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GENERALIZED FRACTIONAL OSTROWSKI TYPE INEQUALITIES VIA $(\alpha,\beta,\gamma,\delta)$–CONVEX FUNCTIONS

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Abstract. We are introducing very first time a generalized class named it the class of $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind. This generalized class contains many subclasses including class of (α,β)-convex functions in 1^{st} and 2^{nd} kind, (s,r)-convex functions in mixed kind, s-convex functions in 1^{st} and 2^{nd} kind, P-convex function, quasi convex functions and the class of ordinary convex. Also, we would like to state the generalization of the classical Ostrowski inequality via fractional integrals with respect to another function, which is obtained for functions whose first derivative in absolute values is $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind. Moreover we establish some Ostrowski type inequalities via fractional integrals with respect to another function and their particular cases for the class of functions whose derivatives in absolute values at certain powers are $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, standard results would be capture as special cases. Moreover, some applications in terms of special means would also be given.

1. Introduction

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality.

THEOREM 1. [23] *Let $\varsigma:[a,b]\to\mathbb{R}$ be differentiable on (a,b) with the property that $|\varsigma'(t)|\leqslant M\ \forall t\in(a,b)$. Then*

$$\left|\varsigma(x)-\frac{1}{b-a}\int_a^b \varsigma(t)dt\right|\leqslant (b-a)M\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right], \tag{1}$$

$\forall x\in(a,b)$.

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Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [6]–[12] and [16]–[20].

Nowadays, with the increasing demand of researchers for the study of natural phenomena, the use of fractional differential operators and fractional differential equations has become an effective means to achieve this goal, compared with integer order operators, fractional operators, which can simulate natural phenomena better, are a class of operators developed in recent years. This kind of operators have expanded and have been widely used in modeling real-world phenomena such as biomathematics, electrical circuits, medicine, disease transmission and control.

The importance of convex functions for the generalization of integral inequalities due to the variety of their nature the notion have been established. Integral inequalities are satisfied by many convex. Among these, the well known is classical Ostrowski inequality [23]. To generalize the Ostrowski inequality, we need to generalize the concept of convex function, in this way we can easily see the generalizations and its particular cases. From literature, we recall and introduce some definitions for various convex.

DEFINITION 1. [3] The $\tau : I \subset (0,\infty) \to \mathbb{R}$ is convex function, if

$$\tau(tx+(1-t)y) \leqslant t\tau(x)+(1-t)\tau(y),$$

$\forall x,y \in I,\ t \in [0,1]$.

DEFINITION 2. [25] Let $s \in (0,1]$. The $\tau : I \subset (0,\infty) \to [0,\infty)$ is the s-convex function in 1^{st} kind, if

$$\tau(tx+(1-t)y) \leqslant t^s\tau(x)+(1-t^s)\tau(y),$$

$\forall x,y \in I,\ t \in [0,1]$.

DEFINITION 3. [15] The $\tau : I \subset (0,\infty) \to [0,\infty)$ is quasi convex function, if

$$\tau(tx+(1-t)y) \leqslant \max\{\tau(x),\tau(y)\}$$

$\forall x,y \in I,\ t \in [0,1]$.

DEFINITION 4. [25] Let $s \in (0,1]$. The $\tau : I \subset (0,\infty) \to [0,\infty)$ is the s-convex function in 2^{nd} kind, if

$$\tau(tx+(1-t)y) \leqslant t^s\tau(x)+(1-t)^s\tau(y),$$

$\forall x,y \in I,\ t \in [0,1]$.

DEFINITION 5. [3] The $\tau : I \subset (0,\infty) \to [0,\infty)$ is a P-convex function, if $\tau(x) \geqslant 0$ and $\forall x,y \in I$ and $t \in [0,1]$,

$$\tau(tx+(1-t)y) \leqslant \tau(x)+\tau(y).$$

DEFINITION 6. [14] Let $(A,B)\in(0,1]^2$. The $\tau : I\subset(0,\infty)\to[0,\infty)$ is an (A,B)-convex function in 1^{st} kind, if

$$\tau\left(tx+(1-t)y\right)\leqslant t^{\alpha}\tau(x)+(1-t^{\beta})\tau(y),$$

$\forall x,y\in I,\ t\in[0,1]$.

DEFINITION 7. [14] Let $(A,B)\in(0,1]^2$. The $\tau : I\subset(0,\infty)\to[0,\infty)$ is an (A,B)-convex function in 2^{nd} kind, if

$$\tau\left(tx+(1-t)y\right)\leqslant t^{\alpha}\tau(x)+(1-t)^{\beta}\tau(y),$$

$\forall x,y\in I,\ t\in[0,1]$.

DEFINITION 8. [24] The Riemann-Liouville fractional integral operator of order $\vartheta>0$ with $a\geqslant 0$ is defined as

$$J_a^{\vartheta}\varsigma(x)=\frac{1}{\Gamma(\vartheta)}\int_a^x\frac{\varsigma(t)}{(x-t)^{1-\vartheta}}dt,$$
$$J_a^{0}\varsigma(x)=\varsigma(x).$$

In case of $\vartheta=1$, the fractional integral reduces to the classical integral.

DEFINITION 9. [24] The Riemann-Liouville fractional integrals $I_{a^+}^{\vartheta}\varsigma$ and $I_{b^-}^{\vartheta}\varsigma$ of $\varsigma\in L_1([a,b])$ having order $\vartheta>0$ with $a\geqslant 0$, $a<b$ are defined by

$$I_{a^+}^{\vartheta}\varsigma(x)=\frac{1}{\Gamma(\vartheta)}\int_a^x\frac{\varsigma(t)}{(x-t)^{1-\vartheta}}dt,\ x>a$$

and

$$I_{b^-}^{\vartheta}\varsigma(x)=\frac{1}{\Gamma(\vartheta)}\int_x^b\frac{\varsigma(t)}{(t-x)^{1-\vartheta}}dt,\ x<b,$$

respectively. Here $\Gamma(\vartheta)=\int_0^{\infty}e^{-u}u^{\vartheta-1}du$ is the Gamma function and $I_{a^+}^{0}\varsigma(x)$ $=I_{b^-}^{0}\varsigma(x)=\varsigma(x)$.

DEFINITION 10. [24] Let $g:[a,b]:\mathbb{R}$ be an increasing and positive function on $(a,b]$, having a continuous derivatives $g'(x)$ on (a,b). The fractional integrals $I_{a^+,g}^{\vartheta}\varsigma$ and $I_{b^-,g}^{\vartheta}\varsigma$ of ς with respect to the function g on $[a,b]$ of order $\vartheta>0$ are defined by

$$I_{a^+,g}^{\vartheta}\varsigma(x)=\frac{1}{\Gamma(\vartheta)}\int_a^x\frac{g'(t)\varsigma(t)}{(g(x)-g(t))^{1-\vartheta}}dt,\ x>a$$

and

$$I_{b^-,g}^{\vartheta}\varsigma(x)=\frac{1}{\Gamma(\vartheta)}\int_x^b\frac{g'(t)\varsigma(t)}{(g(t)-g(x))^{1-\vartheta}}dt,\ x<b,$$

respectively.

REMARK 1. If $g(t)=t$ the above fractional integrals reduce to the Riemann-Liouville fractional integrals.

THEOREM 2. [12] *Let* $\varsigma : I \to \mathbb{R}$ *be differentiable mapping on* I^0, *with* $a,b \in I$, $a<b$, $\varsigma' \in L_1[a,b]$ *and for* $\vartheta \geqslant 1$, *Montgomery identity for fractional integrals holds:*

$$\varsigma(x)=\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)-J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))+J_a^{\vartheta}(P_1(x,b)\varsigma'(b)),$$

where $P_1(x,t)$ *is defined by:*

$$P_1(x,t)=\begin{cases}\dfrac{t-a}{b-a}\dfrac{\Gamma(\vartheta)}{(b-x)^{\vartheta-1}}, & \text{if} \quad t\in[a,x],\\ \dfrac{t-b}{b-a}\dfrac{\Gamma(\vartheta)}{(b-x)^{\vartheta-1}}, & \text{if} \quad t\in(x,b].\end{cases}$$

THEOREM 3. [12] *Let* $\varsigma : I \to \mathbb{R}$ *be differentiable mapping on* I^0, *with* $a,b \in I$, $a<b$, $\varsigma' \in L_1[a,b]$ *and for* $\vartheta \geqslant 1$, *generalized Montgomery identity for fractional integrals holds:*

$$\begin{aligned}(1-\varepsilon)\varsigma(x)=&\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)-J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))\\&-\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^{0}\varsigma(a)+J_a^{\vartheta}(P_2(x,b)\varsigma'(b)),\end{aligned}$$

where $P_2(x,t)$ *is the generalized fractional Peano Kernel defined by:*

$$P_2(x,t)=\begin{cases}\dfrac{t-\mu}{b-a}\dfrac{\Gamma(\vartheta)}{(b-x)^{\vartheta-1}}, & \text{if} \quad t\in[a,x],\\ \dfrac{t-\nu}{b-a}\dfrac{\Gamma(\vartheta)}{(b-x)^{\vartheta-1}}, & \text{if} \quad t\in(x,b].\end{cases}$$

$\forall x\in[\mu,\nu]$ *where* $\mu=a+\varepsilon\frac{b-a}{2}$ *and* $\nu=b-\varepsilon\frac{b-a}{2}$ *for* $\varepsilon\in[0,1]$.

Throughout this paper, $g:[a,b]\to\mathbb{R}$ is an increasing and positive on $[a,b]$, having a continuous derivative $g'(x)$ on (a,b). In order to prove our results, we need the following Lemma.

LEMMA 1. [21] *Let* $\varsigma:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $\varsigma':[a,b]\to\mathbb{R}$ *be integrable on* $[a,b]$. *Then the identity for generalized fractional integrals*

$$\begin{aligned}&\varsigma(x)-\Gamma(\vartheta+1)\left[\frac{I^{\vartheta}_{x^+,g}\varsigma(b)}{2\left(g(b)-g(x)\right)^{\vartheta}}+\frac{I^{\vartheta}_{x^-,g}\varsigma(a)}{2\left(g(x)-g(a)\right)^{\vartheta}}\right]\\&\quad=\frac{x-a}{2\left(g(x)-g(a)\right)^{\vartheta}}\int_0^1\left(g(tx+(1-t)a)-g(a)\right)^{\vartheta}\varsigma'(tx+(1-t)a)dt\\&\qquad-\frac{b-x}{2\left(g(b)-g(x)\right)^{\vartheta}}\int_0^1\left(g(b)-g(tx+(1-t)b)\right)^{\vartheta}\varsigma'(tx+(1-t)b)dt.\end{aligned}$$

Throughout this paper, we denote

$$
{}_{g}^{\vartheta}\kappa_{a}^{b}(x)=\left[\frac{(x-a)^{\vartheta+1}}{2\left(g(x)-g(a)\right)^{\vartheta}}+\frac{(b-x)^{\vartheta+1}}{2\left(g(b)-g(x)\right)^{\vartheta}}\right],
$$

$$
{}_{\varsigma,g}^{\vartheta}\theta_{a}^{b}(x)=\varsigma(x)-\Gamma(\vartheta+1)\left[\frac{I_{x^{+},g}^{\vartheta}\varsigma(b)}{2\left(g(b)-g(x)\right)^{\vartheta}}+\frac{I_{x^{-},g}^{\vartheta}\varsigma(a)}{2\left(g(x)-g(a)\right)^{\vartheta}}\right].
$$

We also make use of the Euler's beta function, which is for $x,y>0$ defined as

$$
B(x,y)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
$$

Let $[a,b]\subseteq(0,+\infty),$ we may define special means as follows:

(a) The arithmetic mean

$$
A=A(a,b):=\frac{a+b}{2};
$$

(b) The geometric mean

$$
G=G(a,b):=\sqrt{ab};
$$

(c) The harmonic mean

$$
H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}};
$$

(d) The logarithmic mean

$$
L=L(a,b):=\begin{cases} a & \text{if} \quad a=b \\ \dfrac{b-a}{\ln b-\ln a}, & \text{if} \quad a\neq b \end{cases};
$$

(e) The identric mean

$$
I=I(a,b):=\begin{cases} a & \text{if} \quad a=b \\ \dfrac{1}{e}\left(\dfrac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}, & \text{if} \quad a\neq b. \end{cases};
$$

(f) The $p-$logarithmic mean

$$
L_{p}=L_{p}(a,b):=\begin{cases} a & \text{if} \quad a=b \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if} \quad a\neq b. \end{cases};
$$

where $p\in\mathbb{R}\setminus\{0,-1\}$.

The main aim of our study is to present Ostrowski inequality for fractional integrals with respect to another function, which is generalization of the classical Ostrowski inequality (1) via $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind, which is given in Section 2. Moreover we establish some Ostrowski type inequalities via fractional integrals with repect to another function for the class of functions whose derivatives in absolute values at certain powers are $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind by using different techniques including Hölder's inequality [28] and power mean inequality [27]. Also we give the special cases of our results and applications of midpoint inequalities in special means. In the last section gives us conclusion with some remarks and future ideas to generalize the results.

2. Generalizations of fractional Ostrowski inequality via generalized fractional integral operator

The convexity is very simple and ordinary concept, due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real world problems the concept of convexity is very decisive. The problems faced in constrained control and estimation are convex. Geometrically, a real valued function is said to be convex functions if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space. We are introducing the very first time the class of (r,s)-convex and $(\alpha,\beta,\gamma,\delta)$-convex functions function in mixed kind.

DEFINITION 11. Let $(r,s)\in(0,1]^2$. The $\tau: I\subset[0,\infty)\to[0,\infty)$ is (r,s)-convex functions in mixed kind, if

$$\tau(tx+(1-t)y)\leqslant t^{rs}\tau(x)+(1-t^r)^s\tau(y),$$

$\forall x,y\in I, t\in[0,1]$.

DEFINITION 12. Let $(\alpha,\beta,\gamma,\delta)\in(0,1]^4$. The $\tau: I\subset[0,\infty)\to[0,\infty)$ is $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind, if

$$\tau(tx+(1-t)y)\leqslant t^{\alpha\gamma}\tau(x)+(1-t^{\beta})^{\delta}\tau(y), \tag{2}$$

$\forall x,y\in I,\ t\in[0,1]$.

REMARK 2. In Definition 12, we can capture:

1. If $\gamma=\delta=1$ in (2), we get (α,β)-convex functions in 1^{st} kind function.
2. If $\beta=\gamma=1$ in (2), we get (α,β)-convex functions in 2^{nd} kind function.
3. If $\alpha=\delta=s,\ \beta=\gamma=r$, where $s,r\in[0,1]$ in (2), we get (s,r)-convex functions in mixed kind function.
4. If $\alpha=\beta=s$ and $\gamma=\delta=1$ where $s\in[0,1]$ in (2), we get s-convex functions in 1^{st} kind function.

5. If $\alpha = \beta \to 0$, and $\gamma = \delta = 1$, in (2), we get quasi-convex.

6. If $\alpha = \delta = s$, $\beta = \gamma = 1$ where $s \in [0,1]$ in (2), we get s-convex functions in 2^{nd} kind function.

7. If $\alpha = \delta \to 0$, and $\beta = \gamma = 1$, in (2), we get P-convex.

8. If $\alpha = \beta = \gamma = \delta = 1$ in (2), gives us ordinary convex.

THEOREM 4. *Let $\varsigma : [a,b] \to \mathbb{R}$ be differentiable on (a,b), $\varsigma' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $\tau : I \subset \mathbb{R} \to \mathbb{R}$, be a $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed sense, then*

$$\begin{aligned}
&\tau\Bigg[(1-\varepsilon)\varsigma(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^{\vartheta}\varsigma(b) \\
&\quad + J_a^{\vartheta-1}(P_2(x,b)\varsigma(b)) + \frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}} J_a^0 \varsigma(a)\Bigg] \\
&\leqslant (b-x)^{1-\vartheta}\Bigg[\frac{(x-a)^{\alpha\gamma-1}}{(b-a)^{\alpha\gamma}} \int_a^x \tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right] dt \\
&\quad + \frac{\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}}{b-x} \int_x^b \tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right] dt\Bigg],
\end{aligned} \tag{3}$$

$\forall x \in [\mu,\nu]$ *and* $\varepsilon \in [0,1]$.

Proof. Utilizing the generalized Montgomery identity (2) for fractional, we get

$$\begin{aligned}
&(1-\varepsilon)\varsigma(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^{\vartheta}\varsigma(b) \\
&\quad + J_a^{\vartheta-1}(P_2(x,b)\varsigma(b)) + \frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}} J_a^0 \varsigma(a) \\
&= J_a^{\vartheta}(P_2(x,b)\varsigma'(b)) \\
&= \frac{1}{\Gamma(\vartheta)} \int_a^b P_2(x,t) \frac{\varsigma'(t)}{(b-t)^{1-\vartheta}} dt \\
&= \left(\frac{x-\alpha}{b-a}\right)\left[\frac{(b-x)^{1-\vartheta}}{x-\alpha} \int_a^x \frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}} dt\right] \\
&\quad + \left(1-\left(\frac{x-\alpha}{b-a}\right)\right)\left[\frac{(b-x)^{1-\vartheta}}{\beta-x} \int_x^b \frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}} dt\right],
\end{aligned}$$

$\forall x \in [\mu,\nu]$ for $\varepsilon \in [0,1]$. Next by using the $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed sense of $\tau : I \subset [0,\infty) \to \mathbb{R}$, we get

$$\tau\Bigg[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)$$
$$+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\Bigg]$$
$$\leqslant\left(\frac{x-\alpha}{b-a}\right)^{\alpha\gamma}\tau\left[\frac{(b-x)^{1-\vartheta}}{x-a}\int_a^x\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}dt\right]$$
$$+\left(1-\left(\frac{x-\alpha}{b-a}\right)^{\beta}\right)^{\delta}\tau\left[\frac{(b-x)^{1-\vartheta}}{\beta-x}\int_x^b\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}dt\right],$$

$\forall x\in[\mu,\nu]$ and $\varepsilon\in[0,1]$. Applying Jensen's integral inequality [7], we get the Inequality (3). □

REMARK 3. In Theorem 4, If $\varepsilon=0$, in (3), we get

$$\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant(b-x)^{1-\vartheta}\Bigg[\frac{(x-a)^{\alpha\gamma-1}}{(b-a)^{\alpha\gamma}}\int_a^x\tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt$$
$$+\frac{\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}}{b-x}\int_x^b\tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\Bigg].$$

COROLLARY 1. *In Theorem 4, one can see the followings:*

1. *If one put $\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$, in (3), then one has the fractional Ostrowski type inequality for (α,β)-convex functions in 1^{st} kind:*

$$\tau\Bigg[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)$$
$$+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\Bigg]$$
$$\leqslant(b-x)^{1-\vartheta}\Bigg[\frac{(x-a)^{\alpha-1}}{(b-a)^{\alpha}}\int_a^x\tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt$$
$$+\frac{(b-a)^{\beta}-(x-a)^{\beta}}{(b-a)^{\beta}(b-x)}\int_x^b\tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\Bigg].\tag{4}$$

REMARK 4. If $\varepsilon = 0$, in (4), we get

$$\tau\left[\varsigma(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^{\vartheta}\varsigma(b) + J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant (b-x)^{1-\vartheta}\left[\frac{(x-a)^{\alpha-1}}{(b-a)^{\alpha}}\int_a^x \tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-a)^{\beta}-(x-a)^{\beta}}{(b-a)^{\beta}(b-x)}\int_x^b \tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

2. *If one put $\beta = \gamma = 1$, $\alpha \in [0,1]$ and $\delta \in [0,1]$, in (3), then one get fractional Ostrowski type inequality for (α,δ)-convex functions in 2^{nd} kind:*

$$\tau\left[(1-\varepsilon)\varsigma(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^{\vartheta}\varsigma(b)\right.$$
$$\left.+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b)) + \frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}} J_a^0\varsigma(a)\right]$$
$$\leqslant (b-x)^{1-\vartheta}\left[\frac{(x-a)^{\alpha-1}}{(b-a)^{\alpha}}\int_a^x \tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-x)^{\beta-1}}{(b-a)^{\beta}}\int_x^b \tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \tag{5}$$

REMARK 5. If $\varepsilon = 0$, in (5), we get

$$\tau\left[\varsigma(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^{\vartheta}\varsigma(b) + J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant (b-x)^{1-\vartheta}\left[\frac{(x-a)^{\alpha-1}}{(b-a)^{\alpha}}\int_a^x \tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-x)^{\beta-1}}{(b-a)^{\beta}}\int_x^b \tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

3. *If one put $\alpha = \delta = s$, $\beta = \gamma = r$, where $s \in [0,1]$ and $r \in (0,1]$ in (3), then one has the fractional Ostrowski type inequality for (s,r)-convex functions in mixed kind:*

$$\tau\left[(1-\varepsilon)\varsigma(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^{\vartheta}\varsigma(b)\right.$$
$$\left.+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b)) + \frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}} J_a^0\varsigma(a)\right]$$

$$\leqslant \frac{(b-x)^{1-\vartheta}}{(b-a)^{rs}}\left[(x-a)^{rs-1}\int_a^x \tau\left[\frac{\{t-\mu\}\,\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{((b-a)^r-(x-\mu)^r)^s}{b-x}\int_x^b \tau\left[\frac{\{t-\nu\}\,\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \tag{6}$$

REMARK 6. If $\varepsilon = 0$, in (6), we get

$$\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant \frac{(b-x)^{1-\vartheta}}{(b-a)^{rs}}\left[(x-a)^{rs-1}\int_a^x \tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{((b-a)^r-(x-a)^r)^s}{b-x}\int_x^b \tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

4. *If one put* $\alpha=\beta=s$ *and* $\gamma=\delta=1$, *where* $s\in(0,1]$ *in* (3), *then one has the Ostrowski inequality for* s*-convex functions in* 1^{st} *kind:*

$$\tau\left[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)\right.$$
$$\left.+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\right]$$
$$\leqslant \frac{(b-x)^{1-\vartheta}}{(b-a)^{s}}\left[(x-a)^{s-1}\int_a^x \tau\left[\frac{\{t-\mu\}\,\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-a)^s-(x-a)^s}{(b-x)}\int_x^b \tau\left[\frac{\{t-\nu\}\,\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \tag{7}$$

REMARK 7. If $\varepsilon = 0$, in (7), we get

$$\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant \frac{(b-x)^{1-\vartheta}}{(b-a)^{s}}\left[(x-a)^{s-1}\int_a^x \tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-a)^s-(x-a)^s}{(b-x)}\int_x^b \tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

5. *If one put* $\alpha=\beta=0$, *and* $\gamma=\delta=1$, *in* (3), *then one can get Ostrowski in-*

equalities for quasi-convex functions.

$$\begin{aligned}
&\tau\Bigg[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)\\
&+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\Bigg]\\
\leqslant\ &\frac{(b-x)^{1-\vartheta}}{(x-\alpha)}\left[\int_a^x\tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \qquad (8)
\end{aligned}$$

REMARK 8. If $\varepsilon=0$, in (8), we get

$$\begin{aligned}
&\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]\\
\leqslant\ &\frac{(b-x)^{1-\vartheta}}{(x-a)}\left[\int_a^x\tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].
\end{aligned}$$

6. *If one put $\beta=\gamma=1,\ \alpha=\delta=s$ where $s\in[0,1)$ (3), then fractional Ostrowski type inequality for s-convex functions in 2^{nd} kind:*

$$\begin{aligned}
&\tau\Bigg[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)\\
&+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\Bigg]\\
\leqslant\ &\frac{(b-x)^{1-\vartheta}}{(b-a)^s}\Bigg[(x-a)^{s-1}\int_a^x\tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\\
&+(b-x)^{s-1}\int_x^b\tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\Bigg]. \qquad (9)
\end{aligned}$$

REMARK 9. If $\varepsilon=0$, in (9), we get

$$\begin{aligned}
&\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]\\
\leqslant\ &\frac{(b-x)^{1-\vartheta}}{(b-a)^s}\Bigg[(x-a)^{s-1}\int_a^x\tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\\
&+(b-x)^{s-1}\int_x^b\tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\Bigg].
\end{aligned}$$

7. *If one put $\alpha=\delta=0$ and $\beta=\gamma=1$ in (3), then one has the fractional Ostrowski*

type inequality for P-convex functions:

$$\tau\left[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)\right.$$
$$\left.+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\right]$$
$$\leqslant(b-x)^{1-\vartheta}\left[\frac{1}{x-\alpha}\int_a^x\tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{1}{\beta-x}\int_x^b\tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \tag{10}$$

REMARK 10. If $\varepsilon=0$, in (10), we get

$$\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant(b-x)^{1-\vartheta}\left[\frac{1}{x-a}\int_a^x\tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{1}{b-x}\int_x^b\tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

8. *If one put* $\alpha=\beta=\gamma=\delta=1$, *in* (3), *then one has the fractional Ostrowski type inequality for convex functions:*

$$\tau\left[(1-\varepsilon)\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)\right.$$
$$\left.+J_a^{\vartheta-1}(P_2(x,b)\varsigma(b))+\frac{\varepsilon(b-x)^{1-\vartheta}}{2(b-a)^{1-\vartheta}}J_a^0\varsigma(a)\right]$$
$$\leqslant\frac{(b-x)^{1-\vartheta}}{b-a}\left[\int_a^x\tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\int_x^b\tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \tag{11}$$

REMARK 11. If $\varepsilon=0$, in (11), we get

$$\tau\left[\varsigma(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varsigma(b)+J_a^{\vartheta-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leqslant\frac{(b-x)^{1-\vartheta}}{b-a}\left[\int_a^x\tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt+\int_x^b\tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

9. *If one put* $\vartheta = \alpha = \beta = \gamma = \delta = 1$, *and* $\varepsilon = 0$ *in* (3), *then one has the inequality* (2.1) *of Theorem* 7 *in* [7].

THEOREM 5. *If Lemma* 1 *hold. Additionally, assume that* $|\varsigma'|$ *is* $(\alpha,\beta,\gamma,\delta)$-*convex functions on* $[a,b]$ *and* $|\varsigma'(x)| \leqslant M$, $|g'(x)| \leqslant L$, $\forall x \in [a,b]$, g *is Lipschizian function. Then*

$$\left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant ML^{\vartheta} \left(\frac{1}{\vartheta+\alpha\gamma+1} + \frac{B\left(\frac{\vartheta+1}{\beta}, \delta+1\right)}{\beta} \right) {}^{\vartheta}_{g}\kappa_a^b(x). \tag{12}$$

$\forall x \in (a,b)$.

Proof. From the Lemma 1 we have

$$\begin{aligned} \left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant & \frac{x-a}{2\left(g(x)-g(a)\right)^{\vartheta}} \int_0^1 \frac{|\varsigma'(tx+(1-t)a)|}{\left(g(tx+(1-t)a)-g(a)\right)^{-\vartheta}} dt \\ & + \frac{b-x}{2\left(g(b)-g(x)\right)^{\vartheta}} \int_0^1 \frac{|\varsigma'(tx+(1-t)b)|}{\left(g(b)-g(tx+(1-t)b)\right)^{-\vartheta}} dt. \end{aligned} \tag{13}$$

Since g is differentiable and $|g'(x)| \leqslant L$ on $[a,b]$, g is Lipschizian function, i.e.

$$\begin{aligned} g\left(tx+(1-t)a\right) - g(a) &\leqslant Lt(x-a), \\ g(b) - g\left(tx+(1-t)b\right) &\leqslant Lt(b-x). \end{aligned} \tag{14}$$

Using inequalities (14) in (13), we get

$$\begin{aligned} \left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant & L^{\vartheta} \frac{(x-a)^{\vartheta+1}}{2\left(g(x)-g(a)\right)^{\vartheta}} \int_0^1 t^{\vartheta} \left|\varsigma'\left(tx+(1-t)a\right)\right| dt \\ & + L^{\vartheta} \frac{(b-x)^{\vartheta+1}}{2\left(g(b)-g(x)\right)^{\vartheta}} \int_0^1 t^{\vartheta} \left|\varsigma'\left(tx+(1-t)b\right)\right| dt. \end{aligned} \tag{15}$$

Since $|\varsigma'|$ is $(\alpha,\beta,\gamma,\delta)$-convex functions on $[a,b]$ and $|\varsigma'(x)| \leqslant M$, we have

$$\begin{aligned} & \int_0^1 t^{\vartheta} \left(t^{\alpha\gamma} |\varsigma'(x)| + (1-t^{\beta})^{\delta} |\varsigma'(a)| \right) dt \\ & \leqslant M \int_0^1 t^{\vartheta} \left(t^{\alpha\gamma} + (1-t^{\beta})^{\delta} \right) dt \end{aligned} \tag{16}$$

and similarly

$$\begin{aligned} & \int_0^1 t^{\vartheta} \left(t^{\alpha\gamma} |\varsigma'(x)| + (1-t^{\beta})^{\delta} |\varsigma'(b)| \right) dt \\ & \leqslant M \int_0^1 t^{\vartheta} \left(t^{\alpha\gamma} + (1-t^{\beta})^{\delta} \right) dt. \end{aligned} \tag{17}$$

By using inequalities (16) and (17) in (15), we get (12). □

COROLLARY 2. *In Theorem 5, one can see the followings:*

1. *If one put* $\gamma = \delta = 1$, $\alpha \in [0,1]$ *and* $\beta \in (0,1]$, *in inequality* (12), *then one has the Ostrowski inequality for* (α,β)*-convex functions in* 1^{st} *kind via generalized fractional integrals:*

$$\left| {}_{\varsigma,g}^{\vartheta}\theta_a^b(x) \right| \leqslant ML^{\vartheta} \left(\frac{1}{\vartheta+\alpha+1} + \frac{B\left(\frac{\vartheta+1}{\beta},2\right)}{\beta} \right) {}_{g}^{\vartheta}\kappa_a^b(x).$$

2. *If one put* $\beta = \gamma = 1$, $\alpha \in [0,1]$ *and* $\delta \in [0,1]$, *in inequality* (12), *then one has the Ostrowski inequality for* (α,δ)*-convex functions in* 2^{nd} *kind via generalized fractional integrals:*

$$\left| {}_{\varsigma,g}^{\vartheta}\theta_a^b(x) \right| \leqslant ML^{\vartheta} \left(\frac{1}{\vartheta+\alpha+1} + B(\vartheta+1,\delta+1) \right) {}_{g}^{\vartheta}\kappa_a^b(x).$$

3. *If one put* $\alpha = \delta = s$, $\beta = \gamma = r$, *where* $s \in [0,1]$ *and* $r \in (0,1]$ *in inequality* (12), *then one has the Ostrowski inequality for* (s,r)*-convex functions in mixed kind via generalized fractional integrals:*

$$\left| {}_{\varsigma,g}^{\vartheta}\theta_a^b(x) \right| \leqslant ML^{\vartheta} \left(\frac{1}{\vartheta+rs+1} + \frac{B\left(\frac{\vartheta+1}{r},s+1\right)}{r} \right) {}_{g}^{\vartheta}\kappa_a^b(x).$$

4. *If one put* $\alpha = \beta = s$ *and* $\gamma = \delta = 1$, *where* $s \in (0,1]$ *in inequality* (12), *then one has the Ostrowski inequality for* s*-convex functions in* 1^{st} *kind via generalized fractional integrals:*

$$\left| {}_{\varsigma,g}^{\vartheta}\theta_a^b(x) \right| \leqslant ML^{\vartheta} \left(\frac{1}{\vartheta+s+1} + \frac{B\left(\frac{\vartheta+1}{s},2\right)}{s} \right) {}_{g}^{\vartheta}\kappa_a^b(x).$$

5. *If one put* $\alpha = \delta = s$, *and* $\beta = \gamma = 1$, *where* $s \in (0,1]$ *in inequality* (12), *then one has the Ostrowski inequality for* s*-convex functions in* 2^{nd} *kind via generalized fractional integrals:*

$$\left| {}_{\varsigma,g}^{\vartheta}\theta_a^b(x) \right| \leqslant ML^{\vartheta} \left(\frac{1}{\vartheta+s+1} + B(\vartheta+1,s+1) \right) {}_{g}^{\vartheta}\kappa_a^b(x).$$

6. *If one put* $\alpha = \delta = s$, *where* $s \in (0,1]$, $\vartheta = \beta = \gamma = 1$ *and* $g(t) = t$ *in inequality* (12), *then one has the inequality* (2.1) *of Theorem* 2 *in* [1].

7. *If one put* $\alpha = \delta = s$, *where* $s \in (0,1]$, $\beta = \gamma = 1$ *and* $g(t) = t$ *in inequality* (12), *then one has the inequality* (2.6) *of Theorem* 7 *in* [25].

8. *If one put* $\alpha=\delta\to 0$, *and* $\beta=\gamma=1$, *in inequality* (12), *then one has the Ostrowski inequality for* P*-convex functions via generalized fractional integrals:*

$$\left|{}_{\varsigma,g}^{\vartheta}\theta_{a}^{b}(x)\right| \leqslant ML^{\vartheta}\left(\frac{1}{\vartheta+1}+B(\vartheta+1,1)\right){}_{g}^{\vartheta}\kappa_{a}^{b}(x).$$

9. *If one put* $\alpha=\beta=\gamma=\delta=1$, *in inequality* (12), *then one has the Ostrowski inequality for convex functions via generalized fractional integrals:*

$$\left|{}_{\varsigma,g}^{\vartheta}\theta_{a}^{b}(x)\right| \leqslant ML^{\vartheta}\left(\frac{1}{\vartheta+2}+B(\vartheta+1,2)\right){}_{g}^{\vartheta}\kappa_{a}^{b}(x).$$

10. *If one put* $\vartheta=\alpha=\beta=\gamma=\delta=1$ *and* $g(t)=t$ *in inequality* (12), *then one has the Ostrowski inequality* (1) *for convex.*

THEOREM 6. *If Lemma 1 hold. Additionally, assume that* $|\varsigma'|^{q}$ *is* $(\alpha,\beta,\gamma,\delta)$*-convex functions on* $[a,b]$, $q\geqslant 1$ *and* $|\varsigma'(x)|\leqslant M$, $|g'(x)|\leqslant L$, $\forall x\in[a,b]$, g *is Lipschizian function. Then*

$$\left|{}_{\varsigma,g}^{\vartheta}\theta_{a}^{b}(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+\alpha\gamma+1}+\frac{B\left(\frac{\vartheta+1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}{}_{g}^{\vartheta}\kappa_{a}^{b}(x). \tag{18}$$

$\forall x\in(a,b)$.

Proof. From the inequality (15) and using power mean inequality [27], we have

$$\begin{aligned}\left|{}_{\varsigma,g}^{\vartheta}\theta_{a}^{b}(x)\right| \leqslant{}& L^{\vartheta}\frac{(x-a)^{\vartheta+1}}{2(g(x)-g(a))^{\vartheta}}\left(\int_{0}^{1}t^{\vartheta}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t^{\vartheta}\left|\varsigma'(tx+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}\\ &+L^{\vartheta}\frac{(b-x)^{\vartheta+1}}{2(g(b)-g(x))^{\vartheta}}\left(\int_{0}^{1}t^{\vartheta}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t^{\vartheta}\left|\varsigma'(tx+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}.\end{aligned} \tag{19}$$

Since $|\varsigma'|^{q}$ is $(\alpha,\beta,\gamma,\delta)$-convex functions on $[a,b]$ and $|\varsigma'(x)|\leqslant M$, we get

$$\int_{0}^{1}t^{\vartheta}\left|\varsigma'(tx+(1-t)a)\right|^{q}dt\leqslant M^{q}\int_{0}^{1}t^{\vartheta}\left(t^{\alpha\gamma}+(1-t^{\beta})^{\delta}\right)dt \tag{20}$$

and

$$\int_{0}^{1}t^{\vartheta}\left|\varsigma'(tx+(1-t)b)\right|^{q}dt\leqslant M^{q}\int_{0}^{1}t^{\vartheta}\left(t^{\alpha\gamma}+(1-t^{\beta})^{\delta}\right)dt. \tag{21}$$

Using the inequalities (19) – (21), we get (18). □

COROLLARY 3. *In Theorem 6, one can see the followings:*

1. *If one put* $\gamma=\delta=1,\ \alpha\in[0,1]$ *and* $\beta\in(0,1]$, *in inequality* (18), *then one has the Ostrowski inequality for* (α,β)*-convex functions in* 1^{st} *kind via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+\alpha+1}+\frac{B\left(\frac{\vartheta+1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

2. *If one put* $\beta=\gamma=1,\ \alpha\in[0,1]$ *and* $\delta\in[0,1]$, *in inequality* (18), *then one has the Ostrowski inequality for* (α,δ)*-convex functions in* 2^{nd} *kind via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+\alpha+1}+B(\vartheta+1,\beta+1)\right)^{\frac{1}{q}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

3. *If one put* $\alpha=\delta=s,\ \beta=\gamma=r$, *where* $s\in[0,1]$ *and* $r\in(0,1]$ *in inequality* (18), *then one has the Ostrowski inequality for* (s,r)*-convex functions in mixed kind via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+rs+1}+\frac{B\left(\frac{\vartheta+1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

4. *If one put* $\alpha=\beta=s$ *and* $\gamma=\delta=1$, *where* $s\in(0,1]$ *in inequality* (18), *then one has the Ostrowski inequality for* s*-convex functions in* 1^{st} *kind via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+s+1}+\frac{B\left(\frac{\vartheta+1}{s},2\right)}{s}\right)^{\frac{1}{q}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

5. *If one put* $\alpha=\delta=s$, *and* $\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (18), *then one has the Ostrowski inequality for* s*-convex functions in* 2^{nd} *kind via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+s+1}+B(\vartheta+1,s+1)\right)^{\frac{1}{q}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

6. *If one put* $\alpha=\delta=s$, *where* $s\in(0,1],\ \vartheta=\beta=\gamma=1$ *and* $g(t)=t$ *in inequality* (18), *then one has the inequality* (2.3) *of Theorem* 4 *in* [1].

7. *If* $\alpha=\delta=s$, *where* $s\in(0,1],\ \beta=\gamma=1$ *and* $g(t)=t$ *in inequality* (12), *then one has the inequality* (2.8) *of Theorem* 9 *in* [25].

8. *If one put* $\alpha=\delta\rightarrow 0$, *and* $\beta=\gamma=1$, *in inequality* (18), *then one has the Ostrowski inequality for* P*-convex functions via generalized fractional integrals:*

$$\left|{}_{\varsigma,g}^{\vartheta}\theta_a^b(x)\right|\leqslant\frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+1}+B(\vartheta+1,1)\right)^{\frac{1}{q}}\ {}_{g}^{\vartheta}\kappa_a^b(x).$$

9. *If one put* $\alpha=\beta=\gamma=\delta=1$, *in inequality* (18), *then one has the Ostrowski inequality for convex functions via generalized fractional integrals:*

$$\left|{}_{\varsigma,g}^{\vartheta}\theta_a^b(x)\right|\leqslant\frac{ML^{\vartheta}}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+2}+B(\vartheta+1,2)\right)^{\frac{1}{q}}\ {}_{g}^{\vartheta}\kappa_a^b(x).$$

THEOREM 7. *If Lemma 1 hold. Additionally, assume that* $|\varsigma'|^q$ *is* $(\alpha,\beta,\gamma,\delta)$*-convex functions on* $[a,b]$, $q>1$ *such that* $p^{-1}+q^{-1}=1$, $|\varsigma'(x)|\leqslant M$, *and* $|g'(x)|\leqslant L$, $\forall x\in[a,b]$, g *is Lipschizian function. Then*

$$\left|{}_{\varsigma,g}^{\vartheta}\theta_a^b(x)\right|\leqslant\frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\ {}_{g}^{\vartheta}\kappa_a^b(x),\tag{22}$$

$\forall x\in(a,b)$.

Proof. From the inequality (15) and using Hölder's inequality [28], we have

$$\begin{aligned}\left|{}_{\varsigma,g}^{\vartheta}\theta_a^b(x)\right|\leqslant{}&L^{\vartheta}\frac{(x-a)^{\vartheta+1}}{2\left(g(x)-g(a)\right)^{\vartheta}}\left(\int_0^1 t^{\vartheta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\varsigma'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\&+L^{\vartheta}\frac{(b-x)^{\vartheta+1}}{2\left(g(b)-g(x)\right)^{\vartheta}}\left(\int_0^1 t^{\vartheta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\varsigma'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}\tag{23}$$

Since $|\varsigma'|^q$ is $(\alpha,\beta,\gamma,\delta)$-convex functions and $|\varsigma'(x)|\leqslant M$, we have

$$\int_0^1\left|\varsigma'\left(tx+(1-t)a\right)\right|^q dt\leqslant M^q\int_0^1 t^{\alpha\gamma}+(1-t^{\beta})^{\delta}dt,\tag{24}$$

and

$$\int_0^1\left|\varsigma'\left(tx+(1-t)b\right)\right|^q dt\leqslant M^q\int_0^1 t^{\alpha\gamma}+(1-t^{\beta})^{\delta}dt.\tag{25}$$

Using inequalities (23)–(25), we get (22). □

COROLLARY 4. *In Theorem 7, one can see the followings:*

1. *If one put* $\gamma = \delta = 1$, $\alpha \in [0,1]$ *and* $\beta \in (0,1]$, *in inequality* (22), *then one has the Ostrowski inequality for* (α,β)*-convex functions in* 1^{st} *kind via generalized fractional integrals:*

$$\left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}} \left(\frac{1}{\alpha+1} + \frac{B\left(\frac{1}{\beta},2\right)}{\beta} \right)^{\frac{1}{q}} {}^{\vartheta}_{g}\kappa_a^b(x).$$

2. *If one put* $\beta = \gamma = 1$, $\alpha \in [0,1]$ *and* $\delta \in [0,1]$, *in inequality* (22), *then one has the Ostrowski inequality for* (α,δ)*-convex functions in* 2^{nd} *kind via generalized fractional integrals:*

$$\left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}} \left(\frac{1}{\alpha+1} + B(1,\beta+1) \right)^{\frac{1}{q}} {}^{\vartheta}_{g}\kappa_a^b(x).$$

3. *If one put* $\alpha = \delta = s$, $\beta = \gamma = r$, *where* $s \in [0,1]$ *and* $r \in (0,1]$ *in inequality* (22), *then one has the Ostrowski inequality for* (s,r)*-convex functions in mixed kind via generalized fractional integrals:*

$$\left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}} \left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r},s+1\right)}{r} \right)^{\frac{1}{q}} {}^{\vartheta}_{g}\kappa_a^b(x).$$

4. *If one put* $\alpha = \beta = s$ *and* $\gamma = \delta = 1$, *where* $s \in (0,1]$ *in inequality* (22), *then one has the Ostrowski inequality for* s*-convex functions in* 1^{st} *kind via generalized fractional integrals:*

$$\left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}} \left(\frac{1}{s+1} + \frac{B\left(\frac{1}{s},2\right)}{s} \right)^{\frac{1}{q}} {}^{\vartheta}_{g}\kappa_a^b(x).$$

5. *If one put* $\alpha = \delta = s$, *and* $\beta = \gamma = 1$, *where* $s \in (0,1]$ *in inequality* (22), *then one has the Ostrowski inequality for* s*-convex functions in* 2^{nd} *kind via generalized fractional integrals:*

$$\left| {}^{\vartheta}_{\varsigma,g}\theta_a^b(x) \right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}} \left(\frac{1}{s+1} + B(1,s+1) \right)^{\frac{1}{q}} {}^{\vartheta}_{g}\kappa_a^b(x).$$

6. *If one put* $\alpha = \delta = s$, *where* $s \in (0,1]$, $\vartheta = \beta = \gamma = 1$ *and* $g(t) = t$ *in inequality* (22), *then one has the inequality* (2.2) *of Theorem* 3 *in* [1].

7. *If one put* $\alpha = \delta = s$, *where* $s \in (0,1]$, $\beta = \gamma = 1$ *and* $g(t) = t$ *in inequality* (12), *then one has the inequality* (2.7) *of Theorem* 8 *in* [25].

8. *If one put* $\alpha=\delta\to 0$, *and* $\beta=\gamma=1$, *in inequality* (22), *then one has the Ostrowski inequality for* P*-convex functions via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}}\left(1+B(1,1)\right)^{\frac{1}{q}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

9. *If one put* $\alpha=\beta=\gamma=\delta=1$, *in inequality* (22), *then one has the Ostrowski inequality for convex functions via generalized fractional integrals:*

$$\left|{}^{\vartheta}_{\varsigma,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\vartheta}}{(\vartheta p+1)^{\frac{1}{p}}}\ {}^{\vartheta}_{g}\kappa_a^b(x).$$

3. Applications of midpoint inequalities

If ς by $-\varsigma$ and $x=\frac{a+b}{2}$ in Theorem 4, we get

THEOREM 8. *Let* $\varsigma:[a,b]\to\mathbb{R}$ *be differentiable on* (a,b), $\varsigma':[a,b]\to\mathbb{R}$ *be integrable on* $[a,b]$ *and* $\tau: I\subset\mathbb{R}\to\mathbb{R}$, *be a* $(\alpha,\beta,\gamma,\delta)$*-convex functions in mixed sense, then*

$$\begin{aligned}
&\tau\Bigg[(\varepsilon-1)\varsigma\left(\frac{a+b}{2}\right)+\frac{\Gamma(\vartheta)\left(\frac{b-a}{2}\right)^{1-\vartheta}}{b-a}J_a^{\vartheta}\varsigma(b)\\
&-J_a^{\vartheta-1}\left(P_2\left(\frac{a+b}{2},b\right)\varsigma(b)\right)-\frac{\varepsilon}{2^{2-\vartheta}}J_a^0\varsigma(a)\Bigg]\\
&\leqslant\frac{2^{\vartheta-1}}{(\beta-\alpha)^{\vartheta}}\Bigg[\frac{1}{2^{\alpha\gamma-1}}\int_{\frac{a+b}{2}}^{\alpha}\tau\left[\frac{\{t-\mu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\\
&+\frac{\left(2^{\beta}-1\right)^{\delta}}{2^{\beta\delta-1}}\int_{\beta}^{\frac{a+b}{2}}\tau\left[\frac{\{t-\nu\}\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\Bigg],
\end{aligned}\tag{26}$$

$\forall x\in[\mu,\nu]$ *and* $\varepsilon\in[0,1]$.

REMARK 12. In Theorem 8,

1. If $\varepsilon=0$, in (26). we get

$$\begin{aligned}
&\tau\left[\frac{\Gamma(\vartheta)\left(\frac{b-a}{2}\right)^{1-\vartheta}}{b-a}J_a^{\vartheta}\varsigma(b)-\varsigma\left(\frac{a+b}{2}\right)-J_a^{\vartheta-1}\left(P_1\left(\frac{a+b}{2},b\right)\varsigma(b)\right)\right]\\
&\leqslant\frac{2^{\vartheta-1}}{(b-a)^{\vartheta}}\Bigg[\frac{1}{2^{\alpha\gamma-1}}\int_{\frac{a+b}{2}}^{a}\tau\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\\
&+\frac{\left(2^{\beta}-1\right)^{\delta}}{2^{\beta\delta-1}}\int_{b}^{\frac{a+b}{2}}\tau\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\vartheta}}\right]dt\Bigg].
\end{aligned}$$

2. If $\vartheta = 1$ in (26). we get

$$\tau\left((\varepsilon-1)\varsigma\left(\frac{a+b}{2}\right)-\varepsilon\frac{\varsigma(a)+\varsigma(b)}{2}+\frac{1}{b-a}\int_a^b\varsigma(t)dt\right)$$
$$\leqslant\frac{1}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_\alpha^{\frac{a+b}{2}}\tau\left[(\mu-t)\varsigma'(t)\right]dt\right.$$
$$\left.+\frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^\beta\tau\left[(\nu-t)\varsigma'(t)\right]dt\right]. \tag{27}$$

3. If $\varepsilon = 0$, $\vartheta = 1$ in (26). we get

$$\tau\left(\frac{1}{b-a}\int_a^b\varsigma(t)dt-\varsigma\left(\frac{a+b}{2}\right)\right)$$
$$\leqslant\frac{1}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau[(a-t)\varsigma'(t)]dt\right.$$
$$\left.+\frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau[(b-t)\varsigma'(t)]dt\right].$$

REMARK 13. Assume that $\tau : [\alpha,\beta]\to\mathbb{R}$ be an $(\alpha,\beta,\gamma,\delta)$-convex functions function in mixed kind:

1. If $\varsigma(t)=\frac{1}{t}$ in inequality (27) where $t\in[a,b]\subset(0,\infty)$, then we have

$$\tau\left[\frac{A(a,b)+(\varepsilon-1)L(a,b)}{A(a,b)L(a,b)}-\varepsilon\frac{A(a,b)}{G^2(a,b)}\right]$$
$$\leqslant\frac{1}{\beta-\alpha}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{t-\mu}{t^2}\right]dt\right.$$
$$\left.+\frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{t-\nu}{t^2}\right]dt\right].$$

2. If $\varsigma(t)=-\ln t$ in inequality (27), where $t\in[a,b]\subset(0,\infty)$, then we have

$$\tau\left[\ln\left(\frac{\exp[\varepsilon A(\ln a,\ln b)]A^{(1-\varepsilon)}(a,b)}{I(a,b)}\right)\right]$$
$$\leqslant\frac{1}{\beta-\alpha}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{t-\mu}{t}\right]dt\right.$$
$$\left.+\frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{t-\nu}{t}\right]dt\right].$$

3. If $\varsigma(t)=t^p$, $p\in\mathbb{R}\setminus\{0,-1\}$ in inequality (27), where $t\in[a,b]\subset(0,\infty)$, then we have

$$\begin{aligned}&\tau\left[L_p^p(a,b)+(\varepsilon-1)A^p(a,b)-\varepsilon A(a^p,b^p)\right]\\&\leqslant\frac{1}{\beta-\alpha}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{p(\mu-t)}{t^{1-p}}\right]dt\right.\\&\left.+\frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^b\tau\left[\frac{p(\nu-t)}{t^{1-p}}\right]dt\right].\end{aligned}$$

REMARK 14. In Theorem 6,

1. Let $g(t)=t$, $x=\frac{a+b}{2}$, $\vartheta=1$, $0<a<b$, $q\geqslant 1$ and $\varsigma:\mathbb{R}\to\mathbb{R}^+$, $\varsigma(t)=t^n$ in (18). Then

$$\begin{aligned}&|L_n^n(a,b)+(\varepsilon-1)A^n(a,b)-\varepsilon A(a^n,b^n)|\\&\leqslant\frac{M(b-a)(\varepsilon-1)^2}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.\end{aligned}$$

2. Let $g(t)=t$, $x=\frac{a+b}{2}$, $\vartheta=1$, $0<a<b$, $q\geqslant 1$ and $\varsigma:(0,1]\to\mathbb{R}$, $\varsigma(t)=-\ln t$ in (18). Then

$$\begin{aligned}&\left|\ln\left(\frac{\exp[\varepsilon A(\ln a,\ln b)]A^{(1-\varepsilon)}(a,b)}{I(a,b)}\right)\right|\\&\leqslant\frac{M(b-a)(\varepsilon-1)^2}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.\end{aligned}$$

REMARK 15. In Theorem 7,

1. Let $g(t)=t$, $x=\frac{a+b}{2}$, $\vartheta=1$, $0<a<b$, $p^{-1}+q^{-1}=1$ and $\varsigma:\mathbb{R}\to\mathbb{R}^+$, $\varsigma(t)=t^n$ in (22). Then

$$\begin{aligned}&|L_n^n(a,b)+(\varepsilon-1)A^n(a,b)-\varepsilon A(a^n,b^n)|\\&\leqslant\frac{M(b-a)(\varepsilon-1)^2}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.\end{aligned}$$

2. Let $g(t)=t$, $x=\frac{a+b}{2}$, $\vartheta=1$, $0<a<b$, $p^{-1}+q^{-1}=1$ and $\varsigma:(0,1]\to\mathbb{R}$,

$\varsigma(t) = -\ln t$ in (22). Then

$$\left|\ln\left(\frac{\exp\left[\varepsilon A(\ln a, \ln b)\right] A^{(1-\varepsilon)}(a,b)}{I(a,b)}\right)\right|$$

$$\leqslant \frac{M(b-a)(\varepsilon-1)^2}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

4. Conclusion and remarks

4.1. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind, this class of functions contains many important classes including class of (α,β)-convex functions in 1^{st} and 2^{nd} kind [14], (s,r)-convex functions in mixed kind [2], s-convex functions in 1^{st} and 2^{nd} kind [5], P-convex functions [13], quasi convex functions and the class of convex. We have stated our first main result in section 2, the generalization of Ostrowski inequality [23] via fractional integral with respect to another function and others results obtained by using different techniques including Hölder's inequality [28] and power mean inequality [27]. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

4.2. Remarks and future ideas

1. One may do similar work to generalize all results stated in this article by applying weights.

2. One may also do similar work by using various different classes of functions.

3. One may also generalize this work in fractional integral form.

4. One may try to state all results stated in this article for fractional integral with respect to another function.

5. One may also state all results stated in this article for higher order derivatives.

6. One may also state all results stated in this article for multivariable real valued functions.

7. One may also state all results stated in this article for quantum Calculus.

8. One may also state all results stated in this article in time scale domain.

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