GENERALIZED FRACTIONAL OSTROWSKI TYPE
INEQUALITIES VIA $\phi - \lambda$ –CONVEX FUNCTION

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Abstract. In this paper, we would like to state well-known Ostrowski inequality via generalized Montgomery identity for the $\phi - \lambda$ -convex function. Also, we would like to state the generalization of the classical Ostrowski inequality via generalized fractional integrals, which is obtained for functions whose first derivative in absolute values is $\phi - \lambda$ -convex. Moreover we establish some Ostrowski type inequalities via generalized fractional integrals and their particular cases for the class of functions whose derivatives in absolute values at certain powers are $\phi - \lambda$ -convex by using different techniques including Hölder's inequality and power mean inequality. Also, standard results would be capture as special cases. Moreover, some applications in terms of special means would also be given.

1. Introduction

In recent years, the generalization of classical convex function have emerged resulting in applications in the field of Mathematics. From literature, we recall some definitions for different types of convex (concave).

DEFINITION 1. [2] The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex (concave), if

$$\eta(tx + (1-t)y) \leqslant (\geqslant) t\eta(x) + (1-t)\eta(y),$$

$\forall x, y \in I$, $t \in [0,1]$.

DEFINITION 2. [2] The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be MT -convex (concave), if $\eta(x) \geqslant 0$ and

$$\eta(tx + (1-t)y) \leqslant (\geqslant) \frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y),$$

$\forall x, y \in I$, $t \in [0,1]$.

DEFINITION 3. [15] The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a P-convex (concave), if $\eta(x) \geqslant 0$ and $\forall x, y \in I$ and $t \in [0, 1]$ we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant)\eta(x) + \eta(y).$$

DEFINITION 4. [17] The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin convex (concave), if $\eta(x) \geqslant 0$ and $\forall x, y \in I$ and $t \in (0, 1)$ we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant)\frac{1}{t}\eta(x) + \frac{1}{1-t}\eta(y).$$

DEFINITION 5. [3] Let $s \in [0, 1]$. The $\eta : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex (concave) in the 2nd kind, if

$$\eta(tx + (1-t)y) \leqslant (\geqslant)t^s \eta(x) + (1-t)^s \eta(y),$$

$\forall x, y \in I$, $t \in [0, 1]$.

DEFINITION 6. [9] The $\eta : I \subset \mathbb{R} \to [0, \infty)$ is of Godunova-Levin s-convex (concave), with $s \in [0, 1]$, if

$$\eta(tx + (1-t)y) \leqslant (\geqslant)\frac{1}{t^s}\eta(x) + \frac{1}{(1-t)^s}\eta(y),$$

$\forall t \in (0, 1)$ and $x, y \in I$.

DEFINITION 7. [23] Let $h : J \subseteq \mathbb{R} \to [0, \infty)$ with $h \neq 0$. The $\eta : I \subseteq \mathbb{R} \to [0, \infty)$ is an h-convex (concave) if $\forall x, y \in I$, we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant)h(t)\eta(x) + h(1-t)\eta(y),$$

$\forall t \in [0, 1]$.

DEFINITION 8. [10] Let $\phi : (0, 1) \to (0, \infty)$. The $\eta : I \to [0, \infty)$ is a ϕ-convex (concave) on the interval I if for all $x, y \in I$ we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant)t\phi(t)\eta(x) + (1-t)\phi(1-t)\eta(y),$$

$\forall t \in (0, 1)$.

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality.

THEOREM 1. [20] *Let* $\varphi : [\rho_a, \rho_b] \rightarrow \mathbb{R}$ *be differentiable on* (ρ_a, ρ_b) *with the property that* $|\varphi'(t)| \leqslant M$ *for all* $t \in (\rho_a, \rho_b)$. *Then*

$$\left| \varphi(x) - \frac{1}{\rho_b - \rho_a} \int_{\rho_a}^{\rho_b} \varphi(t)dt \right| \leqslant (\rho_b - \rho_a)M \left[\frac{1}{4} + \left(\frac{x - \frac{\rho_a + \rho_b}{2}}{\rho_b - \rho_a} \right)^2 \right], \quad (1)$$

$\forall x \in (\rho_a, \rho_b)$.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [6]–[14].

DEFINITION 9. The Riemann-Liouville integral operator of order $\zeta > 0$ with $\rho_a \geqslant 0$ is defined as

$$J_{\rho_a}^{\zeta} \varphi(x) = \frac{1}{\Gamma(\zeta)} \int_{\rho_a}^{x} (x-t)^{\zeta-1} \varphi(t)dt,$$
$$J_{\rho_a}^{0} \varphi(x) = \varphi(x). \quad (2)$$

In case of $\zeta = 1$, the fractional integral reduces to the classical integral.

DEFINITION 10. [21] The Riemann-Liouville integrals $I_{\rho_a^+}^{\zeta} \varphi$ and $I_{\rho_b^-}^{\zeta} \varphi$ of $\varphi \in L_1([\rho_a, \rho_b])$ having order $\zeta > 0$ with $\rho_a \geqslant 0$, $\rho_a < \rho_b$ are defined by

$$I_{\rho_a^+}^{\zeta} \varphi(x) = \frac{1}{\Gamma(\zeta)} \int_{\rho_a}^{x} (x-t)^{\zeta-1} \varphi(t)dt, \ x > \rho_a$$

and

$$I_{\rho_b^-}^{\zeta} \varphi(x) = \frac{1}{\Gamma(\zeta)} \int_{x}^{\rho_b} (t-x)^{\zeta-1} \varphi(t)dt, \ x < \rho_b,$$

respectively. Here $\Gamma(\zeta) = \int_0^\infty e^{-u} u^{\zeta-1} du$ is the GammThe and $I_{\rho_a^+}^{0} \varphi(x) = I_{\rho_b^-}^{0} \varphi(x) = \varphi(x)$.

DEFINITION 11. [21] Let $g : [\rho_a, \rho_b] : \mathbb{R}$ be an increasing and positive function on (ρ_a, ρ_b), having a continuous derivatives $g'(x)$ on (ρ_a, ρ_b). The fractional integrals $I_{\rho_a^+, g}^{\zeta} \varphi$ and $I_{\rho_b^-, g}^{\zeta} \varphi$ of φ with respect to the function g on $[\rho_a, \rho_b]$ of order $\zeta > 0$ are defined by

$$I_{\rho_a^+, g}^{\zeta} \varphi(x) = \frac{1}{\Gamma(\zeta)} \int_{\rho_a}^{x} (g(x) - g(t))^{\zeta-1} g'(t) \varphi(t)dt, \ x > \rho_a$$

and

$$I_{\rho_b^-, g}^{\zeta} \varphi(x) = \frac{1}{\Gamma(\zeta)} \int_{x}^{\rho_b} (g(t) - g(x))^{\zeta-1} g'(t) \varphi(t)dt, \ x < \rho_b,$$

respectively.

REMARK 1. If we replace $g(x) = x$ the above fractional integrals reduce to the Riemann-Liouville fractional integrals.

THEOREM 2. [21] *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $\rho_a, \rho_b \in I$, $\rho_a < \rho_b$ $\varphi' \in L_1[\rho_a, \rho_b]$ and for $\zeta > 1$, Montgomery identity for fractional integrals holds:*

$$\varphi(x) = \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b) - J_{\rho_a}^{\zeta-1}(P_1(x,\rho_b)\varphi(\rho_b)) + J_{\rho_a}^{\zeta}(P_1(x,\rho_b)\varphi'(\rho_b)),$$

where $P_1(x,t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t) = \begin{cases} \dfrac{t - \rho_a}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}\Gamma(\zeta), & \text{if } t \in [\rho_a, x], \\[2mm] \dfrac{t - \rho_b}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}\Gamma(\zeta), & \text{if } t \in (x, \rho_b]. \end{cases}$$

THEOREM 3. [21] *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $\rho_a, \rho_b \in I$, $\rho_a < \rho_b$ $\varphi' \in L_1[\rho_a, \rho_b]$ and for $\zeta > 1$, Generalized Montgomery identity for fractional integrals holds:*

$$(1 - \varepsilon)\varphi(x) = \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b) - J_{\rho_a}^{\zeta-1}(P_2(x,\rho_b)\varphi(\rho_b))$$

$$- \frac{\varepsilon(\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}}J_{\rho_a}^0\varphi(\rho_a) + J_{\rho_a}^{\zeta}(P_2(x,\rho_b)\varphi'(\rho_b)), \qquad (3)$$

where $P_2(x,t)$ is the fractional Peano Kernel defined by:

$$P_2(x,t) = \begin{cases} \dfrac{t - \mu}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}\Gamma(\zeta), & \text{if } t \in [\rho_a, x], \\[2mm] \dfrac{t - \nu}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}\Gamma(\zeta), & \text{if } t \in (x, \rho_b]. \end{cases}$$

$\forall x \in [\mu, \nu]$ *for $\mu = \rho_a + \varepsilon\frac{\rho_b - \rho_a}{2}$ and $\nu = \rho_b - \varepsilon\frac{\rho_b - \rho_a}{2}$.*

Throughout this paper, we will assume that $g : [\rho_a, \rho_b] \to \mathbb{R}$ is an increasing and positive function on $[\rho_a, \rho_b]$, having a continuous derivative $g'(x)$ on (ρ_a, ρ_b). In order to prove our results, we need the following Lemma.

LEMMA 1. [18] *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be a differentiable mapping on (ρ_a, ρ_b) with $a < b$. If $\varphi' : [\rho_a, \rho_b] \to \mathbb{R}$ be integrable on $[\rho_a, \rho_b]$. Then the identity for fractional*

integrals holds with respect to another function

$$\varphi(x) - \Gamma(\zeta+1) \left[\frac{I^\zeta_{x^+,g}\varphi(\rho_b)}{2\left(g(\rho_b)-g(x)\right)^\zeta} + \frac{I^\zeta_{x^-,g}\varphi(\rho_a)}{2\left(g(x)-g(\rho_a)\right)^\zeta} \right]$$

$$= \frac{x-\rho_a}{2\left(g(x)-g(\rho_a)\right)^\zeta} \int_0^1 \left(g(tx+(1-t)\rho_a)-g(\rho_a)\right)^\zeta \varphi'(tx+(1-t)\rho_a)dt$$

$$- \frac{\rho_b-x}{2\left(g(\rho_b)-g(x)\right)^\zeta} \int_0^1 \left(g(\rho_b)-g(tx+(1-t)\rho_b)\right)^\zeta \varphi'(tx+(1-t)\rho_b)dt.$$

Throughout this paper, we denote

$$\zeta_g \kappa^{\rho_b}_{\rho_a}(x) = \left[\frac{(x-\rho_a)^{\zeta+1}}{2\left(g(x)-g(\rho_a)\right)^\zeta} + \frac{(\rho_b-x)^{\zeta+1}}{2\left(g(\rho_b)-g(x)\right)^\zeta} \right],$$

$$\zeta_{\varphi,g} \theta^{\rho_b}_{\rho_a}(x) = \varphi(x) - \Gamma(\zeta+1) \left[\frac{I^\zeta_{x^+,g}\varphi(\rho_b)}{2\left(g(\rho_b)-g(x)\right)^\zeta} + \frac{I^\zeta_{x^-,g}\varphi(\rho_a)}{2\left(g(x)-g(\rho_a)\right)^\zeta} \right].$$

We also make use of the Euler's betThe, which is for $x,y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The main aim of our study is to present Ostrowski inequality for fractional integrals with respect to another function, which is the generalization of the classical Ostrowski inequality (1) via $\phi - \lambda$-convex. Moreover we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\phi - \lambda$-convex by using different techniques including Hölder's inequality [25] and power mean inequality [24]. Also we give the special cases of our results and applications of midpoint inequalities in special means. In the last section gives us conclusion with some remarks and future ideas to generalize the results.

2. Generalized Fractional Ostrowski inequality via $\phi - \lambda$-convex

In this section, we are introducing very first time the concept of $\phi - \lambda$-convex function, which contain many classes of convex in litrature.

DEFINITION 12. Let $\lambda \in (0,1]$ and $\phi : (0,1) \to (0,\infty)$, The $\eta : I \to [0,\infty)$ is a $\phi - \lambda$-convex (concave) on the interval I if $\forall x,y \in I$ we have

$$\eta(tx+(1-t)y) \leqslant (\geqslant) t^\lambda \phi(t)\eta(x) + (1-t)^\lambda \phi(1-t)\eta(y), \tag{4}$$

$\forall t \in (0,1)$.

REMARK 2. In Definition 12,

1. If $\lambda = 1$ in (4), we get ϕ-convex (concave).

2. If $\lambda = 1$, $l(t) = t$, and by taking $h = l\phi$ in (4), we get h-convex (concave).

3. If $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$ in (4), then we get the class of GL s-convex (concave).

4. If $\lambda = 1$, $\phi(t) = \frac{1}{t^2}$ in (4), then we get the concept of GL convex (concave).

5. If $\lambda = 1$, $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in (4), then we get the concept of s-convex (concave) in 2nd kind.

6. If $\lambda = 1$, $\phi(t) = \frac{1}{t}$ in (4), then we get the concept of P-convex (concave).

7. If $\lambda = 1$, $\phi(t) = 1$ in (4), then we get the concept of ordinary convex (concave).

8. If $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (4), then we get the concept of MT-convex (concave).

THEOREM 4. *If Lemma* 1 *hold. Additionally, assume that* $\lambda \in (0,1]$, $|\varphi'|$ *is* $\phi - \lambda$-*convex function on* $[\rho_a, \rho_b]$ *with* $\phi(t) \neq \frac{1}{t^2}$ *and* $|\varphi'(x)| \leqslant M$, $|g'(x)| \leqslant L$, $x \in [\rho_a, \rho_b]$. *Then for each* $x \in (\rho_a, \rho_b)$ *the following inequality holds:*

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant ML^{\zeta} \left(\int_0^1 \left[t^{\zeta+\lambda}\phi(t) + t^{\zeta}(1-t)^{\lambda}\phi(1-t) \right] dt \right) {}^{\zeta}_{g}\kappa^{\rho_b}_{\rho_a}(x). \qquad (5)$$

Proof. From the Lemma 1 we have

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant \frac{x - \rho_a}{2\left(g(x) - g(\rho_a)\right)^{\zeta}} \int_0^1 \frac{|\varphi'(tx + (1-t)\rho_a)|}{\left(g(tx+(1-t)\rho_a) - g(\rho_a)\right)^{-\zeta}} dt$$
$$+ \frac{\rho_b - x}{2\left(g(\rho_b) - g(x)\right)^{\zeta}} \int_0^1 \frac{|\varphi'(tx+(1-t)\rho_b)|}{\left(g(\rho_b) - g(tx+(1-t)\rho_b)\right)^{-\zeta}} dt. \qquad (6)$$

Since g is differentiable and $|g'(x)| \leqslant L$ on $[\rho_a, \rho_b]$, we get that g is Lipschizian function, i.e.

$$g\left(tx + (1-t)\rho_a\right) - g(\rho_a) \leqslant Lt(x - \rho_a) \qquad (7)$$

$$g(\rho_b) - g\left(tx + (1-t)\rho_b\right) \leqslant Lt(\rho_b - x). \qquad (8)$$

Using inequalities (7) and (8) in (6), we get

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant L^{\zeta} \frac{(x - \rho_a)^{\zeta+1}}{2\left(g(x) - g(\rho_a)\right)^{\zeta}} \int_0^1 t^{\zeta} \left| \varphi'(tx+(1-t)\rho_a) \right| dt$$
$$+ L^{\zeta} \frac{(\rho_b - x)^{\zeta+1}}{2\left(g(\rho_b) - g(x)\right)^{\zeta}} \int_0^1 t^{\zeta} \left| \varphi'(tx+(1-t)\rho_b) \right| dt. \qquad (9)$$

Since $|\phi'|$ is $\phi - \lambda$-convex on $[\rho_a, \rho_b]$ and $|\phi'(x)| \leqslant M$, we have

$$\int_0^1 t^\zeta \left[t^\lambda \phi(t) |\phi'(x)| + (1-t)^\lambda \phi(1-t) |\phi'(\rho_a)| \right] dt$$
$$\leqslant M \int_0^1 t^\zeta \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t) \right] dt. \tag{10}$$

and similarly

$$\int_0^1 t^\zeta \left[t^\lambda \phi(t) |\phi'(x)| + (1-t)^\lambda \phi(1-t) |\phi'(\rho_b)| \right] dt$$
$$\leqslant M \int_0^1 t^\zeta \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t) \right] dt. \tag{11}$$

By using inequalities (10) and (11) in (9), we get

$$\left| \varsigma_{\phi,g} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant ML^\zeta \left(\int_0^1 t^\zeta \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t) \right] dt \right) \, \varsigma_g \kappa_{\rho_a}^{\rho_b}(x). \quad \square$$

COROLLARY 1. *In Theorem* 4,

1. *If $\lambda = 1$ in* (5), *then Fractional Ostrowski type inequality for ϕ-convex:*

$$\left| \varsigma_{\phi,g} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant ML^\zeta \left(\int_0^1 t^\zeta [t\phi(t) + (1-t)\phi(1-t)] dt \right) \, \varsigma_g \kappa_{\rho_a}^{\rho_b}(x).$$

2. *If $\lambda = 1$, $l(t) = t$ and $h = l\phi$ in* (5), *then Fractional Ostrowski type inequality for h-convex:*

$$\left| \varsigma_{\phi,g} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant ML^\zeta \left(\int_0^1 t^\zeta [h(t) + h(1-t)] dt \right) \, \varsigma_g \kappa_{\rho_a}^{\rho_b}(x).$$

3. *If $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1)$ in* (5), *then Ostrowski inequality for Godunova-Levin s-convex:*

$$\left| \varsigma_{\phi,g} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant ML^\zeta \left(\frac{1}{1+\zeta-s} + \frac{\Gamma(1+\zeta)\Gamma(1-s)}{\Gamma(2+\zeta-s)} \right) \, \varsigma_g \kappa_{\rho_a}^{\rho_b}(x).$$

4. *If $\lambda = 1$, $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in* (5), *then Ostrowski inequality for s-convex in 2nd kind:*

$$\left| \varsigma_{\phi,g} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant ML^\zeta \left(\frac{1}{1+\zeta+s} + \frac{\Gamma(1+\zeta)\Gamma(1+s)}{\Gamma(2+\zeta+s)} \right) \, \varsigma_g \kappa_{\rho_a}^{\rho_b}(x).$$

5. *If $\lambda = 1$, $g(x) = x$, $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in inequality* (5), *then inequality* (2.6) *of Theorem 7 in* [22].

6. If $\lambda = 1$, $g(x) = x$, $\zeta = 1$, $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in inequality (5), then inequality (2.1) of Theorem 2 in [1].

7. If $\lambda = 1$, $\phi(t) = \frac{1}{t}$ in inequality (5), then Ostrowski inequality for P-convex via fractional integrals:

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant \frac{2ML^{\zeta}}{1+\zeta} \, {}^{\zeta}_g \kappa^{\rho_b}_{\rho_a}(x)$$

8. If $\lambda = \phi(t) = 1$ in inequality (5), then Ostrowski inequality for convex via fractional integrals:

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant \frac{ML^{\zeta}}{1+\zeta} \, {}^{\zeta}_g \kappa^{\rho_b}_{\rho_a}(x)$$

9. If $g(x) = x$, $\lambda = \phi(t) = 1$ in inequality (5), then Corollary 1 in [22].

10. If $g(x) = x$, $\lambda = \phi(t) = \zeta = 1$ in inequality (5), then one has inequality (1.3) of Theorem 3 in [22].

11. If $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (5), then Fractional Ostrowski type inequality MT-convex:

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant ML^{\zeta} \left(\frac{\sqrt{\pi}\,\Gamma\left[\frac{1}{2}+\zeta\right]}{2\,\Gamma[1+\zeta]} \right) \, {}^{\zeta}_g \kappa^{\rho_b}_{\rho_a}(x).$$

THEOREM 5. *If Lemma* 1 *hold. Additionally, assume that* $\lambda \in (0,1]$, $|\varphi'|^q$ *is* $\phi - \lambda$-*convex function on* $[\rho_a, \rho_b]$, $q \geqslant 1$ *with* $\phi(t) \neq \frac{1}{t^2}$ *and* $|\varphi'(x)| \leqslant M$, $|g'(x)| \leqslant L$, $x \in [\rho_a, \rho_b]$. *Then* $\forall x \in (\rho_a, \rho_b)$ *the following inequality holds:*

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant \frac{ML^{\zeta} \, {}^{\zeta}_g \kappa^{\rho_b}_{\rho_a}(x)}{(\zeta+1)^{1-\frac{1}{q}}} \left(\int_0^1 \left[t^{\zeta+\lambda}\phi(t) + t^{\zeta}(1-t)^{\lambda}\phi(1-t) \right] dt \right)^{\frac{1}{q}}. \quad (12)$$

Proof. From the inequality (9) and using power mean inequality [24], we have

$$\left| {}^{\zeta}_{\phi,g}\theta^{\rho_b}_{\rho_a}(x) \right| \leqslant L^{\zeta} \frac{(x-\rho_a)^{\zeta+1}}{2\,(g(x)-g(\rho_a))^{\zeta}} \int_0^1 t^{\zeta} \left| \varphi'(tx+(1-t)\rho_a) \right|^q dt$$

$$+ L^{\zeta} \frac{(\rho_b-x)^{\zeta+1}}{2\,(g(\rho_b)-g(x))^{\zeta}} \int_0^1 t^{\zeta} \left| \varphi'(tx+(1-t)\rho_b) \right|^q dt$$

$$\leqslant L^{\zeta} \frac{(x-\rho_a)^{\zeta+1}}{2\,(g(x)-g(\rho_a))^{\zeta}} \left(\int_0^1 t^{\zeta} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\zeta} \left| \varphi'(tx+(1-t)\rho_a) \right|^q dt \right)^{\frac{1}{q}}$$

$$+ L^{\zeta} \frac{(\rho_b-x)^{\zeta+1}}{2\,(g(\rho_b)-g(x))^{\zeta}} \left(\int_0^1 t^{\zeta} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\zeta} \left| \varphi'(tx+(1-t)\rho_a) \right|^q dt \right)^{\frac{1}{q}}.$$

$$(13)$$

Since $|\varphi'|^q$ is $\phi - \lambda$-convex on $[\rho_a, \rho_b]$. and $|\varphi'(x)| \leqslant M$, we get

$$\int_0^1 t^\zeta \left|\varphi'(tx+(1-t)\rho_a)\right|^q dt \leqslant M^q \int_0^1 t^\zeta \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)\right] dt \quad (14)$$

and

$$\int_0^1 t^\zeta \left|\varphi'(tx+(1-t)\rho_b)\right|^q dt \leqslant M^q \int_0^1 t^\zeta \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)\right] dt. \quad (15)$$

Using the inequalities $(13)-(15)$, we get

$$\left|{}^\zeta_{\phi,g}\theta^{\rho_b}_{\rho_a}(x)\right| \leqslant \frac{ML^\zeta}{(\zeta+1)^{1-\frac{1}{q}}} \left(\int_0^1 t^\zeta \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)\right] dt\right)^{\frac{1}{q}} {}^\zeta_g \kappa^{\rho_b}_{\rho_a}(x). \quad \square$$

COROLLARY 2. *In Theorem* 5,

1. *If* $q = 1$, *one has the Theorem* 4.

2. *If* $\lambda = 1$ *in* (12), *then Fractional Ostrowski type inequality for* ϕ-*convex:*

$$\left|{}^\zeta_{\phi,g}\theta^{\rho_b}_{\rho_a}(x)\right| \leqslant \frac{ML^\zeta {}^\zeta_g \kappa^{\rho_b}_{\rho_a}(x)}{(1+\zeta)^{1-\frac{1}{q}}} \left(\int_0^1 t^\zeta [t\phi(t) + (1-t)\phi(1-t)]dt\right)^{\frac{1}{q}}.$$

3. *If* $\lambda = 1$, $l(t) = t$ *and* $h = l\phi$ *in* (12), *then Fractional Ostrowski type inequality for* h-*convex:*

$$\left|{}^\zeta_{\phi,g}\theta^{\rho_b}_{\rho_a}(x)\right| \leqslant \frac{ML^\zeta {}^\zeta_g \kappa^{\rho_b}_{\rho_a}(x)}{(1+\zeta)^{1-\frac{1}{q}}} \left(\int_0^1 t^\zeta [h(t) + h(1-t)]dt\right)^{\frac{1}{q}}.$$

4. *If* $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ *with* $s \in [0,1)$ *in* (12), *then Ostrowski inequality for Godunova-Levin* s-*convex:*

$$\left|{}^\zeta_{\phi,g}\theta^{\rho_b}_{\rho_a}(x)\right| \leqslant \frac{ML^\zeta {}^\zeta_g \kappa^{\rho_b}_{\rho_a}(x)}{(1+\zeta)^{1-\frac{1}{q}}} \left(\frac{1}{1+\zeta-s} + \frac{\Gamma(1+\zeta)\Gamma(1-s)}{\Gamma(2+\zeta-s)}\right)^{\frac{1}{q}}.$$

5. *If* $\lambda = 1$, $\phi(t) = t^{s-1}$ *with* $s \in [0,1]$ *in inequality* (12), *then Fractional Ostrowski type inequality for* s-*convex in* 2nd *kind:*

$$\left|{}^\zeta_{\phi,g}\theta^{\rho_b}_{\rho_a}(x)\right| \leqslant \frac{ML^\zeta {}^\zeta_g \kappa^{\rho_b}_{\rho_a}(x)}{(\zeta+1)^{1-\frac{1}{q}}} \left(\frac{1}{1+\zeta+s} + \frac{\Gamma(1+\zeta)\Gamma(1+s)}{\Gamma(2+\zeta+s)}\right)^{\frac{1}{q}}.$$

6. *If* $\lambda = 1$, $g(x) = x$, *and* $\phi(t) = t^{s-1}$ *with* $s \in [0,1]$ *in inequality* (12), *then inequality* (2.8) *of Theorem 9 in* [22].

7. If $g(x) = x$, $\lambda = \zeta = 1$ and $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in inequality (12), then inequality (2.3) of Theorem 4 in [1].

8. If $\lambda = 1$, $\phi(t) = \frac{1}{t}$ in inequality (12), then Ostrowski inequality for P-convex via fractional integrals:

$$\left| {}_{\phi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{2^{\frac{1}{q}} M L^{\zeta}}{(1+\zeta)^{1-\frac{1}{q}}} {}_{g}^{\zeta} \kappa_{\rho_a}^{\rho_b}(x).$$

9. If $\lambda = \phi(t) = 1$ in inequality (12), then Ostrowski inequality for convex via fractional integrals:

$$\left| {}_{\phi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{M L^{\zeta}}{(1+\zeta)^{1-\frac{1}{q}}} {}_{g}^{\zeta} \kappa_{\rho_a}^{\rho_b}(x).$$

10. If $g(x) = x$, $\lambda = \phi(t) = 1$ in inequality (12), then inequality of Corollary 3 in [22].

11. If $g(x) = x$, $\lambda = \phi(t) = \zeta = 1$ in inequality (12), then one has inequality (1.5) of Theorem 5 in [22].

12. If $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (12), then Fractional Ostrowski type inequality MT-convex:

$$\left| {}_{\phi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{M L^{\zeta}}{(1+\zeta)^{1-\frac{1}{q}}} \left(\frac{\sqrt{\pi} \, \Gamma\left[\frac{1}{2}+\zeta\right]}{2 \, \Gamma[1+\zeta]} \right)^{\frac{1}{q}} {}_{g}^{\zeta} \kappa_{\rho_a}^{\rho_b}(x).$$

THEOREM 6. *If Lemma* 1 *hold. Additionally, assume that* $\lambda \in (0,1]$, $|\varphi'|^q$ *is* $\phi - \lambda$-*convex function on* $[\rho_a, \rho_b]$, $q > 1$ *with* $\phi(t) \neq \frac{1}{t^2}$ *and* $|\varphi'(x)| \leqslant M$, $|g'(x)| \leqslant L$, $x \in [\rho_a, \rho_b]$. *Then for each* $x \in (\rho_a, \rho_b)$ *the following inequality holds:*

$$\left| {}_{\phi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{M L^{\zeta} \, {}_{g}^{\zeta} \kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}} \left(\int_0^1 \left[t^{\lambda} \phi(t) + (1-t)^{\lambda} \phi(1-t) \right] dt \right)^{\frac{1}{q}}, \qquad (16)$$

where $p^{-1} + q^{-1} = 1$.

Proof. From the inequality (9) and and using Hölder's inequality [25], we have

$$\left| {}_{\phi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant L^{\zeta} \frac{(x - \rho_a)^{\zeta+1}}{2 \left(g(x) - g(\rho_a) \right)^{\zeta}} \left(\int_0^1 t^{\zeta p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| \varphi'(tx + (1-t)\rho_a) \right|^q dt \right)^{\frac{1}{q}}$$

$$+ L^{\zeta} \frac{(\rho_b - x)^{\zeta+1}}{2 \left(g(\rho_b) - g(x) \right)^{\zeta}} \left(\int_0^1 t^{\zeta p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| \varphi'(tx + (1-t)\rho_b) \right|^q dt \right)^{\frac{1}{q}}.$$
$$\tag{17}$$

Since $|\varphi'|^q$ is $\phi - \lambda$-convex and $|\varphi'(x)| \leqslant M$, we have

$$\int_0^1 |\varphi'(tx + (1-t)\rho_a)|^q \, dt \leqslant M^q \int_0^1 \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t) \right] dt \qquad (18)$$

and

$$\int_0^1 |\varphi'(tx + (1-t)\rho_b)|^q \, dt \leqslant M^q \int_0^1 \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t) \right] dt. \qquad (19)$$

Using inequalities $(17)-(19)$, we get

$$\left| {}_{\varphi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{ML^\zeta \, {}_g^\zeta \kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}} \left(\int_0^1 \left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t) \right] dt dt \right)^{\frac{1}{q}}. \quad \square$$

COROLLARY 3. *In Theorem* 6,

1. *If* $\lambda = 1$, *$l(t) = t$ and $h = l\phi$ in* (16), *then Fractional Ostrowski type inequality for h-convex:*

$$\left| {}_{\varphi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{ML^\zeta \, {}_g^\zeta \kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}} \left(\int_0^1 [h(t) + h(1-t)] dt \right)^{\frac{1}{q}}$$

2. *If* $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ *with $s \in [0,1)$ in* (16), *then Ostrowski inequality for Godunova-Levin s-convex:*

$$\left| {}_{\varphi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{ML^\zeta \, {}_g^\zeta \kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}} \left(\frac{2}{1-s} \right)^{\frac{1}{q}}.$$

3. *If* $\lambda = 1$, $\phi(t) = t^{s-1}$ *with $s \in [0,1]$ in inequality* (16), *then Fractional Ostrowski type inequality for s-convex in 2nd kind:*

$$\left| {}_{\varphi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{ML^\zeta \, {}_g^\zeta \kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}} \left(\frac{2}{1+s} \right)^{\frac{1}{q}}.$$

4. *If* $\lambda = 1$, $g(x) = x$ *and $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in inequality* (16), *then inequality* (2.7) *of Theorem* 8 *in* [22].

5. *If* $\lambda = 1$, $g(x) = x$, $\zeta = 1$ *and $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in inequality* (16), *then inequality* (2.4) *of Theorem* 3 *in* [1].

6. *If* $\lambda = 1$, $\phi(t) = \frac{1}{t}$ *in inequality* (16), *then Ostrowski inequality for P-convex via fractional integrals:*

$$\left| {}_{\varphi,g}^{\zeta} \theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{2^{\frac{1}{q}} ML^\zeta \, {}_g^\zeta \kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}}.$$

7. If $\lambda = \phi(t) = 1$ in inequality (16), then Ostrowski inequality for convex via fractional integrals:

$$\left| {}_{\phi,g}^{\zeta}\theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \frac{ML^{\zeta}\, {}_{g}^{\zeta}\kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}}.$$

8. If $g(x) = x$, $\lambda = \phi(t) = 1$ in inequality (16), then one has Corollary 2 in [22].

9. If $g(x) = x$, $\lambda = \phi(t) = \zeta = 1$ in inequality (16), then one has inequality (1.4) of Theorem 4 in [22].

10. If $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (16), then Fractional Ostrowski type inequality MT-convex:

$$\left| {}_{\phi,g}^{\zeta}\theta_{\rho_a}^{\rho_b}(x) \right| \leqslant \left(\frac{\pi}{2}\right)^{\frac{1}{q}} \frac{ML^{\zeta}\, {}_{g}^{\zeta}\kappa_{\rho_a}^{\rho_b}(x)}{(\zeta p + 1)^{\frac{1}{p}}}.$$

THEOREM 7. *Let $\lambda \in (0,1]$, $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable on (ρ_a, ρ_b), $\varphi' : [\rho_a, \rho_b] \to \mathbb{R}$ be integrable on $[\rho_a, \rho_b]$ and $\eta : I \subset \mathbb{R} \to \mathbb{R}$, be a $\phi - \lambda$-convex (concave), then we have the inequalities*

$$\eta\left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b) \right.$$

$$\left. + J_{\rho_a}^{\zeta-1}(P_2(x,\rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}}J_{\rho_a}^{0}\varphi(\rho_a) \right]$$

$$\leqslant (\geqslant) \frac{(x-\rho_a)^{\lambda-1}(\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)^{\lambda}}\phi\left(\frac{x-\rho_a}{\rho_b - \rho_a}\right)\left[\int_{\rho_a}^{x}\eta\left[\frac{(t-\mu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right]$$

$$+ \frac{(\rho_b - x)^{\lambda-\zeta}}{(\rho_b - \rho_a)^{\lambda}}\phi\left(\frac{\rho_b - x}{\rho_b - \rho_a}\right)\left[\int_{x}^{\rho_b}\eta\left[\frac{(t-\nu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right], \tag{20}$$

$\forall x \in [\mu, \nu]$ *and $\varepsilon \in [0,1]$.*

Proof. Utilizing the generalized Montgomery identity (3) for fractional, we get

$$(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b)$$

$$+ J_{\rho_a}^{\zeta-1}(P_2(x,\rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}}J_{\rho_a}^{0}\varphi(\rho_a)$$

$$= J_{\rho_a}^{\zeta}(P_2(x,\rho_b)\varphi'(\rho_b))$$

$$= \frac{1}{\Gamma(\zeta)}\int_{\rho_a}^{\rho_b}P_2(x,t)\frac{\varphi'(t)}{(\rho_b - t)^{1-\zeta}}dt$$

$$= \left(\frac{x - \rho_a}{\rho_b - \rho_a} \right) \left[\frac{(\rho_b - x)^{1-\zeta}}{x - \rho_a} \int_{\rho_a}^x \frac{\{t - \mu\} \varphi'(t)}{(\rho_b - t)^{1-\zeta}} dt \right]$$

$$+ \left(\frac{\rho_b - x}{\rho_b - \rho_a} \right) \left[\frac{(\rho_b - x)^{1-\zeta}}{\rho_b - x} \int_x^{\rho_b} \frac{\{t - v\} \varphi'(t)}{(\rho_b - t)^{1-\zeta}} dt \right],$$

$\forall x \in [\mu, v]$ and $\varepsilon \in [0, 1]$. Next by using $\eta : I \subset [0, \infty) \to \mathbb{R}$, the $\phi - \lambda$-convex (concave), we get

$$\eta \left[(1 - \varepsilon) \varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^\zeta \varphi(\rho_b) \right.$$

$$\left. + J_{\rho_a}^{\zeta - 1}(P_2(x, \rho_b) \varphi(\rho_b)) + \frac{\varepsilon (\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^0 \varphi(\rho_a) \right]$$

$$\leqslant (\geqslant) \left[\left(\frac{x - \rho_a}{\rho_b - \rho_a} \right)^\lambda \phi \left(\frac{x - \rho_a}{\rho_b - \rho_a} \right) \right] \eta \left[\frac{(\rho_b - x)^{1-\zeta}}{x - \rho_a} \int_{\rho_a}^x \frac{\{t - \mu\} \varphi'(t)}{(\rho_b - t)^{1-\zeta}} dt \right]$$

$$+ \left[\left(\frac{\rho_b - x}{\rho_b - \rho_a} \right)^\lambda \phi \left(\frac{\rho_b - x}{\rho_b - \rho_a} \right) \right] \eta \left[\frac{(\rho_b - x)^{1-\zeta}}{\rho_b - x} \int_x^{\rho_b} \frac{\{t - v\} \varphi'(t)}{(\rho_b - t)^{1-\zeta}} dt \right],$$

$\forall x \in [\mu, v]$ and $\varepsilon \in [0, 1]$. Applying Jensen's integral inequality [7], we get the inequality (20). $\square$

REMARK 3. In Theorem 7, If $\varepsilon = 0$, in (20) we get

$$\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^\zeta \varphi(\rho_b) + J_{\rho_a}^{\zeta - 1}(P_1(x, \rho_b) \varphi(\rho_b)) \right]$$

$$\leqslant (\geqslant) \frac{(x - \rho_a)^{\lambda - 1}(\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)^\lambda} \phi \left(\frac{x - \rho_a}{\rho_b - \rho_a} \right) \left[\int_{\rho_a}^x \eta \left[\frac{(t - \rho_a) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right]$$

$$+ \frac{(\rho_b - x)^{\lambda - \zeta}}{(\rho_b - \rho_a)^\lambda} \phi \left(\frac{\rho_b - x}{\rho_b - \rho_a} \right) \left[\int_x^{\rho_b} \eta \left[\frac{(t - \rho_b) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right].$$

COROLLARY 4. *In Theorem 7,*

1. *If $\lambda = 1$ in (20), then Fractional Ostrowski type inequality for ϕ-convex (concave):*

$$\eta \left[(1 - \varepsilon) \varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^\zeta \varphi(\rho_b) \right.$$

$$\left. + J_{\rho_a}^{\zeta - 1}(P_2(x, \rho_b) \varphi(\rho_b)) + \frac{\varepsilon (\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^0 \varphi(\rho_a) \right]$$

$$\leqslant (\geqslant) \frac{(\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)} \left[\phi \left(\frac{x - \rho_a}{\rho_b - \rho_a} \right) \int_{\rho_a}^x \eta \left[\frac{(t - \mu) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right.$$

$$\left. + \phi \left(\frac{\rho_b - x}{\rho_b - \rho_a} \right) \int_x^{\rho_b} \eta \left[\frac{(t - v) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right]. \tag{21}$$

REMARK 4. If $\varepsilon = 0$ in (21), we get

$$\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) + J_{\rho_a}^{\zeta-1}(P_1(x, \rho_b)\varphi(\rho_b)) \right]$$

$$\leqslant (\geqslant) \frac{(\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)} \left[\phi\left(\frac{x - \rho_a}{\rho_b - \rho_a}\right) \int_{\rho_a}^{x} \eta \left[\frac{(t - \rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right.$$

$$\left. + \phi\left(\frac{\rho_b - x}{\rho_b - \rho_a}\right) \int_{x}^{\rho_b} \eta \left[\frac{(t - \rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right].$$

2. If $\lambda = 1$, $l(t) = t$ and $h = l\phi$ in (20), then Fractional Ostrowski type inequality for h-convex (concave):

$$\eta \left[(1 - \varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) \right.$$

$$+ J_{\rho_a}^{\zeta-1}(P_2(x, \rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^{0} \varphi(\rho_a) \bigg]$$

$$\leqslant (\geqslant) h\left(\frac{x - \rho_a}{\rho_b - \rho_a}\right) \left[\frac{(\rho_b - x)^{1-\zeta}}{x - \rho_a} \int_{\rho_a}^{x} \eta \left[\frac{(t - \mu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right]$$

$$+ h\left(\frac{\rho_b - x}{\rho_b - \rho_a}\right) \left[\frac{1}{(\rho_b - x)^{\zeta}} \int_{x}^{\rho_b} \eta \left[\frac{(t - \nu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right]. \qquad (22)$$

REMARK 5. If $\varepsilon = 0$ in (22), we get

$$\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) + J_{\rho_a}^{\zeta-1}(P_1(x, \rho_b)\varphi(\rho_b)) \right]$$

$$\leqslant (\geqslant) h\left(\frac{x - \rho_a}{\rho_b - \rho_a}\right) \left[\frac{(\rho_b - x)^{1-\zeta}}{x - \rho_a} \int_{\rho_a}^{x} \eta \left[\frac{(t - \rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right]$$

$$+ h\left(\frac{\rho_b - x}{\rho_b - \rho_a}\right) \left[\frac{1}{(\rho_b - x)^{\zeta}} \int_{x}^{\rho_b} \eta \left[\frac{(t - \rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right].$$

3. If $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$ in (20), then Ostrowski inequality for Godunova-Levin s-convex (concave):

$$\eta \left[(1 - \varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) \right.$$

$$+ J_{\rho_a}^{\zeta-1}(P_2(x, \rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^{0} \varphi(\rho_a) \bigg]$$

$$\leqslant (\geqslant) \frac{(\rho_b - \rho_a)^{s}(\rho_b - x)^{1-\zeta}}{(x - \rho_a)^{1+s}} \int_{\rho_a}^{x} \eta \left[\frac{(t - \mu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt$$

$$+ \frac{(\rho_b - \rho_a)^{s}}{(\rho_b - x)^{\zeta+s}} \int_{x}^{\rho_b} \eta \left[\frac{(t - \nu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt. \qquad (23)$$

REMARK 6. If $\varepsilon = 0$ in (23), we get

$$\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) + J_{\rho_a}^{\zeta-1} (P_1(x, \rho_b) \varphi(\rho_b)) \right]$$

$$\leqslant (\geqslant) \frac{(\rho_b - \rho_a)^s (\rho_b - x)^{1-\zeta}}{(x - \rho_a)^{1+s}} \int_{\rho_a}^{x} \eta \left[\frac{(t - \rho_a) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt$$

$$+ \frac{(\rho_b - \rho_a)^s}{(\rho_b - x)^{\zeta+s}} \int_{x}^{\rho_b} \eta \left[\frac{(t - \rho_b) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt.$$

4. If $\lambda = 1$, $\phi(t) = \frac{1}{t^2}$ in (20), then Fractional Ostrowski type inequality for Godunova-Levin convex (concave):

$$\eta \left[(1 - \varepsilon) \varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) \right.$$

$$\left. + J_{\rho_a}^{\zeta-1} (P_2(x, \rho_b) \varphi(\rho_b)) + \frac{\varepsilon (\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^{0} \varphi(\rho_a) \right]$$

$$\leqslant (\geqslant) \frac{(\rho_b - \rho_a)(\rho_b - x)^{1-\zeta}}{(x - \rho_a)^2} \int_{\rho_a}^{x} \eta \left[\frac{(t - \mu) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt$$

$$+ \frac{(\rho_b - \rho_a)}{(\rho_b - x)^{\zeta+1}} \int_{x}^{\rho_b} \eta \left[\frac{(t - \nu) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt. \tag{24}$$

REMARK 7. If $\varepsilon = 0$ in (24), we get

$$\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) + J_{\rho_a}^{\zeta-1} (P_1(x, \rho_b) \varphi(\rho_b)) \right]$$

$$\leqslant (\geqslant) \frac{(\rho_b - \rho_a)(\rho_b - x)^{1-\zeta}}{(x - \rho_a)^2} \int_{\rho_a}^{x} \eta \left[\frac{(t - \rho_a) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt$$

$$+ \frac{(\rho_b - \rho_a)}{(\rho_b - x)^{\zeta+1}} \int_{x}^{\rho_b} \eta \left[\frac{(t - \rho_b) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt.$$

5. If $\lambda = 1$, $\phi(t) = t^{s-1}$ with $s \in [0, 1]$ in (20), then Fractional Ostrowski type inequality for s-convex (concave) in 2nd kind:

$$\eta \left[(1 - \varepsilon) \varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a} (\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) \right.$$

$$\left. + J_{\rho_a}^{\zeta-1} (P_2(x, \rho_b) \varphi(\rho_b)) + \frac{\varepsilon (\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^{0} \varphi(\rho_a) \right]$$

$$\leqslant (\geqslant) \frac{(x - \rho_a)^{s-1} (\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)^s} \int_{\rho_a}^{x} \eta \left[\frac{(t - \mu) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt$$

$$+ \frac{(\rho_b - x)^{s-\zeta}}{(\rho_b - \rho_a)^s} \int_{x}^{\rho_b} \eta \left[\frac{(t - \nu) \varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt. \tag{25}$$

REMARK 8. If $\varepsilon = 0$ in (25), we get

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b) + J_{\rho_a}^{\zeta-1}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$

$$\leqslant (\geqslant)\frac{(x-\rho_a)^{s-1}(\rho_b-x)^{1-\zeta}}{(\rho_b-\rho_a)^s}\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt$$

$$+\frac{(\rho_b-x)^{s-\zeta}}{(\rho_b-\rho_a)^s}\int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt.$$

6. If $\lambda = 1$, $\phi(t) = \frac{1}{t}$ in (20), then Fractional Ostrowski type inequality for P-convex (concave):

$$\eta\left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b)\right.$$

$$\left.+J_{\rho_a}^{\zeta-1}(P_2(x,\rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b-x)^{1-\zeta}}{2(\rho_b-\rho_a)^{1-\zeta}}J_{\rho_a}^{0}\varphi(\rho_a)\right]$$

$$\leqslant (\geqslant)\frac{(\rho_b-x)^{1-\zeta}}{(x-\rho_a)}\int_{\rho_a}^{x}\eta\left[\frac{(t-\mu)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt + \frac{1}{(\rho_b-x)^{\zeta}}\int_{x}^{\rho_b}\eta\left[\frac{(t-\nu)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt.$$

$$(26)$$

REMARK 9. If $\varepsilon = 0$ in (26), we get

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b) + J_{\rho_a}^{\zeta-1}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$

$$\leqslant (\geqslant)\frac{(\rho_b-x)^{1-\zeta}}{(x-\rho_a)}\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt + \frac{1}{(\rho_b-x)^{\zeta}}\int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt.$$

7. If $\lambda = \phi(t) = 1$ in (20), then Fractional Ostrowski type inequality for convex (concave):

$$\eta\left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b)\right.$$

$$\left.+J_{\rho_a}^{\zeta-1}(P_2(x,\rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b-x)^{1-\zeta}}{2(\rho_b-\rho_a)^{1-\zeta}}J_{\rho_a}^{0}\varphi(\rho_a)\right]$$

$$\leqslant (\geqslant)\frac{(\rho_b-x)^{1-\zeta}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{x}\eta\left[\frac{(t-\mu)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt + \int_{x}^{\rho_b}\eta\left[\frac{(t-\nu)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right].$$

$$(27)$$

REMARK 10. If $\varepsilon = 0$ in (27), we get

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J_{\rho_a}^{\zeta}\varphi(\rho_b) + J_{\rho_a}^{\zeta-1}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$

$$\leqslant (\geqslant)\frac{(\rho_b-x)^{1-\zeta}}{\rho_b-\rho_a}\left[\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt + \int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right].$$

8. If $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (20), then Fractional Ostrowski type inequality MT -convex (concave):

$$
\eta \left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) \right.
$$

$$
\left. + J_{\rho_a}^{\zeta-1}(P_2(x,\rho_b)\varphi(\rho_b)) + \frac{\varepsilon(\rho_b - x)^{1-\zeta}}{2(\rho_b - \rho_a)^{1-\zeta}} J_{\rho_a}^{0} \varphi(\rho_a) \right]
$$

$$
\leqslant (\geqslant) \frac{(\rho_b - x)^{\frac{1}{2}-\zeta}}{2\sqrt{(x-\rho_a)}} \left[\int_{\rho_a}^{x} \eta \left[\frac{(t-\mu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt + \int_{x}^{\rho_b} \eta \left[\frac{(t-v)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right].
$$

(28)

REMARK 11. If $\varepsilon = 0$ in (28), we get

$$
\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J_{\rho_a}^{\zeta} \varphi(\rho_b) + J_{\rho_a}^{\zeta-1}(P_1(x,\rho_b)\varphi(\rho_b)) \right]
$$

$$
\leqslant (\geqslant) \frac{(\rho_b - x)^{\frac{1}{2}-\zeta}}{2\sqrt{(x-\rho_a)}} \left[\int_{\rho_a}^{x} \eta \left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt + \int_{x}^{\rho_b} \eta \left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right].
$$

3. Applications of midpoint inequalties

If we replace φ by $-\varphi$ and $x = \frac{\rho_a + \rho_b}{2}$ in Theorem 7, we get

THEOREM 8. *Let* $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ *be differentiable on* (ρ_a, ρ_b), $\varphi' : [\rho_a, \rho_b] \to \mathbb{R}$ *be integrable on* $[\rho_a, \rho_b]$ *and* $\eta : I \subset \mathbb{R} \to \mathbb{R}$, *be a* $\phi - \lambda$ *-convex (concave), then*

$$
\eta \left[\frac{\Gamma(\zeta)\left(\frac{\rho_b - \rho_a}{2}\right)^{1-\zeta}}{\rho_b - \rho_a} J_{\bar{a}}^{\zeta} \varphi(\rho_b) - \varphi\left(\frac{\rho_a + \rho_b}{2}\right) \right.
$$

$$
\left. - J_{\bar{a}}^{\zeta-1}\left(P_2\left(\frac{\rho_a + \rho_b}{2}, b\right)\varphi(\rho_b)\right) - \frac{\varepsilon}{2^{2-\zeta}} J_{\rho_a}^{0} \varphi(\rho_a) \right]
$$

$$
\leqslant (\geqslant) \frac{2^{\zeta-\lambda}\phi\left(\frac{1}{2}\right)}{(\rho_b - \rho_a)^{\zeta}} \left[\int_{\frac{\rho_a + \rho_b}{2}}^{\rho_a} \eta \left[\frac{(t-\mu)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt + \int_{\rho_b}^{\frac{\rho_a + \rho_b}{2}} \eta \left[\frac{(t-v)\varphi'(t)}{(\rho_b - t)^{1-\zeta}} \right] dt \right].
$$

(29)

$\forall \varepsilon \in [0,1]$.

REMARK 12. In Theorem 8,

1. If $\varepsilon = 0$, in (29) we get

$$\eta\left[\frac{\Gamma(\zeta)\left(\frac{\rho_b-\rho_a}{2}\right)^{1-\zeta}}{\rho_b-\rho_a}J_a^\zeta\varphi(\rho_b)-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-J_a^{\zeta-1}\left(P_1\left(\frac{\rho_a+\rho_b}{2},b\right)\varphi(\rho_b)\right)\right]$$

$$\leqslant(\geqslant)\frac{2^{\zeta-\lambda}\phi\left(\frac{1}{2}\right)}{(\rho_b-\rho_a)^\zeta}\left[\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_a}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt+\int_{\rho_b}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right].$$

2. If $\zeta = 1$ in (29) we get

$$\eta\left[(\varepsilon-1)\varphi\left(\frac{\rho_a+\rho_b}{2}\right)-\varepsilon\frac{\varphi(a)+\varphi(\rho_b)}{2}+\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right]$$

$$\leqslant(\geqslant)\frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(v-t)\varphi'(t)\right]dt\right].$$

3. If $\varepsilon = 0$, $\zeta = 1$ in (29) we get

$$\eta\left[\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt-\varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right]$$

$$\leqslant(\geqslant)\frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[(\rho_a-t)\varphi'(t)\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[(\rho_b-t)\varphi'(t)\right]dt\right].$$

REMARK 13. Assume that $\eta:I\subset[0,\infty)\to\mathbb{R}$ be an ϕ-convex (concave):

1. If $\zeta = 1$, $\varphi(t) = \frac{1}{t}$ in inequality (30) where $t\in[\rho_a,\rho_b]\subset(0,\infty)$, then we have

$$\eta\left[\frac{A(\rho_a,\rho_b)+(\varepsilon-1)L(\rho_a,\rho_b)}{A(\rho_a,\rho_b)L(\rho_a,\rho_b)}-\varepsilon\frac{A(\rho_a,\rho_b)}{G^2(\rho_a,\rho_b)}\right]$$

$$\leqslant(\geqslant)\frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{t-\mu}{t^2}\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{t-v}{t^2}\right]dt\right].$$

2. If $\zeta = 1$, $\varphi(t) = -\ln t$ in inequality (30), where $t\in[\rho_a,\rho_b]\subset(0,\infty)$, then we have

$$\eta\left[\ln\left(\frac{\exp[\varepsilon A(\ln\rho_a,\ln\rho_b)]A^{(1-\varepsilon)}(\rho_a,\rho_b)}{I(\rho_a,\rho_b)}\right)\right]$$

$$\leqslant(\geqslant)\frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{\rho_b-\rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{t-\mu}{t}\right]dt+\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{t-v}{t}\right]dt\right].$$

3. If $\zeta = 1$, $\varphi(t) = t^p$, $p \in \mathbb{R} - \{0, -1\}$ in inequality (30), where $t \in [\rho_a, \rho_b] \subset (0, \infty)$, then we have

$$\eta\left[L_p^p(\rho_a, \rho_b) - (\varepsilon - 1)A^p(\rho_a, \rho_b) - \varepsilon A(a^p, b^p)\right]$$
$$\leqslant (\geqslant) \frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{\rho_b - \rho_a}\left[\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}} \eta\left[\frac{p(\mu - t)}{t^{1-p}}\right] dt + \int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b} \eta\left[\frac{p(\nu - t)}{t^{1-p}}\right] dt\right].$$

REMARK 14. In Theorem 5,

1. Let $g(x) = x$, $\zeta = 1$, $x = \frac{\rho_a+\rho_b}{2}$, $0 < \rho_a < \rho_b$, $q \geqslant 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(x) = x^n$ in (12). Then

$$|A(\rho_a, \rho_b) - L_n^n(\rho_a, \rho_b)| \leqslant \frac{M(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1\left[t^{1+\lambda}\phi(t) + t(1-t)^\lambda\phi(1-t)\right] dt\right)^{\frac{1}{q}}.$$

2. Let $g(x) = x$, $\zeta = 1$, $x = \frac{\rho_a+\rho_b}{2}$, $0 < \rho_a < \rho_b$, $q \geqslant 1$ and $\varphi : (0,1] \to \mathbb{R}$, $\varphi(x) = -\ln x$ in (12). Then

$$|\ln I(\rho_a, \rho_b) - \ln A(\rho_a, \rho_b)| \leqslant \frac{M(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1\left[t^{1+\lambda}\phi(t) + t(1-t)^\lambda\phi(1-t)\right] dt\right)^{\frac{1}{q}}.$$

REMARK 15. In Theorem 6,

1. Let $g(x) = x$, $\zeta = 1$, $x = \frac{\rho_a+\rho_b}{2}$, $0 < \rho_a < \rho_b$, $q \geqslant 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(x) = x^n$ in (16). Then

$$|A(\rho_a, \rho_b) - L_n^n(\rho_a, \rho_b)| \leqslant \frac{M(\rho_b - \rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left[t^\lambda\phi(t) + (1-t)^\lambda\phi(1-t)\right] dt\right)^{\frac{1}{q}}.$$

2. Let $g(x) = x$, $\zeta = 1$, $x = \frac{\rho_a+\rho_b}{2}$, $0 < \rho_a < \rho_b$, $q \geqslant 1$ and $\varphi : (0,1] \to \mathbb{R}$, $\varphi(x) = -\ln x$ in (16). Then

$$|\ln I(\rho_a, \rho_b) - \ln A(\rho_a, \rho_b)| \leqslant \frac{M(\rho_b - \rho_a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left[t^\lambda\phi(t) + (1-t)^\lambda\phi(1-t)\right] dt\right)^{\frac{1}{q}}.$$

4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, the generalization of Fractional Ostrowski inequality via generalized Montgomery identity [14] with $\phi - \lambda$-convex. This class of functions contains many important classes including class of ϕ-convex [10], h-convex [23], Godunova-Levin s-convex [9], s-convex in the 2nd kind

[3] and hence contains class of convex and *MT*-convex [2]. It also contains class of *P*-convex [15] and class of Godunova-Levin functions [17]. We have stated our main result in section 2, which is the generalization of Ostrowski inequality via generalized Montgomery identity by generalized fractional integrals for $\phi - \lambda$-convex. Further, we used different techniques including Hölder's inequality [25] and power mean inequality [24] for generalization of Ostrowski inequality. In second last section we have given some applications in terms of special means including arithmetic, geometric, harmonic, logarithmic, identric and *p*-logarithmic means by using the midpoint inequalities.

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