Fractional
Differential
Calculus
Volume 12, Number 1 (2022), 1–12
doi:10.7153/fdc-2022-12-01

EXISTENCE OF MILD SOLUTIONS FOR SEMILINEAR EVOLUTION EQUATION USING HILFER FRACTIONAL DERIVATIVES

BANDITA ROY AND SWAROOP NANDAN BORA*

(*Communicated by C. Cuevas*)

Abstract. In this article, we discuss the existence of solutions of a class of Cauchy-type fractional semilinear evolution equation involving Hilfer fractional derivative. A nonlocal Cauchy problem is also discussed for evolution equations. The results are obtained by using various results of fractional calculus and fixed point theorems in the weighted space of continuous functions. Examples are presented to illustrate the derived theory.

1. Introduction

The theory of fractional differential equations is considered as an important branch of differential equation and hence it has been emerging as an important area of investigation in the last few decades due to its growing number of applications in various areas of applied sciences and engineering [11, 12, 14, 15]. This subject is as old as classical calculus, that is, around the time (17th century) when Newton and Leibnitz independently developed differential and integral calculus.

The primary advantage of fractional calculus over classical calculus very well lies in the fact that fractional derivatives provide an excellent tool for describing memory and hereditary properties of various materials and processes, and make the fractional order models more realistic than the integer order models. Recently, many models have been reformulated and expressed in terms of fractional differential equations so that their physical meaning can be incorporated in the mathematical models more realistically. For quite some time now, fractional differential equations are being considered as an alternative model to the nonlinear differential equations.

Many processes can be described accurately by using systems of differential equations containing different types of fractional derivatives. There are many possible generalizations of the n-th order operator $\frac{d^n}{dx^n}$ to the case when n is not an integer, named as Riemann-Liouville, Caputo, Grünwald-Letnikov, Hadamard's etc. These operators interpolate between integer order differential operators. The most popular among them

Mathematics subject classification (2020): 26A33, 34A08, 34G20.
Keywords and phrases: Fractional derivative, Hilfer derivative, existence of solution, fixed point theorems.

* Corresponding author.

are the derivatives expressed in the Riemann-Liouville and Caputo sense. In [11], Hilfer proposed a new definition of fractional derivative, called Hilfer fractional derivative, which includes both Riemann-Liouville fractional derivative and Caputo fractional derivative.

In [7], Furati et al. took up an initial value problem for a class of nonlinear fractional differential equation involving Hilfer fractional derivative. Wang and Zhang [17] investigated the following nonlocal initial value problem:

$$D_{a^+}^{\alpha,\beta}x(t) = f(t,x(t)), \;\; \alpha \in (0,1), \;\; \beta \in [0,1], \;\; t \in (a,b],$$
$$I_{a^+}^{1-\gamma}x(a^+) = \sum_{i=1}^{m} \lambda_i x(\tau_i), \;\; \gamma = \alpha + \beta - \alpha\beta, \;\; \tau_i \in (a,b],$$

where $D_{0^+}^{\alpha,\beta}$ denotes the Hilfer derivative of order α and type β which will be defined in the next section. Some more problems involving Hilfer derivative can be found in [1, 2, 16].

Hilfer derivative is notably more general than Riemann-Liouville and Caputo fractional derivatives and so the results that are obtained are also more general than the known results. The facts that fractional differential equations encompass more attributes and that Hilfer derivative is more general in nature motivate us to pursue studies in this area.

In this article, we study the existence and uniqueness of mild solutions of the following semilinear evolution equation:

$$D_{0^+}^{\alpha,\beta}x(t) = A(t)x(t) + f(t,x(t)),$$
$$\alpha \in [0,1], \;\; \beta \in (0,1), \;\; t \in (0,b] = J',$$

with initial condition

$$I_{0^+}^{1-\gamma}x(0) = x_0,$$

and nonlocal condition

$$I_{0^+}^{1-\gamma}x(0) - g(x) = x_0,$$

where $A(t)$ is a bounded linear operator on $\mathbb{R}$ for each $t \in J = [0,b]$, $1-\gamma = (1-\alpha)(1-\beta)$ and $x_0 \in \mathbb{R}$. $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given nonlinear function and g is a given function satisfying some assumptions which will be specified later. For more details related to the above class of differential equations, the readers are referred to [3, 10, 13].

Nonlocal conditions are introduced to extend the study of classical initial value problems. The study of nonlocal conditions was initiated by Byszewski [4] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski and Laksmikantham [5], the nonlocal conditions can be more useful than the standard initial condition in describing some physical phenomena.

2. Preliminaries

In this section we present the following definitions and theorems which will be used in establishing our results.

DEFINITION 1. [7, 17] Let $-\infty < a < b < \infty$ and $C[a,b]$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm $\|f\|_C = \sup_{t\in[a,b]} |f(t)|$. For $0 \leqslant 1-\gamma < 1$, the weighted space $C_{1-\gamma}[a,b]$ of continuous functions f on $(a,b]$ is defined as

$$C_{1-\gamma}[a,b] = \{f : (a,b] \to \mathbb{R} : (t-a)^{1-\gamma} f(t) \in C[a,b]\}.$$

Then $C_{1-\gamma}[a,b]$ is a Banach space with the norm

$$\|f\|_{C_{1-\gamma}} = \|(t-a)^{1-\gamma} f(t)\|_C,\ C_0[a,b] = C[a,b].$$

By $L^1[a,b]$, we denote the space of all Lebesgue-integrable functions $f : [a,b] \to \mathbb{R}$ with the norm

$$\|f\|_1 = \int_a^b |f(t)| dt.$$

DEFINITION 2. [8] The left-sided Riemann-Liouville fractional integral $I^{\alpha}_{a+} f$ of order $\alpha > 0$ is defined by

$$(I^{\alpha}_{a+} f)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,\ t > a,$$

provided that the integral exists. Here $\Gamma(.)$ is the gamma function. When $\alpha = 0$, we define $I^0_{a+} f = f$.

DEFINITION 3. [8] The left-sided Riemann-Liouville fractional derivative $D^{\alpha}_{a+} f$ of order α $(0 < \alpha < 1)$ is defined by

$$(D^{\alpha}_{a+} f)(t) := \frac{d}{dt} (I^{1-\alpha}_{a+} f)(t),$$

provided the right-hand side exists.

DEFINITION 4. [6, 18] The left-sided Caputo fractional derivative ${}^C D^{\alpha}_{a+} f$ of order $\alpha \geqslant 0$ is defined by

$${}^C D^{\alpha}_{a+} f = I^{n-\alpha}_{a+} D^n f,$$

where n is the least integer $\geqslant \alpha$, whenever $D^n f \in L^1[a,b]$.

DEFINITION 5. [9] The left-sided Hilfer fractional derivative of order $0 \leqslant \alpha \leqslant 1$ and type $0 < \beta < 1$ of a function $f(t)$ with lower limit a is defined as

$$D^{\alpha,\beta}_{a+} f(t) = I^{\alpha(1-\beta)}_{a+} D I^{(1-\alpha)(1-\beta)}_{a+} f(t), \quad \text{where } D := \frac{d}{dt}.$$

In order to solve our problem, the following function spaces are considered:

$$C^{\alpha,\beta}_{1-\gamma}[0,b] = \{f \in C_{1-\gamma}[0,b], D^{\alpha,\beta}_{0+} f \in C_{1-\gamma}[0,b]\}$$

and

$$C^{\gamma}_{1-\gamma}[0,b] = \{f \in C_{1-\gamma}[0,b], D^{\gamma}_{0+} f \in C_{1-\gamma}[0,b]\}.$$

Since $D^{\alpha,\beta}_{0+} f = I^{\alpha(1-\beta)}_{0+} D^{\gamma}_{0+} f$, it follows from [7], Theorem 11 that

$$C^{\gamma}_{1-\gamma}[0,b] \subset C^{\alpha,\beta}_{1-\gamma}[0,b].$$

THEOREM 1. (Banach fixed point theorem) [12] *Let (X,d) be a nonempty complete metric space. Let $T: X \to X$ be a map such that for any $x,y \in X$,*

$$d(Tx,Ty) \leqslant kd(x,y), \quad 0 \leqslant k < 1$$

holds. Then the operator T has a unique fixed point $x^* \in X$.

THEOREM 2. (Krasnoselskii's fixed point theorem) [3] *Let S be a nonempty closed, convex subset of a Banach space X. Let P and Q be two operators such that*

(1) $Px + Qy \in S$ for every pair $x,y \in S$,
(2) P is a contraction mapping,
(3) Q is compact and continuous.
Then there exists $z \in S$ such that $z = Pz + Qz$.

THEOREM 3. (Schauder's fixed point theorem) [18] *Let X be a Banach space and $S \subset X$ a convex, closed and bounded set. If $T: S \to S$ is a continuous operator such that $TS \subset X$ is relatively compact, then T has at least one fixed point in S.*

3. Semilinear evolution equation

Consider the fractional semilinear evolution equation

$$\left. \begin{array}{l} D^{\alpha,\beta}_{0+} x(t) = A(t)x(t) + f(t,x(t)), \ \alpha \in [0,1], \ \beta \in (0,1), \ t \in (0,b], \\ I^{1-\gamma}_{0+} x(0) = x_0, \end{array} \right\} \tag{1}$$

where $A(t)$ is a bounded linear operator on $\mathbb{R}$ and $x_0 \in \mathbb{R}$.

THEOREM 4. *Assume that*
(i) $A(.)x(.) \in C_{1-\gamma}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$,
(ii) $f(.,x(.)) \in C_{1-\gamma}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$, hold.
Then $x \in C^{\gamma}_{1-\gamma}[0,b]$ is a solution of the Cauchy problem (1) if and only if x satisfies the integral equation

$$x(t) = \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [A(s)x(s) + f(s,x(s))] ds, \ t \in J'. \tag{2}$$

Proof. We refer to Theorem 23 in [7] which has been modified and presented above in conformation with our proposed problem. The proof follows in a similar pattern. □

Now we proceed to state our problem and establish the result.

First we introduce the following assumptions:

$(H1)$ $A(t)$ is a bounded linear operator on $\mathbb{R}$ for each $t \in [0,b]$. The function $t \to A(t)$ is continuous in the uniform operator topology.

$(H2)$ $A(.)x(.) \in C_{1-\gamma}^{\alpha(1-\beta)}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$,

$(H3)$ $f : (0,b] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(.,x(.)) \in C_{1-\gamma}^{\alpha(1-\beta)}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$. For all $x,y \in \mathbb{R}$, there exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leqslant L|x-y|.$$

Let $M = \sup_{t\in[0,b]} \|A(t)\|$ and set $F(t) = f(t,0)$.

The following existence result for problem (1) will be established by using Banach fixed point theorem.

THEOREM 5. *Assume that* $(H1)$–$(H3)$ *hold. If*

$$\xi_1 = (M+L)\frac{B(\gamma,\beta)}{\Gamma(\beta)}b^{\beta} < 1,$$

then there exists a unique solution for the Cauchy type problem (1) *in* $C_{1-\gamma}^{\gamma}[0,b] \subset C_{1-\gamma}^{\alpha,\beta}[0,b]$.

Proof. According to Theorem 4, it is sufficient to prove the existence result for the equivalent integral equation (2). Define $T_1 : C_{1-\gamma}[0,b] \to C_{1-\gamma}[0,b]$ by

$$(T_1x)(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[A(s)x(s) + f(s,x(s))]ds, \quad t \in (0,b].$$

Let $\phi_1 = \frac{|x_0|}{\Gamma(\gamma)} + \frac{B(\gamma,\beta)}{\Gamma(\beta)}b^{\beta}\|F\|_{C_{1-\gamma}}$. Choose $r \geqslant \frac{\phi_1}{1-\xi_1}$. Then we can show that $T_1B_r \subset B_r$ where $B_r = \{x \in C_{1-\gamma}[0,b] : \|x\|_{C_{1-\gamma}} \leqslant r\}$.

Let $x \in B_r$. Then we get

$$\begin{aligned} t^{1-\gamma}(T_1x)(t) &= \frac{x_0}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}A(s)x(s)ds \\ &\quad + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,x(s))ds. \end{aligned}$$

Therefore,

$$\begin{aligned}
&|t^{1-\gamma}(T_1x)(t)|\\
&\leqslant \frac{|x_0|}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|A(s)\||x(s)|ds\\
&\quad+\frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[|f(s,x(s))-f(s,0)|+|f(s,0)|]ds\\
&\leqslant \frac{|x_0|}{\Gamma(\gamma)}+\frac{Mt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)|ds+\frac{Lt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)|ds+\frac{t^{1-\gamma}}{\Gamma(\beta)}\\
&\quad\times\int_0^t (t-s)^{\beta-1}|F(s)|ds\\
&\leqslant r.
\end{aligned}$$

Now, take $x,y\in C_{1-\gamma}[0,b]$. Then we get

$$\begin{aligned}
&|t^{1-\gamma}((T_1x)(t)-(T_1y)(t))|\\
&\leqslant \frac{Mt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)-y(s)|ds+\frac{Lt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)-y(s)|ds\\
&\leqslant \frac{M+L}{\Gamma(\beta)}t^{1-\gamma}\int_0^t (t-s)^{\beta-1}|x(s)-y(s)|ds\\
&\leqslant \frac{M+L}{\Gamma(\beta)}B(\gamma,\beta)b^{\beta}\|x-y\|_{C_{1-\gamma}},
\end{aligned}$$

which gives

$$\|T_1x-T_1y\|_{C_{1-\gamma}}\leqslant \frac{(M+L)}{\Gamma(\beta)}B(\gamma,\beta)b^{\beta}\|x-y\|_{C_{1-\gamma}}.$$

Thus, T_1 is a contraction mapping on $C_{1-\gamma}[0,b]$. By applying Banach fixed point theorem, we know that the operator T_1 has a unique fixed point on $C_{1-\gamma}[0,b]$. Then by repeating the process of the proof carried out in Theorem 25 of [7], one can show that the solution is actually in $C^{\gamma}_{1-\gamma}[0,b]$. □

4. Nonlocal problem

In this section, we discuss the existence of solution of the nonlocal problem

$$\left.\begin{aligned}
D^{\alpha,\beta}_{0^+}x(t)=A(t)x(t)+f(t,x(t)),\ \ \alpha\in[0,1],\ \ \beta\in(0,1),\ \ t\in(0,b],\\
I^{1-\gamma}_{0^+}x(0)-g(x)=x_0,\qquad\qquad\qquad\qquad\qquad\qquad\qquad
\end{aligned}\right\}\tag{3}$$

where $g:C_{1-\gamma}[0,b]\to\mathbb{R}$ is a continuous function satisfying the following condition:

$(H4)$ there exists a constant $N>0$ such that

$$|g(x)-g(y)|\leqslant N\|x-y\|_{C_{1-\gamma}}\ \text{for all}\ x,y\in C_{1-\gamma}[0,b].$$

THEOREM 6. *Assume that* $(H1)$–$(H4)$ *hold. If*

$$\xi_2 = \frac{N}{\Gamma(\gamma)} + (M+L)\frac{B(\gamma,\beta)}{\Gamma(\beta)}b^{\beta} < 1,$$

then there exists a unique solution for equation (3) *in* $C^{\gamma}_{1-\gamma}[0,b] \subset C^{\alpha,\beta}_{1-\gamma}[0,b]$.

Proof. Define $T_2 : C_{1-\gamma}[0,b] \to C_{1-\gamma}[0,b]$ by

$$(T_2x)(t) = \frac{x_0 + g(x)}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[A(s)x(s) + f(s,x(s))]ds, \quad t \in (0,b].$$

Choose $r \geqslant \frac{\phi_2}{1-\xi_2}$, where $\phi_2 = \frac{|x_0|+|g(0)|}{\Gamma(\gamma)} + \frac{B(\gamma,\beta)}{\Gamma(\beta)}b^{\beta}\|F\|_{C_{1-\gamma}}$. Then we can show that $T_1B_r \subset B_r$ where $B_r = \{x \in C_{1-\gamma}[0,b] : \|x\|_{C_{1-\gamma}} \leqslant r\}$.

Let $x \in B_r$. Then we get

$$t^{1-\gamma}(T_2x)(t) = \frac{x_0 + g(x)}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}A(s)x(s)ds + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} \times f(s,x(s))ds.$$

Therefore,

$$\begin{aligned}
&|t^{1-\gamma}(T_2x)(t)| \\
&\leqslant \frac{|x_0| + |g(x)|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|A(s)\||x(s)|ds + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[|f(s,x(s)) \\
&\quad - f(s,0)| + |f(s,0)|]ds \\
&\leqslant \frac{|x_0|}{\Gamma(\gamma)} + \frac{|g(x) - g(0)| + |g(0)|}{\Gamma(\gamma)} + \frac{Mt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)|ds + \frac{Lt^{1-\gamma}}{\Gamma(\beta)} \\
&\quad \times \int_0^t (t-s)^{\beta-1}|x(s)|ds + \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|F(s)|ds \\
&\leqslant \frac{|x_0|}{\Gamma(\gamma)} + \frac{N\|x\|_{C_{1-\gamma}}}{\Gamma(\gamma)} + \frac{|g(0)|}{\Gamma(\gamma)} + \frac{M+L}{\Gamma(\beta)}\|x\|_{C_{1-\gamma}}B(\gamma,\beta)t^{\beta} + \frac{\|F\|_{C_{1-\gamma}}}{\Gamma(\beta)}B(\gamma,\beta)t^{\beta} \\
&\leqslant r.
\end{aligned}$$

Let $x, y \in C_{1-\gamma}[0,b]$. Then

$$\begin{aligned}
|t^{1-\gamma}((T_2x)(t) - (T_2y)(t))| &\leqslant \frac{|g(x) - g(y)|}{\Gamma(\gamma)} + \frac{Mt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s) - y(s)|ds \\
&\quad + \frac{Lt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s) - y(s)|ds \\
&\leqslant \frac{N}{\Gamma(\gamma)}\|x-y\|_{C_{1-\gamma}} + \frac{M+L}{\Gamma(\beta)}t^{1-\gamma}\int_0^t (t-s)^{\beta-1}|x(s) - y(s)|ds \\
&\leqslant \left[\frac{N}{\Gamma(\gamma)} + \frac{M+L}{\Gamma(\beta)}B(\gamma,\beta)b^{\beta}\right]\|x-y\|_{C_{1-\gamma}}.
\end{aligned}$$

By applying Banach fixed point theorem, we get the desired result. $\square$

Our next result will be established by employing Krasnoselskii's fixed point theorem. Here, we replace $(H3)$ by $(H3)'$ with the following linear growth condition:

$(H3)'$ $f:(0,b]\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(.,x(.))\in C_{1-\gamma}^{\alpha(1-\beta)}[0,b]$ for any $x\in C_{1-\gamma}[0,b]$. There exist constants $L>0$ and $K\geqslant 0$ such that

$$|f(t,x)|\leqslant L|x|+K,\ \text{ for all } x\in\mathbb{R}.$$

THEOREM 7. *Assume that the hypotheses* $(H1)$, $(H2)$, $(H3)'$ *and* $(H4)$ *are satisfied and* $\xi_2<1$. *Then nonlocal problem* (3) *has at least one solution in* $C_{1-\gamma}^{\gamma}[0,b]\subset C_{1-\gamma}^{\alpha,\beta}[0,b]$.

Proof. Choose

$$r\geqslant\frac{\phi_3}{1-\xi_2},\qquad \text{where }\ \phi_3=\frac{|x_0|+|g(0)|}{\Gamma(\gamma)}+\frac{K}{\Gamma(\beta)}b^{\beta+1-\gamma},$$

and define the operators P and Q on B_r as

$$(Px)(t)=\frac{x_0+g(x)}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}A(s)x(s)ds,$$

and

$$(Qx)(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f(s,x(s))ds.$$

We subdivide the proof into several steps.

Step 1: To show $Px+Qy\in B_r$ for every $x,y\in B_r$.

For $x\in B_r$, we get

$$\|Px\|_{C_{1-\gamma}}\leqslant\frac{|x_0|+|g(0)|+N\|x\|_{C_{1-\gamma}}}{\Gamma(\gamma)}+\frac{MB(\gamma,\beta)}{\Gamma(\beta)}b^{\beta}\|x\|_{C_{1-\gamma}},$$

and

$$\|Qx\|_{C_{1-\gamma}}\leqslant\frac{LB(\gamma,\beta)}{\Gamma(\beta)}b^{\beta}\|x\|_{C_{1-\gamma}}+\frac{Kb^{\beta+1-\gamma}}{\Gamma(\beta+1)}.$$

Therefore, for every $x,y\in B_r$,

$$\|Px+Qy\|_{C_{1-\gamma}}\leqslant\|Px\|_{C_{1-\gamma}}+\|Qy\|_{C_{1-\gamma}}\leqslant r.$$

Step 2: To show that P is a contraction mapping.

It can be easily shown that, for any $x,y\in B_r$,

$$\|Px-Py\|_{C_{1-\gamma}}\leqslant\xi_2\|x-y\|_{C_{1-\gamma}}.$$

Step 3: To show that Q is compact and continuous.

Q is continuous: Let (x_n) be a sequence such that $x_n \to x$ in $C_{1-\gamma}[0,b]$. Then for each $t \in (0,b]$,

$$\begin{aligned}|t^{1-\gamma}((Qx_n)(t)-(Qx)(t))| &\leqslant \frac{t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|f(s,x_n(s))-f(s,x(s))|ds \\ &\leqslant \frac{b^\beta}{\Gamma(\beta)}B(\gamma,\beta)\|f(.,x_n(.))-f(.,x(.))\|_{C_{1-\gamma}},\end{aligned}$$

which implies

$$\|Qx_n-Qx\|_{C_{1-\gamma}} \leqslant \frac{b^\beta B(\gamma,\beta)}{\Gamma(\beta)}\|f(.,x_n(.))-f(.,x(.))\|_{C_{1-\gamma}}.$$

Therefore, Q is continuous.

Q maps bounded sets into bounded sets in $C_{1-\gamma}[0,b]$: It is enough to show that, for any $r^*>0$, there exists a $s^*>0$ such that for each $x \in B_{r^*}$, we have $Qx \in B_{s^*}$.

We have,

$$\|Qx\|_{C_{1-\gamma}} \leqslant \frac{LB(\gamma,\beta)}{\Gamma(\beta)}b^\beta\|x\|_{C_{1-\gamma}} + \frac{Kb^{\beta+1-\gamma}}{\Gamma(\beta+1)} \leqslant \frac{LB(\gamma,\beta)}{\Gamma(\beta)}b^\beta r^* + \frac{Kb^{\beta+1-\gamma}}{\Gamma(\beta+1)} := s^*.$$

Q maps bounded sets into equicontinuous sets of $C_{1-\gamma}[0,b]$: Let $0<t_1<t_2\leqslant b$, and $x \in B_r$, then we have

$$\begin{aligned}&|t_2^{1-\gamma}(Qx)(t_2)-t_1^{1-\gamma}(Qx)(t_1))| \\ &= \Big|\frac{t_2^{1-\gamma}}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}f(s,x(s))ds - \frac{t_1^{1-\gamma}}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}f(s,x(s))ds\Big| \\ &\leqslant \Big|\frac{1}{\Gamma(\beta)}\int_0^{t_2}[t_2^{1-\gamma}(t_2-s)^{\beta-1}-t_1^{1-\gamma}(t_1-s)^{\beta-1}]f(s,x(s))ds \\ &\quad + \frac{t_1^{1-\gamma}}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_1-s)^{\beta-1}f(s,x(s))ds\Big|\end{aligned}$$

which tends to zero as $t_2 \to t_1$. So by Arzelà-Ascoli theorem, Q is compact. Hence by Krasnoselskii's fixed point theorem, the problem defined by equation (3) has at least one solution in $C_{1-\gamma}[0,b]$. □

Our next result is based on Schauder fixed point theorem. Here we replace $(H4)$ by the following condition:

$(H4)'$ there exists a constant $N'>0$ such that

$$|g(x)| \leqslant N' \text{ for each } x \in C_{1-\gamma}[0,b].$$

THEOREM 8. *Assume that* $(H1)$, $(H2)$, $(H3)'$ *and* $(H4)'$ *hold. If* $\xi_1<1$, *then* (3) *has at least one solution in* $C^\gamma_{1-\gamma}[0,b] \subset C^{\alpha,\beta}_{1-\gamma}[0,b]$.

Proof. Choose

$$r \geqslant \frac{\phi_4}{1-\xi_1}, \quad \text{where } \phi_4 = \frac{|x_0| + N'}{\Gamma(\gamma)} + \frac{K}{\Gamma(\beta+1)} b^{\beta+1-\gamma}.$$

Then by using the techniques of Theorem 7, it can be easily shown that $T : C_{1-\gamma}[0,b] \to C_{1-\gamma}[0,b]$ defined by

$$(Tx)(t) = \frac{x_0 + g(x)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [A(s)x(s) + f(s,x(s))] ds, \quad t \in (0,b]$$

is continuous and TB_r is relatively compact. Hence, it follows from Schauder's fixed point theorem that (3) has a solution in $C_{1-\gamma}[0,b]$. □

5. Examples

For evolution equation:
Consider

$$\left.\begin{aligned} &D_{0+}^{\frac{1}{5},\frac{3}{5}} x(t) = \frac{1}{10} e^{-t} x(t) + \left(t^{-8/25} + \frac{|x(t)|}{2}\right), \quad t \in (0,1], \\ &I_{0+}^{\frac{8}{25}} x(0) = x_0. \end{aligned}\right\} \tag{4}$$

Here

$$f(t,x(t)) = t^{-8/25} + \frac{|x(t)|}{2} \ \text{ for } t \in (0,1], \quad A(t) = \frac{1}{10} e^{-t} I,$$

where I is the identity operator. It is obvious that $A(.)x(.), f(.,x(.)) \in C_{\frac{8}{25}}[0,1]$. Moreover, $|f(t,x) - f(t,y)| \leqslant \frac{1}{2}|x-y|$ for all $x, y \in \mathbb{R}$. Hence $(H3)$ holds with $L = \frac{1}{2}$. Here, $M = \frac{1}{10}$ and it can be found, after some elementary computation, that

$$\xi_1 = \left(\frac{1}{10} + \frac{1}{2}\right) \frac{B(\frac{17}{25}, \frac{3}{5})}{\Gamma(\frac{3}{5})} \approx 0.88654 < 1.$$

Here $B(.,.)$ denotes beta function.

Thus, all the assumptions in Theorem 5 are satisfied and therefore, we can conclude that (4) has a unique solution in $C_{\frac{8}{25}}[0,1]$.

For nonlocal condition:

Now in equation (4), if we replace the initial condition $I_{0+}^{1-\gamma} x(0) = x_0$ by the nonlocal condition $I_{0+}^{1-\gamma} x(0) - g(x) = x_0$ (from equation (3)), where $g(x) = cx(\frac{1}{2})$, $c \in \mathbb{R}$, then g satisfies $(H4)$ with $N = |c| 2^{\frac{8}{25}}$. By choosing c small enough so that $\xi_2 < 1$ holds, Theorem 6 ensures the existence of solution in $C_{\frac{8}{25}}[0,1]$.

6. Conclusion

The existence of solutions of a class of Cauchy-type fractional semilinear evolution equation involving Hilfer fractional derivative is discussed. A nonlocal Cauchy problem is also discussed for evolution equations. The results are obtained by using various results of fractional calculus and fixed point theorems in the weighted space of continuous functions. Examples are constructed to illustrate the derived theory for fractional semilinear evolution equation and for nonlocal condition.

Our future work will be devoted to the study of the following non-autonomous evolution equation:

$$\begin{aligned} &D_{0^+}^{\alpha,\beta}x(t) = A(t)x(t) + f(t,x(t)), \quad t \in (0,b] = J', \\ &I_{0^+}^{1-\gamma}x(0) = x_0, \end{aligned}$$

where $\alpha \in [0,1]$, $\beta \in (0,1)$, $\{A(t)\}_{t\in[0,b]}$ is a family of closed linear operators defined on a dense domain $D(A)$ (independent of t) and is such that $R(\lambda, A(t))$, the resolvent of $A(t)$, exists for any λ with $\Re(\lambda) \geqslant 0$ and f is a nonlinear function. Sufficient conditions for the existence of solutions shall be established by using fractional calculus and fixed point theorems.

Acknowledgements. The first author expresses her gratitude to Indian Institute of Technology Guwahati for providing her fellowship to pursue her PhD. The authors are thankful to the learned Reviewer for his insightful comments for improving the work. The authors express their gratitude to the Associate Editor Prof. Maja Andrić for allowing a revision.

REFERENCES

[1] S. ABBAS, M. BENCHOHRA AND J. R. GRAEF, *Weak solutions to implicit differential equations involving the Hilfer fractional derivative*, Nonlinear Dyn. Syst. Theory **18**, 1 (2018), 1–11.

[2] S. ABBAS, M. BENCHOHRA AND S. SIVASUNDARAM, *Dynamics and Ulam stability for Hilfer type fractional differential equations*, Nonlinear Stud. **23**, 4 (2016), 627–637.

[3] K. BALACHANDRAN AND J. Y. PARK, *Nonlocal Cauchy problem for abstract fractional semilinear evolution equations*, Nonlinear Anal. **71**, 10 (2009), 4471–4475.

[4] L. BYSZEWSKI, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162**, 2 (1991), 494–505.

[5] L. BYSZEWSKI AND V. LAKSHMIKANTHANAM, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. **40**, 1 (1991), 11–19.

[6] K. DIETHELM, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer-Verlag Berlin, 2004.

[7] K. M. FURATI, M. D. KASSIM AND N. E-. TATAR, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Compt. Math. Appl. **64**, 6 (2012), 1616–1626.

[8] K. M. FURATI, M. D. KASSIM AND N. E-. TATAR, *Non-existence of global solutions for a differential equation involving Hilfer fractional derivative*, Electron. J. Differential Equations **2013**, 235 (2013), 1–10.

[9] H. GU AND J. J. TRUJILLO, *Existence of mild solution for evolution equation with Hilfer fractional derivative*, Appl. Math. Comput. **257**, (2015), 344–354.

[10] G. M. N'GUÉRÉKATA, *A Cauchy problem for some fractional abstract differential equation with nonlocal conditions*, Nonlinear Anal. **70**, (2009), 1873–1876.

[11] R. HILFER, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[12] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier B.V., Amsterdam, 2006.
[13] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.
[14] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, New York, 1999.
[15] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives - Theory and Applications*, Gordon and Breach, Russia, 1993.
[16] D. VIVEK, K. KANAGARAJAN AND S. SIVASUNDARAM, *Dynamics and stability of pantograph equations via Hilfer fractional derivative*, Nonlinear Stud. **23**, 4 (2016), 685–698.
[17] J. R. WANG AND Y. ZHANG, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput. **266**, (2015), 850–859.
[18] Y. ZHOU, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

(Received December 9, 2020)

Bandita Roy
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati 781039, India
e-mail: bandita.roy@iitg.ac.in

Swaroop Nandan Bora
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati 781039, India
e-mail: swaroop@iitg.ac.in