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EXISTENCE RESULTS FOR A CAPUTO–HADAMARD TYPE FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. The main purpose of this paper is to establish the existence and uniqueness of mild solutions for a Caputo-Hadamard type fractional boundary value problem. Existence and uniqueness results are based on the Krasnoselskii fixed point theorem and the Banach contraction mapping principle. Finally, two examples are given to illustrate this work.

1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations have received the attention of many authors, see [1]–[20], [22]–[25] and the references therein.

In [18], Niyom et al. studied the existence and uniqueness of solutions of the following boundary value problem with four Riemann-Liouville fractional derivatives

$$\begin{cases} \left(\lambda D^{\alpha}+(1-\lambda)D^{\beta}\right)x(t)=f(t,x(t)),\ t\in(0,T),\\ x(0)=0,\ \mu D^{\gamma_1}x(T)+(1-\mu)D^{\gamma_2}x(T)=\gamma_3, \end{cases}$$

where D^{ϕ} is the Riemann-Liouville fractional derivative of order $\phi\in\{\alpha,\beta,\gamma_1,\gamma_2\}$ such that $1<\alpha,\beta<2$ and $0<\gamma_1,\gamma_2<\alpha-\beta$, $\gamma_3\in\mathbb{R}$, $0<\lambda\leqslant 1$, $0\leqslant\mu\leqslant 1$ are given constants and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. By using the Banach fixed point theorem, the Krasnoselskii fixed point theorem and the Leray-Schauder nonlinear alternative, the existence and uniqueness of solutions have been established.

The nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t)=f(t,x(t))+{}^{C}D^{\alpha-1}g(t,x(t)),\ 0<t\leqslant T,\\ x(0)=\theta_1>0,\ x'(0)=\theta_2>0, \end{cases}$$

has been investigated in [10], where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $1<\alpha\leqslant 2$, $g,f:[0,T]\times[0,\infty)\to[0,\infty)$ are given continuous functions, g is

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non-decreasing on x and $\theta_2 \geqslant g(0,\theta_1)$. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

Ahmad and Ntouyas in [1] discussed the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\mathfrak{D}_1^{\beta}u(t)-g(t,u_t)\right)=f(t,u_t),\ t\in[1,b],\\ u(t)=\phi(t),\ t\in[1-r,1],\\ \mathfrak{D}_1^{\beta}u(1)=\eta\in\mathbb{R}, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ and $\mathfrak{D}_1^{\beta}$ are the Caputo-Hadamard fractional derivatives, $0<\alpha,\beta<1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

Inspired and motivated by the works mentioned above, we study the existence and uniqueness of mild solutions for the following Caputo-Hadamard type fractional boundary value problem

$$\begin{cases} \left(\lambda\mathfrak{D}_1^{\alpha}+(1-\lambda)\mathfrak{D}_1^{\beta}\right)x(t)=f(t,x(t)),\ t\in(1,T),\\ x(1)=0,\ \mu\mathfrak{D}_1^{\gamma_1}x(T)+(1-\mu)\mathfrak{D}_1^{\gamma_2}x(T)=\gamma_3, \end{cases} \tag{1}$$

where $\mathfrak{D}_1^{\phi}$ is the Caputo-Hadamard fractional derivative of order $\phi\in\{\alpha,\beta,\gamma_1,\gamma_2\}$ such that $1<\alpha,\beta\leqslant 2$ and $0<\gamma_1,\gamma_2<\alpha-\beta$, $\gamma_3\in\mathbb{R}$, $0<\lambda\leqslant 1$, $0\leqslant\mu\leqslant 1$ are given constants and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. To show the existence and uniqueness of mild solutions, we transform (1) into an integral equation and then use the Krasnoselskii and Banach fixed point theorems.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and lemmas, and state some preliminaries results needed in later sections. Also, we present the Banach and Krasnoselskii fixed point theorems. For details on the Banach and Krasnoselskii theorems we refer the reader to [21]. In Section 3, we prove the existence and uniqueness of mild solutions for (1). Finally, two examples are given in Section 4 to illustrate our main results.

2. Preliminaries

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [13, 20].

DEFINITION 2.1. ([13]) The Hadamard fractional integral of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbb{R}$ is defined as

$$\mathfrak{I}_1^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},\ \alpha>0.$$

DEFINITION 2.2. ([13]) The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a continuous function $x : [1,+\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_1^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} (\delta^n x)(s) \frac{ds}{s}, \; n-1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$, $n \in \mathbb{N}$.

LEMMA 2.1. ([13]) *Let $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$ and $x \in C^n([1,T])$. Then*

$$(\mathfrak{I}_1^{\alpha} \mathfrak{D}_1^{\alpha} x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{(\delta^k x)(1)}{\Gamma(k+1)} (\log t)^k.$$

LEMMA 2.2. ([13]) *For all $\mu > 0$ and $\nu > -1$,*

$$\mathfrak{I}_1^{\mu} (\log t)^{\nu} = \frac{1}{\Gamma(\mu)} \int_1^t \left(\log \frac{t}{s}\right)^{\mu-1} (\log s)^{\nu} \frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\log t)^{\mu+\nu}.$$

LEMMA 2.3. *For $\omega \in C([1,T])$, the boundary value problem*

$$\begin{cases} \left(\lambda \mathfrak{D}_1^{\alpha} + (1-\lambda)\mathfrak{D}_1^{\beta}\right) x(t) = \omega(t), \; t \in (1,T), \\ x(1) = 0, \; \mu \mathfrak{D}_1^{\gamma_1} x(T) + (1-\mu)\mathfrak{D}_1^{\gamma_2} x(T) = \gamma_3, \end{cases} \tag{2}$$

has a unique mild solution given by

$$\begin{aligned} x(t) = {} & \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} x(s) \frac{ds}{s} + \frac{1}{\lambda\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \omega(s) \frac{ds}{s} \\ & + \frac{\log t}{\Lambda} \left(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} x(s) \frac{ds}{s} \right. \\ & - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\gamma_1-1} \omega(s) \frac{ds}{s} \\ & - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1} x(s) \frac{ds}{s} \\ & \left. - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-\gamma_2-1} \omega(s) \frac{ds}{s} \right), \; t \in J = [1,T], \end{aligned} \tag{3}$$

where the non zero constant Λ is defined by

$$\Lambda = \frac{\mu(\log T)^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{(1-\mu)(\log T)^{1-\gamma_2}}{\Gamma(2-\gamma_2)}. \tag{4}$$

Proof. From the first equation of (2), we get

$$\mathfrak{D}_1^{\alpha} x(t) = \frac{\lambda-1}{\lambda} \mathfrak{D}_1^{\beta} x(t) + \frac{1}{\lambda} \omega(t), \; t \in J. \tag{5}$$

Taking the Hadamard fractional integral of order α to both sides of (5), we obtain

$$x(t)=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}x(s)\frac{ds}{s}+\frac{1}{\lambda\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\omega(s)\frac{ds}{s}+C_1+C_2\log t,$$

for $C_1,C_2\in\mathbb{R}$. The boundary condition of (2) implies that $C_1=0$. Hence

$$x(t)=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}x(s)\frac{ds}{s}+\frac{1}{\lambda\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\omega(s)\frac{ds}{s}+C_2\log t. \tag{6}$$

Applying the Caputo-Hadamard fractional derivative of order $\psi\in\{\gamma_1,\gamma_2\}$ such that $0<\psi<\alpha-\beta$ to (6), we have

$$D^{\psi}x(t)=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta-\psi)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-\psi-1}x(s)\frac{ds}{s}+\frac{1}{\lambda\Gamma(\alpha-\psi)}\int_1^t(\log\frac{t}{s})^{\alpha-\psi-1}\omega(s)\frac{ds}{s}+C_2\frac{1}{\Gamma(2-\psi)}(\log t)^{1-\psi}.$$

Substituting the values $\psi=\gamma_1$ and $\psi=\gamma_2$ in the above relation and using the second condition of (2), we obtain

$$\begin{aligned}\gamma_3=&\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1}x(s)\frac{ds}{s}\\&+\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_1-1}\omega(s)\frac{ds}{s}+\frac{\mu\,(\log T)^{1-\gamma_1}}{\Gamma(2-\gamma_1)}C_2\\&+\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1}x(s)\frac{ds}{s}\\&+\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1}\omega(s)\frac{ds}{s}+\frac{(1-\mu)\,(\log T)^{1-\gamma_2}}{\Gamma(2-\gamma_2)}C_2,\end{aligned}$$

which leads to

$$\begin{aligned}C_2=\frac{1}{\Lambda}\Bigg[\gamma_3&-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1}x(s)\frac{ds}{s}\\&-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_1-1}\omega(s)\frac{ds}{s}\\&-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1}x(s)\frac{ds}{s}\\&-\frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1}\omega(s)\frac{ds}{s}\Bigg].\end{aligned}$$

Substituting the value of the constant C_2 in (6), we obtain the unique mild solution (3). This completes the proof. □

DEFINITION 2.3. A function $x \in C([1,T])$ is said to be a mild solution of the problem (1) if x satisfies the following associated integral equation of (1)

$$\begin{aligned} x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}x(s)\frac{ds}{s} + \frac{1}{\lambda\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s} \\ &+ \frac{\log t}{\Lambda}\left(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1}x(s)\frac{ds}{s}\right. \\ &- \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_1-1}f(s,x(s))\frac{ds}{s} \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1}x(s)\frac{ds}{s} \\ &\left.- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1}f(s,x(s))\frac{ds}{s}\right), \; t \in J. \end{aligned}$$

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a solution of (1).

DEFINITION 2.4. Let $(X, \|.\|)$ be a Banach space and $\mathscr{P} : X \to X$. The operator $\mathscr{P}$ is a contraction operator if there is an $\rho \in (0,1)$ such that $x, y \in X$ imply

$$\|\mathscr{P}x - \mathscr{P}y\| \leqslant \rho\,\|x-y\|.$$

THEOREM 2.1. (Banach [21]) *Let $\mathscr{K}$ be a nonempty closed convex subset of a Banach space X and $\mathscr{P} : \mathscr{K} \to \mathscr{K}$ be a contraction operator. Then there is a unique $x \in \mathscr{K}$ with $\mathscr{P}x = x$.*

THEOREM 2.2. (Krasnoselskii fixed point theorem [21]) *If $\mathscr{K}$ is a nonempty bounded, closed and convex subset of a Banach space X, $\mathscr{P}_1$ and $\mathscr{P}_2$ two operators defined on $\mathscr{K}$ with values in X such that*

i) $\mathscr{P}_1 x + \mathscr{P}_2 y \in \mathscr{K}$, for all $x, y \in \mathscr{K}$,

ii) $\mathscr{P}_2$ is continuous and compact,

iii) $\mathscr{P}_1$ is a contraction.

Then there exists $z \in \mathscr{K}$ such that $z = \mathscr{P}_1 z + \mathscr{P}_2 z$.

3. Existence and uniqueness results

Let $C := C([1,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[1,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t \in [1,T]\}$.

Our first result is based on the Krasnoselskii fixed point theorem.

THEOREM 3.1. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*
(H1) There exists a function $v \in C(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \leqslant v(t), \text{ for a.e. } t \in J \text{ and each } u \in \mathbb{R}.$$

(H2)

$$\Omega_1 = \frac{(\log T)^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{(\log T)^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1+1)} + \frac{(\log T)^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2+1)} < 1. \tag{7}$$

Then, the problem (1) has at least one mild solution on J.

Proof. Let $B_r = \{x \in C : \|x\| \leqslant r\}$ be a closed bounded and convex subset of C, where r is a fixed constant. Consider the operator $\mathscr{P} : C \to C$ defined by

$$\begin{aligned}(\mathscr{P}x)(t) = {} & \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} x(s)\frac{ds}{s} \\ & + \frac{1}{\lambda\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} \\ & + \frac{\log t}{\Lambda}\left(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} x(s)\frac{ds}{s}\right. \\ & - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma_1-1} f(s,x(s))\frac{ds}{s} \\ & - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1} x(s)\frac{ds}{s} \\ & \left. - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1} f(s,x(s))\frac{ds}{s}\right), \ t \in J.\end{aligned}$$

Let us define $\mathscr{P}_1, \mathscr{P}_2 : C \to C$ by

$$\begin{aligned}(\mathscr{P}_1x)(t) = {} & \frac{(\lambda-1)}{\lambda\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} x(s)\frac{ds}{s} \\ & - \frac{\log t}{\Lambda}\left(\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} x(s)\frac{ds}{s}\right. \\ & \left. + \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1} x(s)\frac{ds}{s}\right),\end{aligned}$$

and

$$\begin{aligned}(\mathscr{P}_2x)(t) = &\frac{1}{\lambda\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\\ &+\frac{\log t}{\Lambda}\left(\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma_1-1} f(s,x(s))\frac{ds}{s}\right.\\ &\left.-\frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1} f(s,x(s))\frac{ds}{s}\right).\end{aligned}$$

Clearly

$$(\mathscr{P}x)(t) = (\mathscr{P}_1x)(t) + (\mathscr{P}_2x)(t),\ t\in J. \tag{8}$$

Obviously the operator $\mathscr{P}$ has a fixed point is equivalent to $\mathscr{P}_1+\mathscr{P}_2$ has one, so it turns to prove that $\mathscr{P}_1+\mathscr{P}_2$ has a fixed point. We shall show that the operators $\mathscr{P}_1$ and $\mathscr{P}_2$ satisfy all conditions of the Krasnoselskii fixed point theorem. The proof will be given in several steps.

Step 1. $\mathscr{P}B_r \subset B_r$. Let us select

$$r \geqslant \frac{\|v\|\Omega_2 + |\gamma_3|(\log T)/\Lambda}{1-\Omega_1},$$

where Ω_1 defined by (H2) and

$$\Omega_2 = \frac{(\log T)^{\alpha}}{\lambda\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda\Gamma(\alpha-\gamma_1+1)} + \frac{(\log T)^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda\Gamma(\alpha-\gamma_2+1)}. \tag{9}$$

For any $x\in B_r$, we have

$$\begin{aligned}|(\mathscr{P}x)(t)| \leqslant &\left|\frac{(\lambda-1)}{\lambda\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} x(s)\frac{ds}{s}\right.\\ &+\frac{1}{\lambda\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\\ &-\frac{(\log t)\mu(\lambda-1)}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} x(s)\frac{ds}{s}\\ &-\frac{(\log t)(1-\mu)(\lambda-1)}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1} x(s)\frac{ds}{s}\\ &+\frac{\log t}{\Lambda}\left(\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} f(s,x(s))\frac{ds}{s}\right.\\ &\left.\left.-\frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1} f(s,x(s))\frac{ds}{s}\right)\right|\end{aligned}$$

$$\leqslant \|x\| \left(\frac{(\log T)^{\alpha-\beta} |\lambda - 1|}{\lambda \Gamma(\alpha-\beta+1)} + \frac{(\log T)^{\alpha-\beta-\gamma_1+1} \mu |\lambda-1|}{\lambda \Lambda \Gamma(\alpha-\beta-\gamma_1+1)} \right.$$
$$\left. + \frac{(\log T)^{\alpha-\beta-\gamma_2+1} (1-\mu)|\lambda-1|}{\lambda \Lambda \Gamma(\alpha-\beta-\gamma_2+1)} \right) + \frac{|\gamma_3| \log T}{\Lambda}$$
$$+ \|v\| \left(\frac{(\log T)^{\alpha}}{\lambda \Gamma(\alpha+1)} + \frac{(\log T)^{\alpha-\gamma_1+1} \mu}{\lambda \Lambda \Gamma(\alpha-\gamma_1+1)} + \frac{(\log T)^{\alpha-\gamma_2+1} (1-\mu)}{\lambda \Lambda \Gamma(\alpha-\gamma_2+1)} \right)$$
$$\leqslant r\Omega_1 + \|v\|\Omega_2 + \frac{|\gamma_3| \log T}{\Lambda} \leqslant r,$$

which implies that $\mathscr{P} B_r \subset B_r$.

Step 2. $\mathscr{P}_2$ is compact and continuous. Observe that the operator P_2 is uniformly bounded in view of Step 1. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then we obtain

$$|(\mathscr{P}_2 x)(t_2) - (\mathscr{P}_2 x)(t_1)|$$
$$\leqslant \frac{1}{\lambda \Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\alpha-1} - \left(\log \frac{t_1}{s} \right)^{\alpha-1} \right] v(s) \frac{ds}{s}$$
$$+ \frac{1}{\lambda \Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha-1} v(s) \frac{ds}{s}$$
$$+ \frac{|\log t_2 - \log t_1|}{\Lambda} \left(|\gamma_3| + \frac{\mu}{\lambda \Gamma(\alpha-\gamma_1)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\gamma_1-1} v(s) \frac{ds}{s} \right.$$
$$\left. + \frac{(1-\mu)}{\lambda \Gamma(\alpha-\gamma_2)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\gamma_2-1} v(s) \frac{ds}{s} \right)$$
$$\leqslant \frac{\|v\|}{\lambda \Gamma(\alpha+1)} \left((\log t_2)^{\alpha} - (\log t_1)^{\alpha} \right)$$
$$+ \left(|\gamma_3| + \frac{\mu (\log T)^{\alpha-\gamma_1} \|v\|}{\lambda \Gamma(\alpha-\gamma_1+1)} + \frac{(1-\mu)(\log T)^{\alpha-\gamma_2} \|v\|}{\lambda \Gamma(\alpha-\gamma_2+1)} \right) \frac{|\log t_2 - \log t_1|}{\Lambda},$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathscr{P}_2(B_r)$ is equicontinuous. So, by the Arzelá-Ascoli theorem, $\mathscr{P}_2(B_r)$ is a relatively compact set. Hence $\mathscr{P}_2$ is compact. Moreover, the continuity of f implies that $\mathscr{P}_2$ is continuous.

Step 3. $\mathscr{P}_1$ is contraction. Let $x, y \in B_r$. Then, we have

$$|(\mathscr{P}_1 x)(t) - (\mathscr{P}_1 y)(t)|$$
$$\leqslant \frac{|\lambda-1|}{\lambda \Gamma(\alpha-\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-1} |x(s) - y(s)| \frac{ds}{s}$$
$$+ \frac{(\log T) \mu |\lambda-1|}{\lambda \Lambda \Gamma(\alpha-\beta-\gamma_1)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-\gamma_1-1} |x(s) - y(s)| \frac{ds}{s}$$
$$+ \frac{(\log T)(1-\mu)|\lambda-1|}{\lambda \Lambda \Gamma(\alpha-\beta-\gamma_2)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha-\beta-\gamma_2-1} |x(s) - y(s)| \frac{ds}{s}$$

$$\leqslant \left\{ \frac{(\log T)^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{(\log T)^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1+1)} + \frac{(\log T)^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2+1)} \right\} \|x-y\|$$
$$= \Omega_1\|x-y\|,$$

which is contraction, since $\Omega_1 < 1$.

From the above three steps, we conclude by the Krasnoselskii fixed point theorem that $\mathscr{P}$ has a fixed point which is a mild solution of the problem (1). □

Our second result is based on the Banach fixed point theorem.

THEOREM 3.2. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that (H3) There exists a constant $L > 0$ such that*

$$|f(t,x) - f(t,y)| \leqslant L|x-y|, \text{ for } t \in J,\ x,y \in \mathbb{R}.$$

If

$$L\Omega_2 + \Omega_1 < 1, \tag{10}$$

where Ω_1 and Ω_2 are given by (7) and (9), respectively, then the problem (1) has a unique mild solution on J.

Proof. Choosing

$$R \geqslant \frac{N\Omega_2 + |\gamma_3|(\log T)/\Lambda}{1 - L\Omega_2 - \Omega_1},$$

where $N = \sup_{t\in J}|f(t,0)|$ and Λ is given by (4). We prove that $\mathscr{P}B_R \subset B_R$, where

$$B_R = \{x \in C : \|x\| \leqslant R\}.$$

For any $x \in B_R$, we have

$$|(\mathscr{P}x)(t)| \leqslant \left| \frac{(\lambda-1)}{\lambda\Gamma(\alpha-\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} x(s)\frac{ds}{s} + \frac{1}{\lambda\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} - \frac{(\log t)\mu(\lambda-1)}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} x(s)\frac{ds}{s} - \frac{(\log t)(1-\mu)(\lambda-1)}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1} x(s)\frac{ds}{s} + \frac{\log t}{\Lambda}\left(\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1} f(s,x(s))\frac{ds}{s}\right.$$

$$
\begin{aligned}
&\left.\left. -\frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1} f(s,x(s))\frac{ds}{s}\right)\right| \\
&\leqslant \|x\|\left(\frac{(\log T)^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{(\log T)^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1+1)}\right. \\
&\quad\left.+\frac{(\log T)^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2+1)}\right)+\frac{|\gamma_3|\log T}{\Lambda} \\
&\quad+(L\|x\|+N)\left(\frac{(\log T)^{\alpha}}{\lambda\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda\Gamma(\alpha-\gamma_1+1)}+\frac{(\log T)^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda\Gamma(\alpha-\gamma_2+1)}\right) \\
&\leqslant (\Omega_1+L\Omega_2)R+N\Omega_2+\frac{|\gamma_3|\log T}{\Lambda}\leqslant R.
\end{aligned}
$$

This means that $\|\mathscr{P}x\|\leqslant R$, which leads to $\mathscr{P}B_R\subset B_R$.

Next, we let $x,y\in C$. Then, for $t\in J$, we have

$$
\begin{aligned}
&|(\mathscr{P}x)(t)-(\mathscr{P}y)(t)| \\
&\leqslant \frac{(\lambda-1)}{\lambda\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}|x(s)-y(s)|\frac{ds}{s} \\
&\quad+\frac{1}{\lambda\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,x(s))-f(s,y(s))|\frac{ds}{s} \\
&\quad+\frac{(\log t)\mu(\lambda-1)}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1}|x(s)-y(s)|\frac{ds}{s} \\
&\quad+\frac{(\log t)(1-\mu)(\lambda-1)}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_2-1}|x(s)-y(s)|\frac{ds}{s} \\
&\quad+\frac{\log t}{\Lambda}\left(\gamma_3-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\beta-\gamma_1-1}|f(s,x(s))-f(s,y(s))|\frac{ds}{s}\right. \\
&\quad\left.+\frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma_2-1}|f(s,x(s))-f(s,y(s))|\frac{ds}{s}\right) \\
&\leqslant L\|x-y\|\left[\frac{(\log T)^{\alpha}}{\lambda\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda\Gamma(\alpha-\gamma_1+1)}+\frac{(\log T)^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda\Gamma(\alpha-\gamma_2+1)}\right] \\
&\quad+\|x-y\|\left[\frac{(\log T)^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)}+\frac{(\log T)^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_1+1)}\right. \\
&\quad\left.+\frac{(\log T)^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda\Gamma(\alpha-\beta-\gamma_2+1)}\right] \\
&\leqslant (L\Omega_2+\Omega_1)\|x-y\|,
\end{aligned}
$$

which implies that

$$
\|\mathscr{P}x-\mathscr{P}y\|\leqslant(L\Omega_2+\Omega_1)\|x-y\|.
$$

From (10), it follows that $\mathscr{P}$ is a contraction. Therefore, by the Banach fixed point theorem, we see that $\mathscr{P}$ has a fixed point which is a unique mild solution of the problem (1). □

4. Two examples

In this section, we give two examples to illustrate our main results.

EXAMPLE 4.1. Consider the following fractional boundary value problem

$$\begin{cases} \frac{35}{38}\mathfrak{D}_1^{8/5}x(t)+\frac{3}{38}\mathfrak{D}_1^{6/5}x(t)=\dfrac{\sin x(t)}{7+t}+\dfrac{4}{5},\ t\in[1,e],\\ x(1)=0,\ \frac{14}{30}\mathfrak{D}_1^{1/4}x(e)+\frac{16}{30}\mathfrak{D}_1^{1/5}x(e)=\frac{4}{5}. \end{cases} \tag{11}$$

Here $\lambda=35/38$, $\alpha=8/5$, $\beta=6/5$, $\mu=14/30$, $\gamma_1=1/4$, $\gamma_2=1/5$, $\gamma_3=4/5$, $T=e$ and $f(t,x)=\frac{\sin x}{7+t}+4/5$. Observe that $0<\gamma_1,\gamma_2<2/5=\alpha-\beta$. It is obvious that

$$|f(t,x)|\leqslant\frac{1}{7+t}+\frac{4}{5}=v(t),$$

which satisfies the condition (H1) of Theorem 3.1. In addition, we get

$$\Omega_1\simeq 0.757<1.$$

Hence, by Theorem 3.1, the fractional boundary value problem (11) has at least one mild solution. Also, we have

$$|f(t,x)-f(t,y)|\leqslant\frac{1}{8}|x-y|,$$

so, $L=1/8$, $\Omega_2\simeq 1.581$ and

$$L\Omega_2+\Omega_1\simeq 0.955<1.$$

Then by Theorem 3.2, the problem (11) has a unique mild solution.

EXAMPLE 4.2. Consider the following fractional boundary value problem

$$\begin{cases} \frac{43}{45}\mathfrak{D}_1^{5/3}x(t)+\frac{2}{45}\mathfrak{D}_1^{4/3}x(t)=\dfrac{x(t)+\cos x(t)}{1+3t}+\dfrac{5}{7},\ t\in[1,e],\\ x(1)=0,\ \frac{17}{40}\mathfrak{D}_1^{1/6}x(e)+\frac{23}{40}\mathfrak{D}_1^{1/7}x(e)=\frac{6}{7}. \end{cases} \tag{12}$$

Here $\lambda=43/45$, $\alpha=5/3$, $\beta=4/3$, $\mu=17/40$, $\gamma_1=1/6$, $\gamma_2=1/7$, $\gamma_3=6/7$, $T=e$ and $f(t,x)=\frac{x+\cos x}{1+3t}+5/7$. Observe that $0<\gamma_1,\gamma_2<1/3=\alpha-\beta$. We have

$$|f(t,x)-f(t,y)|\leqslant\frac{1}{2}|x-y|,$$

which satisfies the condition (H3) of Theorem 3.2 with $L=1/2$. Also, we obtain $\Omega_1\simeq 0.0997$, $\Omega_2\simeq 1.4321$ and

$$L\Omega_2+\Omega_1\simeq 0.8157<1.$$

Hence, by Theorem 3.2, the problem (12) has a unique mild solution.

5. Conclusion

The Caputo-Hadamard type fractional boundary value problem is considered. So, we have studied the existence and uniqueness of mild solutions. The main tool of this work is the Krasnoselskii and Banach fixed point theorems. However, by introducing new fixed mappings, we obtain new existence conditions. Besides, two examples are exhibited to validate the effectiveness of our results. The obtained results have a contribution to the related literature, and they extend the results in [18] from the case of Riemann-Liouville type fractional boundary value problems to that case with Caputo-Hadamard type fractional boundary value problems.

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