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ON THE BACKWARD PROBLEMS IN TIME FOR TIME–FRACTIONAL SUBDIFFUSION EQUATIONS

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Abstract. The backward problem for subdiffusion equation with the fractional Riemann-Liouville time-derivative of order $\rho \in (0,1)$ and an arbitrary positive self-adjoint operator A is considered. This problem is ill-posed in the sense of Hadamard due to the lack of stability of the solution. Nevertheless, we will show that if we consider sufficiently smooth current information, then the solution exists and it is unique. Using this result, we study the inverse problem of initial value identification for subdiffusion equation. The results obtained differ significantly from the corresponding results for the classical diffusion equation (i.e. $\rho = 1$) and even for the subdiffusion equation with the Caputo derivative. A list of examples of operator A is discussed, including linear systems of fractional differential equations, differential models with involution, fractional Sturm-Liouville operators, and many others.

1. Main results

The phenomenon of diffusion that does not satisfy the classical Newton's laws is called anomalous diffusion (subdiffusion or superdiffusion). Since integer-order diffusion equations cannot accurately describe anomalous diffusion phenomena in different fields, and fractional derivatives have the advantages of memory, they can more accurately describe these anomalous diffusion phenomena (see, for example, [1]–[5]). Modeling of subdiffusion processes is carried out by replacing the first time-derivative by a fractional one (of the order of $\alpha \in (0,1)$) in the classical diffusion equations, and the resulting equation is called the subdiffusion equation.

When considering subdiffusion equation as model equation in analyzing anomalous diffusion processes, the data of the model such as the initial data, the diffusion coefficient, the source term or even the order of derivative α are not all known, this leads to study a fractional inverse problem. For this reason, an additional measurement data is required to deal with this type of problems. It should be noted, that numerous contributions are introduced to resolve various fractional inverse problems (see, for example, [6]–[17]).

In this paper, we first consider the backward problem for subdiffusion equation with Riemann-Liouville fractional derivative in time and an arbitrary positive selfadjoint operator, having a discrete spectrum. At the end of the article, we will make

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the necessary remarks that allow us to consider self-adjoint operators with an arbitrary spectrum. It may be worth mentioning that this problem is ill-posed in the sense of Hadamard due to the lack of stability of the solution. Nevertheless, we will show that if we consider sufficiently smooth current information, then the solution exists and is unique. Using this result, we study the inverse problem of initial value identification for subdiffusion equation.

Let us move on to an accurate description of the research objects and formulate the main results of the work.

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $||\cdot||$ and $A: H \to H$ be an arbitrary unbounded positive selfadjoint operator in H. Suppose that A has a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of nonnegative eigenvalues λ_k. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0 < \lambda_1 \leqslant \lambda_2 \cdots \to +\infty$.

Using the definitions of a strong integral and a strong derivative, fractional analogues of integrals and derivatives can be determined for vector-valued functions (or simply functions) $h: \mathbb{R}_+ \to H$, while the well-known formulae and properties are preserved (see, for example, [18]). Recall that the fractional integration of order $\sigma < 0$ of the function $h(t)$ defined on $[0,\infty)$ has the form

$$\partial_t^\sigma h(t) = \frac{1}{\Gamma(-\rho)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler's gamma function. Using this definition one can define the Riemann-Liouville fractional derivative of order ρ, $0 < \rho < 1$, as

$$\partial_t^\rho h(t) = \frac{d}{dt} \partial_t^{\rho-1} h(t).$$

If in this definition we interchange differentiation and fractional integration, then we get the definition of the regularized derivative, that is, the definition of the fractional derivative in the sense of Caputo:

$$D_t^\rho h(t) = \partial_t^{\rho-1} \frac{d}{dt} h(t).$$

Note that if $\rho = 1$, then fractional derivatives coincides with the ordinary classical derivative of the first order: $\partial_t h(t) = D_t h(t) = \frac{d}{dt} h(t)$.

Let $\rho \in (0,1)$ be a fixed number and let $C((a,b);H)$ stand for a set of continuous functions $u(t)$ of $t \in (a,b)$ with values in H. The space $C^\infty((a,b);H)$ is defined similarly.

Consider the Cauchy type problem with inverse time:

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = f(t), \quad 0 < t < T; \\ u(T) = \Phi, \end{cases} \tag{1}$$

where $\Phi \in H$ and $f(t) \in C((0,T);H)$ are given vectors. This problem is called *the backward problem* (see, for example, [14]–[16]).

DEFINITION 1. A function $u(t) \in C((0,T];H)$ with the properties $\partial_t^\rho u(t), Au(t) \in C((0,T);H)$ and satisfying conditions (1) is called *the solution* of the backward problem (1).

The standard formulation of the Cauchy problem for equation (1) has the form:

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = f(t), \quad 0 < t < T; \\ \lim\limits_{t\to 0} \partial_t^{\rho-1} u(t) = \varphi, \end{cases} \tag{2}$$

where $\varphi \in H$ is a given vector. This problem will be called *the forward problem*. The solution to this problem is defined similarly to the solution to the backward problem. In order to investigate the backward problem, one usually uses the properties of the solution to the forward problem.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^\tau g = \sum_{k=1}^{\infty} \lambda_k^\tau g_k v_k,$$

where g_k is the Fourier coefficients of a function $g \in H$: $g_k = (g, v_k)$. Obviously, the domain of this operator has the form

$$D(A^\tau) = \{g \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |g_k|^2 < \infty\}.$$

For elements of $D(A^\tau)$ we introduce the norm

$$||g||_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} |g_k|^2 = ||A^\tau g||^2.$$

We first prove the existence and uniqueness of a solution of problem (1).

THEOREM 1. *Let $f(t) \equiv 0$. Then for any $\Phi \in D(A^2)$ problem (1) has a unique solution. Moreover there exist constants $C_1, C_2 > 0$, such that*

$$C_1 \lim_{t\to 0} ||\partial_t^{\rho-1} u(t)|| \leqslant ||u(T)||_2 \leqslant C_2 \lim_{t\to 0} ||\partial_t^{\rho-1} u(t)||. \tag{3}$$

REMARK 1. If, for example, A is an elliptic operator of the second order, then in order for a solution to the backward problem (1) to exist, Φ must have four derivatives.

The backward problems for the diffusion process are of great importance in engineering fields and are aimed at determining the previous state of a physical field (for example, at $t = 0$) based on its current information (see, for example, [15]). However, regardless of the fact that the Riemann-Liouville or Caputo derivative is taken into the equation, this problem is ill-possed in the sense of Hadamard. In other words, a small change of $u(T)$ in the norm of space H leads to large changes in the initial data. As can be seen from the above theorem, the situation changes if we take the norm of space

$D(A^2)$ instead of the norm in H. It should also be noted, that for the backward problem of the classical diffusion equation (that is $\rho = 1$) estimates of the type (3) on the scales of spaces $D(A^a)$ are generally impossible (see, for example, Chapter 8.2 of [23]).

In case of the Caputo derivative D_t^ρ the problem (1) for various elliptic differential operators A has been considered by a number of authors. Let us mention only some of these works. For the case of the second order symmetric elliptic operator A, Sakamoto and Yamamoto [15] establish the unique existence of weak solutions and the asymptotic behavior as time t goes to ∞. They also prove the stability in the backward problem in time and the uniqueness in determining an initial value. Nonsymmetric case was considered in Florida, Li, Yamamoto [16]. Since the problem is ill-possed, many authors have considered various regularization options for finding the initial condition (see, for one-dimensional elliptical part, Liu and Yamamoto [14], for the nonlinear case, Tuan, Huynh, Ngoc, and Zhou [17]). In particular, as for numerical approaches, see Tuan, Long and Tatar [19], Wang and Liu [20] and the references therein.

For backward problem (1) with non-homogeneous term we have the following result (for the Caputo derivative D_t^ρ see the above mentioned work [16]):

THEOREM 2. *Let $t^{1-\rho} f(t) \in C([0,T]; D(A^{1+\varepsilon}))$ with some $\varepsilon > 0$. Then for any $\Phi \in D(A^2)$ problem (1) has a unique solution. Moreover there exists a constant $C > 0$, such that*

$$\lim_{t\to 0} ||\partial_t^{\rho-1} u(t)|| \leqslant C\Big(||u(T)||_2 + \max_{t\in[0,T]} ||t^{1-\rho} f(t)||_{1+\varepsilon}\Big). \tag{4}$$

The remainder of this paper is composed of three sections. Section 2 is devoted to the study of forward problem (2). In Section 3, we show that problem (1) is ill-possed in the sense of Hadamard and prove Theorems 1 and 2. In the last section, examples of the operator A are presented. In addition, the necessary remarks are given, with the help of which all the statements formulated can be translated to the case when the operator A has an arbitrary spectrum.

Finally we note, that to investigate the forward and backward problems we borrow some original ideas from papers [15], [16], where authors studied the similar problems for equation (1) with the Caputo derivative.

2. Forward problem

THEOREM 3. *Let $\varphi \in H$ and $t^{1-\rho} f(t) \in C([0,T]; D(A^\varepsilon))$ for some $\varepsilon \in (0,1)$. Then there exists a unique solution to the forward problem, such that*

$$\begin{cases} ||u(t)||_1 + ||\partial_t^\rho u(t)|| \leqslant C_\varepsilon\Big(t^{-1-\rho}||A^{-1}\varphi|| + \max\limits_{t\in[0,T]} ||t^{1-\rho} f(t)||_\varepsilon + ||f(t)||\Big), \quad t > 0, \\ \max\limits_{t\in[0,T]} ||t^{1-\rho} u(t)|| \leqslant C(||\varphi|| + \max\limits_{t\in[0,T]} ||t^{1-\rho} f(t)||), \end{cases} \tag{5}$$

where C is an absolute constant and C_ε is a constant depending on ε.

Moreover, if $t^{1-\rho}f(t) \in C([0,T];D(A^{1+\varepsilon}))$, *then there exists a constant* C, *depending on* ε *such that*

$$||u(t)||_2 \leqslant C(t^{-1-\rho}||\varphi|| + \max_{t\in[0,T]} ||t^{1-\rho}f(t)||_{1+\varepsilon}), \quad t > 0. \tag{6}$$

Here is an obvious consequence of estimate (6):

COROLLARY 1. *Let* $\varphi \in H$ *and* $t^{1-\rho}f(t) \in C([0,T];D(A^{1+\varepsilon}))$. *Then there exists a constant* C, *depending on* T *and* ε *such that*

$$||u(T)||_2 \leqslant C(||\varphi|| + \max_{t\in[0,T]} ||t^{1-\rho}f(t)||_{1+\varepsilon}). \tag{7}$$

THEOREM 4. *Let* $\varphi \in H$ *and* $f \equiv 0$. *The the unique solution to the forward problem is infinitely differentiable with respect to the variable* t, *i.e.*

$$u(t) \in C^\infty((0,\infty);H),$$

and there exists a constant C *such that the following estimates are valid*

$$\begin{cases} ||u(t)|| \leqslant \dfrac{Ct^{\rho-1}}{1+(\lambda_1 t^\rho)^2} \cdot ||\varphi||, \quad t > 0, \\ ||\partial_t^m u(t)|| \leqslant Ct^{\rho-1-m} \cdot ||\varphi||, \quad t > 0, \quad m \in \mathbb{N}. \end{cases} \tag{8}$$

We note at once that the solution to equation (1), generally speaking, is not continuous at the point $t = 0$ (see (5)). Of course, we can consider a continuous function at the point $t = 0$ as the right-hand side of the equation, but we assume that $f(t)$ would have a singularity at this point in order to cover a more general case (see (5) and (6)).

Initial-boundary value problems for various subdiffusion equations have been investigated by many specialists. Let us mention only some of these works. In the book of A.A. Kilbas et al. [2] (Chapter 6) there is a survey of works published before 2006. The case of one spatial variable $x \in \mathbb{R}$ and subdiffusion equation with ”the elliptical part” u_{xx} were considered for example in the monograph of A. V. Pskhu [24] (Chapter 4, see references thesein). The paper Gorenflo, Luchko and Yamamoto [25] is devoted to the study of subdiffusion equations in Sobelev spaces. In the paper by Kubica and Yamamoto [26], initial-boundary value problems for equations with time-dependent coefficients are considered. In the multidimensional case ($x \in \mathbb{R}^N$), instead of the differential expression u_{xx}, authors considered either the second order elliptic operator ([27]–[29]) or elliptic pseudodifferential operators with constant coefficients in the whole space $\mathbb{R}^N$ (Umarov [30]). In the paper of Yu. Luchko [28] the author constructed solutions by the eigenfunction expansion in the case of $f = 0$ and discussed the unique existence of the generalized solution to problem (2) with the Caputo derivative. The authors of the recent paper [31] considered initial-boundary value problems for subdiffusion equations with arbitrary elliptic differential operators in bounded domains.

The formulated results for equation (1) with the Caputo derivative were previously proved in [15] (in the case when A is a symmetric elliptic operator of the second order) and [16] (in the case when A is not symmetric).

Let us compare our results (equation (1) with the Riemann-Liouville derivative) with the results of [15] and [16] (equation (1) with the Caputo derivative) and standard results for the case of $\rho = 1$.

In our case we have no smoothing properties like the case of the Caputo and the classical diffusion equation (i.e. $\rho = 1$). In Theorem 5 (estimate (6) there is the smoothing property in space with order 2 which means that $u(t) \in D(A^2)$ for any $t > 0$ and any $\varphi \in H$. For example, if A is an elliptic operator of order two, defined in N-dimensional domain Ω, then the condition $\varphi \in L_2(\Omega)$ guarantee that the solution to problem (2) is in the classical Sobolev space $W_2^4(\Omega)$. Nevertheless, as it is proved in Theorem 4 the regularity in time immediately becomes stronger in t, and is of infinity order (i.e., u is of C^∞ for $t > 0$). In Theorem 1, it is showed that the smoothing in $D(A^2)$ is the best possible and the solution cannot be smoother than $D(A^2)$ at $t > 0$ if $\varphi \in H$.

For the case of the Caputo derivative, in papers [15] and [16] it is proved that the best possible smoothing property is of order 1, i.e. $u(t) \in D(A)$ for any $t > 0$ and any $\varphi \in H$, while in the classical case ($\rho = 1$) the solution is infinitely differentiable both with respect to spatial variables and t with any $t > 0$ and $\varphi \in H$.

The first estimate (8) shows the decay of solution with order $t^{-\rho-1}$ as $t \to \infty$, which is slower than the exponential decay in the case of $\rho = 1$, but faster than the Caputo case with the decay of $t^{-\rho}$ (see [15]).

Proof of Theorem 5. Let us introduce the following formal series

$$u(t) = \sum_{k=1}^{\infty} \Big[t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)\varphi_k + \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta)\,d\eta\Big] v_k, \qquad (9)$$

where φ_k and $f_k(t)$ are the Fourier coefficients of φ and $f(t)$ correspondingly, $E_{\rho,\mu}(t)$ – the Mittag-Leffler function:

$$E_{\rho,\mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)}.$$

By virtue of the formula ([24], p. 104)

$$\lim_{t\to+0} \partial_t^{\alpha-1} u(t) = \Gamma(\alpha) \lim_{t\to+0} t^{1-\alpha} u(t). \qquad (10)$$

one can easily verify that the function (9) formally satisfies the conditions of problem (2) (see, for example, [33], p. 173). In order to prove that function (9) is actually a solution to the problem, it remains to substantiate this formal statement, i.e. show that the operators A and ∂_t^ρ can be applied term by term to the series (9).

To do this we need the asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument. The well known estimate has the form (see, for example, [32], p. 136)

$$|E_{\rho,\mu}(-t)| \leqslant \frac{C}{1+t}, \quad t > 0,$$

where μ is an arbitrary complex number. But for $E_{\rho,\rho}(-t)$ one can get a better estimate. Indeed, using the asymptotic estimate (see, for example, [32], p. 134)

$$E_{\rho,\rho}(-t) = -\frac{t^{-2}}{\Gamma(-\rho)} + O(t^{-3}). \tag{11}$$

and the fact that $E_{\rho,\rho}(t)$ is real analytic, we can obtain the following inequality

$$|E_{\rho,\rho}(-t)| \leqslant \frac{C}{1+t^2}, \quad t>0. \tag{12}$$

We will also use a coarser estimate with positive eigenvalues λ_k and $0<\varepsilon<1$:

$$|t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)| \leqslant \frac{Ct^{\rho-1}}{1+(\lambda_k t^\rho)^2} \leqslant C\lambda_k^{\varepsilon-1}t^{\varepsilon\rho-1}, \quad t>0, \tag{13}$$

which is easy to verify. Indeed, let $t^\rho\lambda_k<1$, then $t<\lambda_k^{-1/\rho}$ and

$$t^{\rho-1} = t^{\rho-\varepsilon\rho}t^{\varepsilon\rho-1} < \lambda_k^{\varepsilon-1}t^{\varepsilon\rho-1}.$$

If $t^\rho\lambda_k \geqslant 1$, then $\lambda_k^{-1}\leqslant t^\rho$ and

$$\lambda_k^{-2}t^{-\rho-1} = \lambda_k^{-1+\varepsilon}\lambda_k^{-1-\varepsilon}t^{-\rho-1} \leqslant \lambda_k^{\varepsilon-1}t^{\varepsilon\rho-1}.$$

Let $S_j(t)$ be the partial sum of series (9). Then

$$AS_j(t) = \sum_{k=1}^{j}\Big[t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)\varphi_k + \int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta\Big]\lambda_k v_k.$$

Due to the Parseval equality we may write

$$||AS_j(t)||^2 = \sum_{k=1}^{j}\lambda_k^2\Big|t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)\,\varphi_k + \int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta\Big|^2.$$

Using estimate (12) and the inequality $\lambda_j t^{\rho-1}(1+(\lambda_k t^\rho)^2)^{-1} < \lambda_k^{-1}t^{-\rho-1}$ we obtain

$$\sum_{k=1}^{j}\lambda_k^2|t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)\,\varphi_k|^2 \leqslant Ct^{-2(1+\rho)}\sum_{k=1}^{j}\lambda_k^{-2}|\varphi_k|^2 = Ct^{-2(1+\rho)}||A^{-1}\varphi||^2.$$

On the other hand, by inequality (13) for $0<\varepsilon<1$ one has

$$\sum_{k=1}^{j}\lambda_k^2\Big|\int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta\Big|^2 \leqslant C\sum_{k=1}^{j}\Big[\int_0^t \eta^{\varepsilon\rho-1}\lambda_k^\varepsilon|f_k(t-\eta)|d\eta\Big]^2$$

(by virtue of the generalized Minkowski inequality)

$$\leqslant C\Big[\int_0^t \eta^{\varepsilon\rho-1}(t-\eta)^{\rho-1}\Big(\sum_{k=1}^{j}|\lambda_k^\varepsilon(t-\eta)^{(1-\rho)}f_k(t-\eta)|^2\Big)^{\frac{1}{2}}d\eta\Big]^2$$

$$\leqslant C_\varepsilon \max_{t\in[0,T]}||t^{1-\rho}f(t)||_\varepsilon^2.$$

Therefore

$$||AS_j(t)||^2 \leqslant Ct^{-2(1+\rho)}||A^{-1}\varphi||^2 + C_\varepsilon \max_{t\in[0,T]} ||t^{1-\rho} f(t)||_\varepsilon^2, \quad t>0. \tag{14}$$

Hence, we obtain $Au(t) \in C((0,T];H)$.

Further, from equation (1) one has $\partial_t^\rho S_j(t) = -AS_j(t) + \sum_{k=1}^{j} f_k(t) v_k$. Therefore, from above reasoning, we have $\partial_t^\rho u(t) \in C((0,T];H)$ and

$$||\partial_t^\rho S_j(t)||^2 \leqslant Ct^{-2(1+\rho)}||A^{-1}\varphi||^2 + C_\varepsilon \max_{t\in[0,T]} ||t^{1-\rho} f(t)||_\varepsilon^2 + ||f(t)||^2, \quad t>0. \tag{15}$$

Thus, we have completed the rationale that (9) is a solution to the forward problem. Inequalities (14) and (15) imply the first estimate (5). The second estimate (5) is proved using similar reasoning.

Now let $t^{1-\rho} f(t) \in C([0,T];D(A^{1+\varepsilon}))$. Then using estimate (12) and the inequality $\lambda_j^2 t^{\rho-1}(1+(\lambda_k t^\rho)^2)^{-1} < t^{-\rho-1}$ we obtain

$$\sum_{k=1}^{j} \lambda_k^4 \left| t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)\, \varphi_k \right|^2 \leqslant Ct^{-2(\rho+1)}||\varphi||, \quad t>0.$$

On the other hand, by inequality (13) we have

$$\sum_{k=1}^{j} \lambda_k^4 \left| \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right|^2 \leqslant C \sum_{k=1}^{j} \left[\int_0^t \eta^{\varepsilon\rho-1} \lambda_k^{1+\varepsilon} |f_k(t-\eta)| d\eta \right]^2$$
$$\leqslant C_\varepsilon \max_{t\in[0,T]} ||t^{1-\rho} f(t)||_{1+\varepsilon}^2.$$

Therefore

$$||A^2 S_j(t)|| \leqslant Ct^{-2(\rho+1)}||\varphi|| + C_\varepsilon \max_{t\in[0,T]} ||t^{1-\rho} f(t)||_{1+\varepsilon}^2, \quad t>0.$$

This implies estimate (6).

We now turn to the proof of the the uniqueness of the forward problem's solution.

Suppose that problem (1) has two solutions $u_1(t)$ and $u_2(t)$. Our aim is to prove that $u(t) = u_1(t) - u_2(t) \equiv 0$. Since the problem is linear, then we have the following homogenous problem for $u(t)$:

$$\partial_t^\rho u(t) + Au(t) = 0, \quad t>0; \tag{16}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(t) = 0. \tag{17}$$

Set

$$w_k(t) = (u(t), v_k).$$

It follows from (16) that for any $k \in \mathbb{N}$

$$\partial_t^\rho w_k(t) = (\partial_t^\rho u(t), v_k) = -(Au(t), v_k) = -(u(t), Av_k) = -\lambda_k w_k(t).$$

Therefore, we have the following Cauchy problem for $w_k(t)$ (see (17)):

$$\partial_t^\rho w_k(t) + \lambda_k w_k(t) = 0, \quad t > 0; \quad \lim_{t\to 0} \partial_t^{\rho-1} w_k(t) = 0.$$

This problem has the unique solution (see, for example, [33], p. 173, [21] and [22]). Therefore, $w_k(t) = 0$ for $t > 0$ and for all $k \geqslant 1$. Then by the Parseval equation we obtain $u(t) = 0$ for all $t > 0$. Hence uniqueness of the solution is proved.

Thus the proof of Theorem 5 is complete. □

Proof of Theorem 4. It is sufficient to prove the estimates (8) since the infinitely differentiability of the solution follows from the second estimate (8).

The estimate of the Mittag-Leffler function (12) implies for $t > 0$ the first inequality (8):

$$||u(t)||^2 = \sum_{k=1}^\infty [t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)\varphi_k]^2 \leqslant \sum_{k=1}^\infty |\varphi_k|^2 \left(\frac{Ct^{\rho-1}}{1+(\lambda_k t^\rho)^2}\right)^2 \leqslant \left(\frac{Ct^{\rho-1}}{1+(\lambda_1 t^\rho)^2}||\varphi||\right)^2.$$

To prove the second estimate we remind the following differentiation formula for the Mittag-Leffler function (see, for example, [33], formula (4.3.1))

$$\left(\frac{d}{dz}\right)^m \left[z^{\mu-1} E_{\alpha,\mu}(z^\alpha)\right] = z^{\mu-1-m} E_{\alpha,\mu-m}(z^\alpha), \quad z \neq 0, \quad m \in \mathbb{N},$$

which is an immediate consequence of the definition of the Mittag-Leffler function $E_{\alpha,\mu}$. Therefore one has

$$\left(\frac{d}{dt}\right)^m \left[t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)\right] = t^{\rho-1-m} E_{\rho,\rho-m}(-\lambda_k t^\rho), \quad t > 0, \quad m \in \mathbb{N},$$

By this formula, we have (note $\partial_t^m = d^m/dt^m$)

$$\partial_t^m u(t) = \sum_{k=1}^\infty t^{\rho-1-m} E_{\rho,\rho-m}(-\lambda_k t^\rho)\varphi_k v_k, \quad t > 0,$$

for $m \in \mathbb{N}$ and estimate (12) implies

$$||\partial_t^m u(t)|| \leqslant Ct^{\rho-1-m} \cdot ||\varphi||, \quad t > 0, \quad m \in \mathbb{N}.$$

The proof of Theorem 4 is completed. □

3. Backward problem

In the case $\rho = 1$ problem (1) is called (see, for example, [23], p. 214) the inverse heat conduction problem with inverse time (retrospective inverse problem). This problem is ill-posed. Usually, when investigating such inverse problems, they are reduced by changing the variables $\tau = T - t$ to an equivalent forward problem, where the sign changes at the time derivative ([23], p. 214). However, this approach cannot be implemented for problem (1) since the simple property of derivatives $\frac{d}{dt} = -\frac{d}{d\tau}$ does not hold for fractional derivatives: $\partial_t^\rho \neq -\partial_\tau^\rho$.

Problem (1) is also ill-posed in the sense of Hadamard because of the same reason as the classical one ($\rho = 1$): a small variation of $u(T)$ in the norm of space H may cause arbitrarily large variations in the initial data. Indeed, let $f(t) \equiv 0$ and $u(T) = \lambda_k^{-2+\varepsilon} v_k$, $\varepsilon > 0$, in problem (1). Then the unique solution of the problem is

$$u(t) = \lambda_k^{-2+\varepsilon} \cdot \frac{t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)}{T^{\rho-1} E_{\rho,\rho}(-\lambda_k T^\rho)} \cdot v_k$$

and

$$\lim_{t\to+0} \partial_t^{\rho-1} u(t) = \lambda_k^{-2+\varepsilon} \cdot \frac{1}{T^{\rho-1} E_{\rho,\rho}(-\lambda_k T^\rho)} \cdot v_k.$$

Therefore, on the one hand, $||u(T)|| = \lambda_k^{-2+\varepsilon}$ and it tends to zero as $k \to \infty$ (even $||u(T)||_a \to 0$ for any $a < 2 - \varepsilon$), and on the other, according to the asymptotic estimate (11),

$$|| \lim_{t\to+0} \partial_t^{\rho-1} u(t)|| = \lambda_k^{-2+\varepsilon} \cdot \frac{1}{T^{\rho-1} E_{\rho,\rho}(-\lambda_k T^\rho)} \to \infty \quad \text{when} \quad k \to \infty.$$

However, if we consider the norm of $u(T)$ in space $D(A^2)$, then the situation will change completely; note the norm $||u(T)||_2 = \lambda_k^\varepsilon$ in this example is unbounded as $k \to \infty$.

We now turn to the proof of Theorems 1 and 2.

Proof of Theorem 1. Since function $E_{\rho,\rho}(z)$ has no negative zero (see, for example, [33], p. 74) and $E_{\alpha,\alpha}(0) = \Gamma^{-1}(\rho) > 0$, then

$$E_{\rho,\rho}(-t) > 0, \quad t \geqslant 0. \tag{18}$$

Let $\Phi \in D(A^2)$ and Φ_k be its Fourier coefficients. Then

$$||\Phi||_2 = \sum_{k=1}^{\infty} \lambda_k^4 |\Phi_k|^2 < \infty.$$

By (18) we can set

$$\varphi_k = \frac{\Phi_k}{T^{\rho-1} E_{\rho,\rho}(-\lambda_k T^\rho)}.$$

Then by virtue of the asymptotic estimate (11) one has

$$\sum_{k=1}^{\infty}\varphi_k^2=\sum_{k=1}^{\infty}\frac{\Phi_k^2}{(T^{\rho-1}E_{\rho,\rho}(-\lambda_k T^{\rho}))^2}=\sum_{k=1}^{\infty}T^{2\rho+2}\lambda_k^4\Gamma^2(-\rho)\Phi_k^2\left(\frac{1}{1+O(\lambda_k^{-3}T^{-3\rho})}\right)^2$$
$$\leqslant CT^{2\rho+2}\sum_{k=1}^{\infty}\lambda_k^4|\Phi_k|^2<\infty.$$

Therefore

$$\varphi=\sum_{k=1}^{\infty}\varphi_k v_k\in H,$$

and the following function (see (9))

$$u(t)=\sum_{k=1}^{\infty}t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^{\rho})\varphi_k v_k$$

is the unique solution to forward problem (2) with $f(t)\equiv 0$ and the initial function φ. Moreover $u(T)=\Phi$ and

$$\lim_{t\to 0}||t^{1-\rho}u(t)||=||\varphi||\leqslant C||\Phi||_2=C||u(T)||_2.$$

The second inequality in (3) is already proved in Theorem 5 (estimate (6)).

Theorem 1 is proved. □

Proof of Theorem 2. Consider the following two auxiliary problems (see [16]):

$$\begin{cases}\partial_t^{\rho}v(t)+Av(t)=f(t), & 0<t<T;\\ \lim\limits_{t\to 0}\partial_t^{\rho}v(t)=0,\end{cases}\tag{19}$$

and

$$\begin{cases}\partial_t^{\rho}w(t)+Aw(t)=0, & 0<t<T;\\ w(T)=\Phi-v(T),\end{cases}\tag{20}$$

If $t^{1-\rho}f(t)\in C([0,T];D(A^{1+\varepsilon}))$, then there exists the unique solution to problem (19) and (see Corollary 1)

$$||v(T)||_2\leqslant C\max_{t\in[0,T]}||t^{1-\rho}f(t)||_{1+\varepsilon}.\tag{21}$$

If $\Phi\in D(A^2)$, then there exists the unique solution to problem (20) and (see Theorem 1)

$$\lim_{t\to 0}||\partial_t^{\rho-1}w(t)||\leqslant C||w(T)||_2.\tag{22}$$

Setting $u=v+w$, we see that $u(T)=\Phi-v(T)+v(T)=\Phi$. Then we can verify that u is the unique solution to problem (1) and the estimates (21) and (22) imply

$$\lim_{t\to 0}||\partial_t^{\rho-1}u(t)||=\lim_{t\to 0}||\partial_t^{\rho-1}w(t)||\leqslant||w(T)||_2\leqslant C(||\Phi||_2+||v(T)||_2)$$
$$\leqslant C(||\Phi||_2+\max_{t\in[0,T]}||t^{1-\rho}f(t)||_{1+\varepsilon}).$$

Theorem 2 is proved. □

4. Examples of operator A and further generalization

The setting of an abstract operator A as in this paper allows one to include many models. For example, as a example one may consider any of physical examples, discussed in Section 6 of the paper of M. Ruzhansky et al. [12], including Sturm-Liouville problems, differential models with involution, fractional Sturm-Liouville operators, harmonic and anharmonic oscillators, Landau Hamiltonians, fractional Laplacians, and harmonic and anharmonic operators on the Heisenberg group. It should be noted, that the authors of [12] considered a class of inverse problems for restoring the right-hand side of a subdiffusion equation with the Caputo derivatives of order $0 < \rho \leqslant 1$ for a large class of positive operators with discrete spectrum.

Usually, when studying the subdiffusion equation, an elliptic equation of order two on a N-dimensional bounded domain Ω with classical boundary conditions, such as Dirichlet, is considered as the elliptic part. The system of eigenfunctions of such operators constitutes a complete set in $L_2(\Omega)$, and the spectrum is discrete and rather regular, that is, $N(\lambda) = \sum_{\lambda_k \leqslant \lambda} 1$ – the number of eigenvalues not exceeding λ has the estimate $N(\lambda) = O(\lambda^{N/2})$. However, for example, for the Laplace operator in a bounded domain Ω, boundary conditions can be specified such that the system of eigenfunctions remains complete in $L_2(\Omega)$, but the set $\{\lambda_k\}$ will be dense in $(1,+\infty)$ (see [34]). It should be noted that the theorems formulated above are also valid for such operators.

On the other hand, not all operators important for applications have a discrete spectrum. For example, if A is the Laplace operator $-\triangle$ in $H = L_2(\mathbb{R}^N)$, then the spectrum of this operator is continuous. A natural question arises: is it possible to apply the above reasoning to the case of the operator A with continuous spectrum? The answer to this question is yes and similar theorems are true as above. Moreover, there is no need to make significant changes to the corresponding proofs, except for the proof of the uniqueness of the solution to the forward problem. Below we give a proof of uniqueness in the general case.

Let H be a Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $||\cdot||$ and $A : H \to H$ be an arbitrary semibounded (with the bound $\mu > 0$) selfadjoint operator in H. By von Neumann's spectral theorem, the operator A has a partition $\{P_\lambda\}$ of unity, and can be represented in the form of

$$A = \int_{\mu}^{\infty} \lambda \, dP_\lambda, \quad \mu > 0.$$

The projections P_λ increase monotonically, are continuous on the left, and tend strongly to the unit operator, that is,

$$\lim_{\lambda\to\infty} ||P_\lambda g - g|| = 0, \quad g \in H.$$

For any real number τ the power of operator A is defined as

$$A^\tau g = \int_{\mu}^{\infty} \lambda^\tau \, dP_\lambda g, \quad g \in D(A^\tau) = \{g \in H : \int_{\mu}^{\infty} \lambda^{2\tau} (dP_\lambda g, g) < \infty\}.$$

Consider the forward problem (2) with this operator A. Let us define the space $D(A^\varepsilon)$ as above and let $\varphi \in H$ and $t^{1-\rho} f(t) \in C([0,T];D(A^\varepsilon))$ for some $\varepsilon \in (0,1)$. Then it is not hard to verify that the following function

$$u(t) = \int_\mu^\infty t^{\rho-1} E_{\rho,\rho}(-\lambda t^\rho)\, dP_\lambda \varphi + \int_0^t \int_\mu^\infty \eta^{\rho-1} E_{\rho,\rho}(-\lambda \eta^\rho)\, dP_\lambda f(t-\eta) d\eta$$

is the solution to the forward problem (2).

Next, we will show that problem (2) has a unique solution.

Let U be the spectral representation of H on the direct sum $\bigoplus_{\beta\in B} L_2(\mathbb{R}_\mu, \mu_\beta)$ $(\mathbb{R}_\mu = (\mu, +\infty))$ with respect to self-adjoint operator A (see [35], Chapter XII, Sec. 3, Theorem 5). Operator U is a linear map of H onto all space $\bigoplus_{\beta\in B} L_2(\mathbb{R}_\mu, \mu_\beta)$ and preserves the scalar product, that is, it is a unitary operator.

Note that

$$\xi \in \bigoplus_{\beta\in B} L_2(\mathbb{R}_\mu, \mu_\beta)$$

means that

$$\xi = \{\xi_\beta(\lambda)\}_{\beta\in B}, \quad \text{where} \quad \xi_\beta : \mathbb{R}_\mu \to \mathbb{C}, \quad \text{and} \quad \xi_\beta \in L_2(\mathbb{R}_\mu, \mu_\beta).$$

Therefore,

$$\int_\mu^\infty |\xi_\beta(\lambda)|^2 d\mu_\beta(\lambda) \; < \; +\infty.$$

Besides,

$$(V\xi)_\beta(\lambda) = \lambda\, \xi_\beta(\lambda), \quad \text{where} \quad V = UAU^{-1}, \tag{23}$$

that is, the projection of the operator A onto the space $L_2(\mathbb{R}_\mu, \mu_\beta)$ acts as a product by λ.

Let $w(t)$ be the solution of the homogeneous Cauchy problem (2). Our goal is to show that $w(t) = 0$. Set

$$\xi(t) = Uw(t).$$

Here

$$\xi(t) = \{\xi_\beta(t,\lambda)\}_{\beta\in B}, \quad \text{where} \quad \xi_\beta : \mathbb{R}_0 \times \mathbb{R}_\mu \to \mathbb{C}.$$

Then

$$\begin{aligned} \partial_t^\rho \xi(t) &= \partial_t^\rho Uw(t) = U\partial_t^\rho w(t) = UAw(t) \\ &= UAU^{-1}Uw(t) = VUw(t) = V\xi(t). \end{aligned}$$

Further (see [35], Chapter XII, Sec. 3, Lemma 3 and (23)),

$$\partial_t^\rho \xi_\beta(t,\lambda) = (\partial_t^\rho \xi(t))_\beta(\lambda) = (V\xi(t))_\beta(\lambda) = \lambda \xi_\beta(t,\lambda).$$

It is clear that

$$\xi_\beta(0,\lambda) = 0.$$

Thus, $\xi_\beta(t,\lambda)$ is the solution of the following homogeneous Cauchy problem

$$\partial_t^\rho \xi_\beta(t,\lambda) = \lambda \xi_\beta(t,\lambda),$$

$$\xi_\beta(0,\lambda) = 0.$$

Consequently, $\xi_\beta(t,\lambda) \equiv 0$ for all β. Then $\xi(t) = 0$ and $w(t) = U^{-1}\xi(t) = 0$.

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