)^{p-1}}\left(|c_2| + 2V\left[[c_2]^2 + \left[\frac{c_1(b-a)^p}{\Gamma(p+1)}\right]^2\right] + \frac{2W(b-a)^{p+1}}{\Gamma(p+2)}\right)\end{aligned}$$

for all possible continuous solutions to (1), (2) for $t \in (a,b]$. □

4. Existence results

The next section completes the picture of existence with the next upcoming results being the final key in proving the novel existence of solutions to the fractional BVP (1), (2) where all possible solutions satisfy the *a priori* bounds in Theorem 3 or 4.

THEOREM 5. *If the conditions of Theorem 3 hold then there exists at least one solution to the fractional BVP* (1), (2).

Proof. Let

$$X := \{y \in C([a,b];\mathbb{R}) \mid \|y\| := \sup_{t\in[a,b]} |y(t)| \ : \ y(a) = 0\}.$$

Define the convex space

$$U := \{y \in C([a,b];\mathbb{R}) \mid \sup_{t\in[a,b]} |y(t)| \leqslant R+1 \ : \ y(a) = 0\}$$

where

$$R := \|c_2\| + V\|c_2\|^2 + \frac{W}{\Gamma(p+2)}(b-a)^{p+1}$$

and the continuous operator, $T : \bar{U} \to X$ by

$$Ty := c_2 - \frac{1}{\Gamma(p)} \int_t^b \int_a^s (s-\tau)^{p-1} f(\tau, y(\tau))\, d\tau ds.$$

A consequence is that every fixed point of the operator T is a solution to the fractional BVP by Theorem 1. To show there exists at least one fixed point, we apply the Leray-Schauder Nonlinear Alternative theorem [1]. By a standard argument and it is not too difficult to show that $T : \bar{U} \to X$ is a compact continuous map and this is mainly due to the continuity of f. It now suffices to show that $T(\bar{U}) \subset U$, this is equivalent to showing that for all $\lambda \in (0,1)$, there is no $y \in \partial U$ such that $y = \lambda T(y)$. If $y \in \partial U$ then $\|y\| = R+1$, however, by using the steps in the proof of Theorem 3 then

$$\|Ty(t)\| \leqslant \|c_2\| + V\|c_2\|^2 + \frac{W}{\Gamma(p+2)}(b-a)^{p+1} = R$$

Thus, this proves there is no $y \in \partial U$ such that $y = \lambda Ty$ for all $\lambda \in (0,1)$. In turn, the Nonlinear Alternative Theorem implies there exists at least one fixed point, $y \in C([a,b];\mathbb{R})$ which satisfies the equivalent integral representation and thus is a solution to the fractional BVP. □

The next existence result applies to the case where the fractional derivative on the lower terminal is not necessarily equal to zero.

THEOREM 6. *If the conditions of Theorem 4 hold then there exists at least one solution to the fractional BVP* (1), (2).

Proof. Let

$$X^* := \{y \in C((a,b];\mathbb{R}) \mid \|y\|_* := \sup_{t\in[a,b]} |(t-a)^{1-p} y(t)|\}.$$

Define the convex space

$$U^* := \{y \in C((a,b];\mathbb{R}) \mid \|y(t)\|_* \leqslant R^* + 1\}$$

where

$$R^* := \frac{|c_1|(b-a)}{\Gamma(p+1)} + \frac{1}{(b-a)^{p-1}}\left(|c_2| + 2V\left[[c_2]^2 + \left[\frac{c_1(b-a)^p}{\Gamma(p+1)}\right]^2\right] + \frac{2W(b-a)^{p+1}}{\Gamma(p+2)}\right)$$

and the continuous operator, $T : \bar{U}^* \to X^*$ by

$$Ty := c_1\frac{(t-a)^p}{\Gamma(p+1)} + \frac{(t-a)^{p-1}}{(b-a)^{p-1}}\left(c_2 - \left[c_1\frac{(b-a)^p}{\Gamma(p+1)} + (J_a^{p+1}f(\cdot,y(\cdot)))(b)\right]\right) + J_a^{p+1}f(t,y(t)).$$

The remainder of the proof is similar to the proof of Theorem 5 and where every fixed point of the operator T is a solution to the fractional BVP by Theorem 2. For brevity, a standard argument in analysis shows that $T : \bar{U}^* \to X^*$ is a compact continuous map and this is mainly due to the continuity of f again. The necessary condition to be shown is $T(\bar{U}^*) \subset U^*$, this is equivalent to showing that for all $\lambda \in (0,1)$, there is no $y \in \partial U^*$ such that $y = \lambda T(y)$. If $y \in \partial U^*$ then $\|y\| = R^* + 1$, however, by using the steps in the proof of Theorem 4 then

$$\|Ty(t)\| \leqslant R^*$$

Thus, this proves there is no $y \in \partial U^*$ such that $y = \lambda Ty$ for all $\lambda \in (0,1)$. In turn, the Nonlinear Alternative Theorem implies there exists at least one fixed point, $y \in U^*$ which satisfies the equivalent integral representation and thus is a solution to the fractional BVP. □

The results above technically deal with the scalar Riemann-Liouville fractional derivative, however, they naturally hold for systems of fractional derivatives directly from the work herein with the appropriate changes in the Banach spaces and norms.

THEOREM 7. *If the fractional boundary value problem* (1), (2) *is a system of equations and either the conditions of Theorem* 3 *or* 4 *hold then there exists at least one solution to the fractional BVP system* (1), (2).

5. Examples

This is the final section of the paper where we illustrate the applicability and usefulness of the results, and in particular where the function f is not Lipschitz or bounded.

EXAMPLE 1. Consider the fractional BVP given by

$$(D_a^{p+1}y)(t) = y^3(t) + ty(t), \quad 0 < t < b \tag{11}$$

$$\lim_{t\to 0^+}(D_a^p y)(t) = 0,\ y(b) = 1 \tag{12}$$

where $0 < p < 1$ and $b > 0$. Here $f(t,u) := u^3 + tu$ and is a continuous function. It suffices to show that there exists non-negative constants V, W such that the inequality (10) is satisfied. If we choose $V = \frac{1}{2}, W = \frac{(1+b)}{2}$ then

$$\begin{aligned} \|f(t,u)\| = |u^3 + tu| &= |u|(u^2 + t) \\ &\leqslant u^4 + 1/2 + t(u^2 + 1/2) \\ &= u^4 + tu^2 + 1/2(1+b) = 2Vuf(t,u) + W \end{aligned}$$

Thus, all the conditions of Theorem 5 are satisfied and there exists at least one solution to the BVP (11), (12) with

$$\|y\| \leqslant \frac{3}{2} + \frac{(1+b)b^{p+1}}{2\Gamma(p+2)}.$$

The next example illustrates another BVP that has unrestricted growth and is not Lipschitz.

EXAMPLE 2. Consider the fractional BVP given by

$$(D_a^{p+1}y)(t) = (e^{y(t)} + \sinh(y(t)) - 1), \quad 0 < t < 2 \tag{13}$$

$$\lim_{t \to 0^+} (D_0^p y)(t) = 1, \; y(2) = 20 \tag{14}$$

where $0 < p < 1$. Here $f(t,u) := (e^u + \sinh(u) - 1)$ and is a continuous function. It suffices to show that there exists non-negative constants V, W such that the inequality (10) is satisfied. If we choose $V = 1/2, W = 1 + \sinh(1)$ then

$$\begin{aligned} \|f(t,u)\| = |(e^u + \sinh(u) - 1)| &\leqslant |e^u - 1| + |\sinh(u)| \\ &\leqslant u(e^u - 1) + 1 + u\sinh(u) + \sinh(1) \\ &= 2Vuf(t,u) + W \end{aligned}$$

Thus, all the conditions of Theorem 6 are satisfied and there exists at least one solution to the BVP (13), (14) with

$$\|y\|_* \leqslant 22 + \frac{1}{4}\left[[20 - (2)]^2 + \frac{(2)^2}{p(p+1)} \right] + \frac{1 + \sinh(1)}{\Gamma(p+2)} (2)^{p+1}.$$

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