ON SOLVABILITY OF THE NON–LOCAL PROBLEM FOR THE FRACTIONAL MIXED–TYPE EQUATION WITH BESSEL OPERATOR

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Abstract. The m-point non-local problem is considered for the partial differential equation of mixed-type with singular coefficients, namely fractional wave equation involving the right-hand side bi-ordinal Hilfer derivative and sub-diffusion equation with the regularized Caputo-like counterpart hyper-Bessel differential operator. The main technique of solving the problem is based on the method of separation variables. Also, the connection between the given data and the uniquely solvability of the problem is established and an explicit solution is represented by Fourier-Bessel series in the given domain.

1. Introduction

Fractional differential equations plays a significant role, because of its multiple applications in engineering, chemistry, biology and other parts of science and modeling the real-life problems [32, 28, 22]. Studying boundary value problems for linear and non-linear fractional differential equations with the Riemann-Liouville and the Caputo fractional derivative have becoming interesting targets simultaneously [5, 13, 26, 6].

The quality and the types of articles have been changed when the generalized Riemann-Liouville differential operators (later called Hilfer derivative) used in the scientific field with its interesting application [20, 21]. To tell the truth, the generalization of the Riemann-Lioville fractional differential operators was already announced by M. M. Dzherbashian and A. B. Nersesian [16] in 1968, but because of some reasons it was not familiar with many mathematicians around the globe till the translation of this work published in the journal of Fract. Calc. Appl. An. [15]. We also refer some papers [2, 10] devoted studying some problems with the Dzherbashian-Nersesian differential operator which has the following form

$$D_{0x}^{\sigma_n} g(x) = I_{0x}^{1-\gamma_n} D_{0x}^{\gamma_{n-1}} \dots D_{0x}^{\gamma_1} D_{0x}^{\gamma_0}, \ n \in \mathbb{N}, \ x > 0 \tag{1}$$

where I_{0x}^{α} and D_{0x}^{α} are the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative of order α respectively, $\sigma_n \in (0, n]$ which is defined by

$$\sigma_n = \sum_{j=0}^{n} \gamma_j - 1 > 0, \ \gamma_j \in (0, 1].$$

V. I. Bulavatsky considered specific generalization of Hilfer fractional derivative [12], the particular case of the Dzherbashian-Nersesian differential operator which forms as follows [25]:

$$D_{0\pm}^{(\alpha,\beta)\mu}g(t) = I_{0\pm}^{\mu(n-\alpha)}\left(\pm\frac{d}{dt}\right)^n I_{0\pm}^{(1-\mu)(n-\beta)}g(t) \tag{2}$$

This operator is similar to Hilfer derivative in terms of its interpolation concept between the Riemann-Liouville and Caputo fractional derivatives, specifically, i.e.

$$D_{a+}^{(\alpha,\beta)\mu}g(t) = \begin{cases} D_{a+}^{\beta}g(t), & \mu = 0, \\ {}^{C}D_{a+}^{\alpha}g(t), & \mu = 1. \end{cases}$$

In [25] the Cauchy problem is investigated for the ordinary differential equation involving the right-sided bi-ordinal Hilfer fractional derivative:

$$\begin{cases} D_{0-}^{(\alpha,\beta)\mu}u(t) = \lambda u(t) + g(t), \\ \lim_{t\to 0-} I_{0-}^{2-\gamma}u(t) = \xi_0, \\ \lim_{t\to 0-} \frac{d}{dt}I_{0-}^{2-\gamma}u(t) = \xi_1, \end{cases} \tag{3}$$

where $1 < \alpha, \beta \leqslant 2$, $\gamma = \beta + \mu(2-\beta)$, $\xi_0, \xi_1 \in \mathbb{R}$, $g(t)$ is the given function.

LEMMA 1. *Let $g(t) \in C[-T,0]$, $g'(t) \in L_1(-T,0)$. Then the solution of the problem (3) as follows:*

$$u(t) = \xi_0(-t)^{\gamma-2}E_{\delta,\gamma-1}[\lambda(-t)^{\delta}] - \xi_1(-t)^{\gamma-1}E_{\delta,\gamma}[\lambda(-t)^{\delta}]$$
$$+ \int_t^0 (z-t)^{\delta-1}E_{\delta,\delta}\left[\lambda(z-t)^{\delta}\right]g(z)dz, \tag{4}$$

where $\delta = \beta + \mu(\alpha-\beta)$, $\gamma = \beta + \mu(2-\beta)$.

We notice that while have been investigated the initial-boundary value or non-local problems involving popular class of differential operators like the Riemann-Liouville, Caputo [4,3], Hilfer fractional derivatives, Hadamard [7], Hilfer-Hadamard, Prabhakar, Atangana-Baleanu [11] the interest to another type of the differential operators is increased by many scientists, for instance, the hyper-Bessel differential operator [30,34] is becoming main target of research. The importance of the hyper-Bessel differential operator is increasing since introduced by Dimovski [14] because of its applications in science. For example, in [18] authors used hyper-Bessel differential operator to investigate heat diffusion equation for describing the Brownian motion. In [17] it is investigated fractional relaxation equation the regularized Caputo-like counterpart of the hyper-Bessel operator which has the following form

$$^{C}\left(t^{\theta}\frac{d}{dt}\right)^{\alpha}f(t) = (1-\theta)^{\alpha}t^{-\alpha(1-\theta)}D_{1-\theta}^{-\alpha,\alpha}\left(f(t) - f(0)\right) \tag{5}$$

where

$$D_\beta^{\gamma,\delta} f(t) \prod_{j=1}^n \left(\gamma + j + \frac{t}{\beta} \frac{d}{dt} \right) (I_\beta^{\gamma+\delta,n-\delta} f(t)),$$

is a Erdelyi-Kober fractional derivative and $I_\beta^{\gamma,\delta}$ is a Erdelyi-Kober (E-K) fractional integral defined as [29]

$$I_\beta^{\gamma,\delta} f(t) = \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t (t^\beta - \tau^\beta)^{\delta-1} \tau^{\beta\gamma} f(\tau) d(\tau^\beta),$$

which can be reduced up with weight to $I_{0+}^q f(t)$ Riemann-Liouville fractional integral at $\gamma = 0$ and $\beta = 1$. In [8] Fatma Al-Musalhi, et al. considered direct and inverse problems for the fractional diffusion equation with the regularized Caputo-like counterpart of the hyper-Bessel operator and proved the theorem of existence and uniqueness of the solution.

Several local and nonlocal boundary value problems for mixed-type equations, ie, elliptic-hyperbolic and hyperbolic type equations were published [1, 24, 23]. The interesting point is that the conjugation conditions are taken according to the considered mixed-type equations and domains [27, 9].

2. Formulation of the problem

In the present work, we investigate the following fractional differential equation of mixed type involving the regularized Caputo-like counterpart of the hyper-Bessel operator and the bi-ordinal Hilfer derivative in $\Omega = \Omega_1 \cup \Omega_2 \cup Q$ domain:

$$f(x,t) = \begin{cases} {}^C \left(t^\theta \dfrac{\partial}{\partial t} \right)^{\alpha_1} u(x,t) - \dfrac{1}{x} u_x(x,t) - u_{xx}(x,t), & (x,t) \in \Omega_1, \\ D_{0-}^{(\alpha_2,\beta_2)\mu} u(x,t) - \dfrac{1}{x} u_x(x,t) - u_{xx}(x,t), & (x,t) \in \Omega_2, \end{cases} \tag{6}$$

where $\Omega_1 = \{(x,t) : 0 < x < 1, \, 0 < t < T\}$, $\Omega_2 = \{(x,t) : 0 < x < 1, \, -T < t < 0\}$, $Q = \{(x,t) : 0 < x < 1, t = 0\}$, $0 < \alpha_1 \leqslant 1$, $\theta < 1$, $1 < \alpha_2, \beta_2 \leqslant 2$, $0 \leqslant \mu \leqslant 1$, $D_{0-}^{(\alpha_2,\beta_2)\mu}$ is the right-sided bi-ordinal Hilfer fractional derivative in the form (2) and ${}^C \left(t^\theta \frac{d}{dt} \right)^{\alpha_1}$ is the regularized Caputo-like counterpart of the hyper-Bessel fractional differential operator defined as (5).

PROBLEM. Find a solution of eq. (6) in Ω, which satisfies the following regularity conditions

$$u(x,t) \in C(\overline{\Omega} \setminus Q), \quad u(\cdot,t) \in C^2(\Omega_1 \cup \Omega_2),$$

$${}^C \left(t^\theta \frac{\partial}{\partial t} \right)^{\alpha_1} u(x,t) \in C(\Omega_1), \quad D_{0-}^{(\alpha_2,\beta_2)\mu} u(x,t) \in C(\Omega_2)$$

along with the boundary conditions

$$\lim_{t \to 0+} x u_x(x,t) = 0, \, u(1,t) = 0, \quad -T \leqslant t \leqslant T, \tag{7}$$

non-local condition

$$\sum_{i=1}^{m} \zeta_i I_{0-}^{(1-\mu)(2-\beta_2)} u(x, \xi_i) = u(x, T), \ \ 0 \leqslant x \leqslant 1, \tag{8}$$

and the gluing conditions

$$\lim_{t \to 0-} I_{0-}^{(1-\mu)(2-\beta_2)} u(x,t) = \lim_{t \to 0+} u(x,t), \ \ 0 \leqslant x \leqslant 1, \tag{9}$$

$$\lim_{t \to 0-} \frac{d}{dt} I_{0-}^{(1-\mu)(2-\beta_2)} u(x,t) = \lim_{t \to 0+} t^{1-(1-\theta)\alpha_1} u_t(x,t), \ \ 0 < x < 1, \tag{10}$$

where $-T \leqslant \xi_1 < \xi_2 < \ldots < \xi_m < 0, \ \ f(x,t)$ is a given function.

3. Method of investigation

Using the separation variables method we obtain the following spectral problem

$$X''(x) + \frac{1}{x}X'(x) + \lambda^2 X(x) = 0, \tag{11}$$

$$\lim_{x \to 0+} xX(x) = 0, \ \ X(1) = 0, \tag{12}$$

(9) is a Bessel equation of order zero; furthermore, (9), (10) is a self-adjoint problem and its eigenfunctions are the Bessel functions given as follows

$$X_k(x) = J_0(\lambda_k x), \ \ k = 1, 2, \ldots, \tag{13}$$

and the eigenvalues λ_k, are the positive zeros of $J_0(x)$, i.e.

$$\lambda_k = \pi k - \frac{\pi}{4}.$$

The system of eigenfunctions $\{X_k\}$ forms an orthogonal basis in $L^2(0,1)$ (see [19], p. 40),, hence we can write sought function and given function in the form of series expansions as follows:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) J_0(\lambda_k x), \tag{14}$$

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) J_0(\lambda_k x), \tag{15}$$

where $u_k(t)$ is not known yet, and $f_k(t)$ is the coefficient of Fourier-Bessel series, i.e.

$$f_k(t) = \frac{2}{J_1^2(\lambda_k)} \int_0^1 x f(x,t) J_0(\lambda_k x) dx.$$

Let us introduce new notations:

$$\lim_{t \to 0-} I_{0-}^{(1-\mu)(2-\beta_2)} u(x,t) = \varphi(x), \ 0 \leqslant x \leqslant 1, \tag{16}$$

$$\lim_{t \to 0-} \frac{d}{dt} I_{0-}^{(1-\mu)(2-\beta_2)} u(x,t) = \psi(x), \ 0 < x < 1, \tag{17}$$

$$\lim_{t \to 0+} u(x,t) = \tau(x), \ 0 \leqslant x \leqslant 1, \tag{18}$$

here $\varphi(x)$, $\tau(x)$ and $\psi(x)$ are unknown functions to be found later.

Further, after substituting (14) and (15) into the Eq. (6) and initial conditions (16), (17), (18), we obtain the following problems

$$\begin{cases} {}^{C}\left(t^{\theta}\dfrac{d}{dt}\right)^{\alpha_1} u_k(t) + \lambda_k^2 u_k(t) = f_k(t), \\ \qquad\qquad u_k(0+) = \tau_k, \end{cases} \tag{19}$$

and

$$\begin{cases} D_{0-}^{(\alpha_2,\beta_2)\mu} u_k(t) + \lambda_k^2 u_k(t) = f_k(t), \\ I_{0-}^{(1-\mu)(2-\beta_2)} u_k(0-) = \varphi_k, \\ \dfrac{d}{dt} I_{0-}^{(1-\mu)(2-\beta_2)} u_k(0-) = \psi_k, \end{cases} \tag{20}$$

in Ω_1 and Ω_2 respectively.

The problem (19) was studied in [8] and we can write the solution of the considered problem in Ω_1:

$$u(x,t) = \sum_{k=1}^{\infty} \left[\tau_k E_{\alpha_1,1}\left(-\frac{\lambda_k^2}{p^{\alpha_1}} t^{p\alpha_1} \right) + G_k(t) \right] J_0(\lambda_k x), \tag{21}$$

here $p = 1 - \theta$ and

$$G_k(t) = \frac{1}{p^{\alpha_1}\Gamma(\alpha_1)} \int_0^t (t^p - \tau^p)^{\alpha_1-1} f_k(\tau)d(\tau^p)$$
$$- \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t (t^p - \tau^p)^{2\alpha_1-1} E_{\alpha_1,2\alpha_1}\left[-\frac{\lambda_k^2}{p^{\alpha_1}} (t^p - \tau^p)^{\alpha_1} \right] f_k(\tau)d(\tau^p),$$

where τ_k is not known yet.

As we mentioned above the problem (20) studied in [25] and using the Lemma 1 we can write the solution of (20) and then, considering (14) the solution of considered

problem in Ω_2 domain can be represented in the following form

$$u(x,t) = \sum_{k=1}^{+\infty} \varphi_k(-t)^{\gamma_2-2}E_{\delta_2,\gamma_2-1}[-\lambda_k^2(-t)^{\delta_2}]J_0(\lambda_k x)$$

$$-\sum_{k=1}^{+\infty} \psi_k(-t)^{\gamma_2-1}E_{\delta_2,\gamma_2}[-\lambda_k^2(-t)^{\delta_2}]J_0(\lambda_k x)$$

$$+\sum_{k=1}^{+\infty}\int_t^0 (z-t)^{\delta_2-1}E_{\delta_2,\delta_2}[-\lambda_k^2(z-t)^{\delta_2}]f_k(z)dzJ_0(\lambda_k x), \qquad (22)$$

where $\gamma_2 = \beta_2 + \mu(2-\beta_2)$, $\delta_2 = \beta_2 + \mu(\alpha_2-\beta_2)$ and φ_k, ψ_k are not known yet.

After substituting (21) and (22) into gluing conditions with considering (14), (15) we obtain the following system of equations with respect to τ_k, φ_k and ψ_k:

$$\begin{cases} \psi_k = -\frac{\lambda_k^2}{\Gamma(\alpha_1)}\tau_k, \\ \tau_k = \varphi_k \end{cases} \qquad (23)$$

Considering non-local condition (8) and from (23) we find unknowns as follows

$$\tau_k = \varphi_k = \frac{F_k}{\Delta_k}, \qquad (24)$$

$$\psi_k = \frac{-\lambda_k^2}{p^{\alpha_1}\Gamma(\alpha_1)}\frac{F_k}{\Delta_k}, \qquad (25)$$

where

$$\Delta_k = \sum_{i=1}^m \zeta_i\left[E_{\delta_2,1}(-\lambda_k^2(-\xi_i)) + \frac{\lambda_k^2\xi_i}{p^{\alpha_1}\Gamma(\alpha_1)}E_{\delta_2,2}(-\lambda_k^2(-\xi_i))\right] - E_{\alpha_1,1}\left(-\frac{\lambda_k^2}{p^{\alpha_1}}T^{\alpha_1 p}\right)$$

$$F_k = G_k(T) - \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 (s-\xi_i)^{\delta_2-\gamma_2+1}E_{\delta_2,\delta_2-\gamma_2+2}(-\lambda_k^2(s-\xi_i)^{\delta_2})f_k(s)ds,$$

$$G_k(T) = \frac{1}{p^{\alpha_1}\Gamma(\alpha_1)}\int_0^T (T^p-\tau^p)^{\alpha_1-1}f_k(\tau)d(\tau^p)$$

$$-\frac{\lambda_k^2}{p^{2\alpha_1}}\int_0^T (T^p-\tau^p)^{2\alpha_1-1}E_{\alpha_1,2\alpha_1}\left[-\frac{\lambda_k^2}{p^{\alpha_1}}(T^p-\tau^p)^{\alpha_1}\right]f_k(\tau)d(\tau^p).$$

If $\Delta_k \neq 0$, then we can find τ_k, φ_k, ψ_k unknowns uniquely.

First we show that $\Delta_k \neq 0$ for sufficiently large k. For this intention we use the following lemma obtained from the properties of Wright-type function studied by A. Pskhu in [33].

LEMMA 2. [33] *If* $\pi \geqslant |argz| > \frac{\pi\alpha}{2} + \varepsilon$, $\varepsilon > 0$, *then the following relations are valid for* $z \to \infty$:

$$\lim_{|z|\to\infty} E_{\alpha,\beta}(z) = 0,$$

$$\lim_{|z|\to\infty} zE_{\alpha,\beta}(z) = -\frac{1}{\Gamma(\beta - \alpha)}.$$

By using the Lemma 2, we can calculate the behavior of Δ_k at $k \to \infty$

$$\lim_{k\to+\infty} \Delta_k = \lim_{|z_1|\to+\infty} \sum_{i=1}^{m} \zeta_i \left[E_{\delta_2,1}(z_1) + \frac{1}{\Gamma(\alpha_1)p^{\alpha_1}}z_1 E_{\delta_2,2}(z_1)\right] - \lim_{|z_2|\to+\infty} E_{\alpha_1,1}(z_2)$$

$$= \sum_{i=1}^{m} \frac{\zeta_i}{\Gamma(\alpha_1)p^{\alpha_1}\Gamma(2-\delta_2)},$$

where $z_1 = -\lambda_k^2(-\xi_i)$, $z_2 = -\frac{\lambda_k^2}{p^{\alpha_1}}T^{\alpha_1}p$, $\lambda_k = \pi k - \frac{\pi}{4}$.

If $\sum_{i=1}^{m} \frac{\zeta_i}{\Gamma(\alpha_1)p^{\alpha_1}\Gamma(2-\delta_2)} > 0$ and from the last equality it is seen that $\Delta_k > 0$ for sufficiently large k.

3.1. Uniqueness of the solution

To show the uniqueness of the solution, it is enough to prove that homogeneous problem has a trivial solution.

Let us first consider the following integral

$$u_k(t) = \frac{2}{J_1^2(\lambda_k)} \int_0^1 xu(x,t)J_0(\lambda_k x)dx, \ k = 1,2,3,\ldots, \tag{26}$$

Then we introduce another function based on (26)

$$v_\varepsilon(t) = \frac{2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} xu(x,t)J_0(\lambda_k x)dx, \ k = 1,2,3,\ldots, \tag{27}$$

Applying $^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1}$ and $D_{0-}^{(\alpha_2,\beta_2)\mu}$ to (27) and using the equation (6) in homogeneous case with respect to t

$$^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1} v_\varepsilon(t) = \frac{2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} {}^C\left(t^\theta \frac{\partial}{\partial t}\right)^{\alpha_1} u(x,t)xJ_0(\lambda_k x)dx$$

$$= \frac{2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} \left[u_{xx}(x,t) + \frac{1}{x}u_x(x,t)\right]xJ_0(\lambda_k x)dx$$

$$= \frac{-2\lambda_k^2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} u(x,t)xJ_0(\lambda_k x)dx$$

$$D_{0-}^{(\alpha_2,\beta_2)\mu}v_\varepsilon(t) = \frac{2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} D_{0-}^{(\alpha_2,\beta_2)\mu} u(x,t)xJ_0(\lambda_k x)dx$$

$$= \frac{2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} \left[u_{xx}(x,t) + \frac{1}{x}u_x(x,t) \right]xJ_0(\lambda_k x)dx$$

$$= \frac{-2\lambda_k^2}{J_1^2(\lambda_k)} \int_\varepsilon^{1-\varepsilon} u(x,t)xJ_0(\lambda_k x)dx$$

and integrating by parts twice the right sides of the equalities on $t \in (0,T)$ and $t \in (-T,0)$, respectively, and passing to the limit on $\varepsilon \to +0$ yield

$$\begin{cases} {}^C\left(t^\theta \frac{d}{dt}\right)^{\alpha_1} u_k(t) + \lambda^2 u_k(t) = 0, \ t > 0, \\ D_{0-}^{(\alpha_2,\beta_2)\mu} u_k(t) + \lambda^2 u_k(t) = 0, \ t < 0. \end{cases} \tag{28}$$

Considering conditions in (19), (20) in homogeneous case, (28) has a solution $u_k(t) = 0$ if $\Delta_k \neq 0$. Then from (26) and the completeness of the system $X_k(x)$ in the space $L^2(0,1)$, $u(x,t) \equiv 0$ in $\overline{\Omega}$. This completes the prove of uniqueness of the solution of the main problem.

3.2. Existence of the solution

In the below, we present the well-known lemma about the upper bound of the Mittag-Leffler function and the theorem related to Fourier-Bessel series for showing the existence of the solution.

LEMMA 3. [28] *Let* $\alpha < 2$, $\beta \in R$ *and* $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, $M^* > 0$. *Then,the following estimate hold*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{M^*}{1+|z|}, \ \mu \leqslant |argz| \leqslant \pi, \ |z| \geqslant 0.$$

THEOREM 1. [31] *Let* $f(x)$ *be a function defined on the interval* $[0,1]$ *such that* $f(x)$ *is differentiable* $2s$ *times* $(s \geqslant 1)$ *and*

- $f(0) = f'(0) = \ldots = f^{(2s-1)}(0) = 0$

- $f^{(2s)}(x)$ *is bounded (this derivative may not exist at certain points)*

- $f(1) = f'(1) = \ldots = f^{(2s-2)}(1) = 0$

then the following inequalities satisfied by the Fourier-Bessel coefficients of $f(x)$:

$$|f_k| \leqslant \frac{M}{\lambda_k^{2s-\frac{1}{2}}}.$$

Considering above Lemma 3 and Theorem 1, we show the upper bound of $G_k(t)$:

$$|G_k(t)| \leqslant \int_0^t |t^p - \tau^p|^{\alpha_1 - 1} |f_k(\tau)| d(\tau^p)$$

$$+ \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t |t^p - \tau^p|^{2\alpha_1 - 1} \left| E_{\alpha_1, 2\alpha_1} \left[-\frac{\lambda_k^2}{p^{\alpha_1}} (t^p - \tau^p)^{\alpha_1} \right] \right| |f_k(\tau)| d(\tau^p)$$

$$\leqslant \frac{1}{p^{\alpha_1} \Gamma(\alpha_1)} \int_0^t |t^p - \tau^p|^{\alpha_1 - 1} \frac{M}{\lambda_k^{7/2}} d(\tau^p)$$

$$+ \frac{\lambda_k^2}{p^{2\alpha_1}} \int_0^t \frac{p^{\alpha_1} |t^p - \tau^p|^{2\alpha_1 - 1} M^*}{p^{\alpha_1} + \lambda_k^2 |t^p - \tau^p|^{\alpha_1}} \frac{M}{\lambda_k^{7/2}} d(\tau^p)$$

$$\leqslant \left[\frac{M}{\lambda_k^{7/2} p^{\alpha_1} \Gamma(\alpha_1)} + \frac{M^* M}{p^{\alpha_1} \lambda_k^{7/2}} \right] \int_0^t |t^p - \tau^p|^{\alpha_1 - 1} d(\tau^p)$$

$$\leqslant \frac{1}{\lambda_k^{7/2}} \left[\frac{M}{p^{\alpha_1} \Gamma(\alpha_1)} + \frac{M^* M}{p^{\alpha_1}} \right] \frac{|t|^{\alpha_1 p}}{\alpha_1}$$

$$\leqslant \frac{M_1}{\lambda_k^{7/2}} |t|^{\alpha_1 p} \leqslant \frac{M_1}{\lambda_k^{7/2}} |T|^{\alpha_1 p},$$

$$M_1 = \frac{M}{p^{\alpha_1} \Gamma(\alpha_1 + 1)} + \frac{M^* M}{\alpha_1 p^{\alpha_1}},$$

By using the last inequality, Lemma 3 and Theorem 1, we can write the upper bound of F_k:

$$|F_k| \leqslant |G_k(T)| + \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 |s - \xi_i|^{\delta_2 - \gamma_2 + 1} |E_{\delta_2, \delta_2 - \gamma_2 + 2}(-\lambda_k^2 (s - \xi_i)^{\delta_2})| |f_k(s)| ds$$

$$\leqslant \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} + \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 |s - \xi_i|^{\delta_2 - \gamma_2 + 1} \frac{M^*}{1 + \lambda_k^2 |s - \xi_i|^{\delta_2}} \frac{M}{\lambda_k^{7/2}} ds$$

$$\leqslant \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} + \sum_{i=1}^m \zeta_i \int_{\xi_i}^0 |s - \xi_i|^{1 - \gamma_2} \frac{M M^*}{\lambda_k^{11/2}}$$

$$\leqslant \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}} + \sum_{i=1}^m \zeta_i \frac{(-\xi_i)^{2 - \gamma_2}}{(2 - \gamma_2)} \frac{M M^*}{\lambda_k^{11/2}}$$

$$\leqslant \frac{M_2}{\lambda_k^{7/2}},$$

$$M_2 = M_1 T^{\alpha_1 p} + \sum_{i=1}^m \frac{\zeta_i M M^* (-\xi_i)^{2 - \gamma_2}}{(2 - \gamma_2) \lambda_k^2},$$

or

$$|F_k| \leqslant \frac{M_2}{\lambda_k^{7/2}}, \tag{29}$$

Now considering (29) and $\Delta_k \neq 0$ then, we write upper bounds of τ_k, φ_k, ψ_k in (24) and (25).

$$|\tau_k| = |\varphi_k| \leqslant \left|\frac{1}{\Delta_k}\right||F_k| \leqslant \frac{M_2}{|\Delta_k|\lambda_k^{7/2}}, \tag{30}$$

$$|\psi_k| = \left|\frac{-\lambda_k^2}{p^{\alpha_1}\Gamma(\alpha_1)}\right|\left|\frac{F_k}{\Delta_k}\right| \leqslant \frac{M_2}{p^{\alpha_1}\Gamma(\alpha_1)|\Delta_k|\lambda_k^{3/2}}. \tag{31}$$

For proving the existence of the solution, we need to show uniform convergence of series representations of $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha u(x,t)$ and $D_{0-}^{(\alpha_2,\beta_2)\mu}u(x,t)$ by using the solution (21) and (22) in Ω_1 and Ω_2 respectively.

According to the last inequality and Theorem 1, then we can present the existence of the solutions in both domains.

$$|u(x,t)| = \sum_{k=1}^{\infty}|u_k(t)||J_0(\lambda_k x)| \leqslant \sum_{k=1}^{\infty}|u_k(t)|$$

$$\leqslant \sum_{k=1}^{\infty}\left[|\tau_k||E_{\alpha_1,1}(-\frac{\lambda_k^2}{p^{\alpha_1}}t^{\alpha_1 p})| + |G_k(t)||\right]$$

$$\leqslant \sum_{k=1}^{\infty}\left(\frac{p^{\alpha_1}}{p^{\alpha_1} + \lambda_k^2|t^{p\alpha_1}|}\frac{M_2}{|\Delta_k|\lambda_k^{7/2}} + \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}}\right).$$

One can shows that the series representation of $u(x,t)$ is bounded by convergent numerical series and by Weierstrass M-test, the series of $u(x,t)$ converges uniformly in Ω_1.

Now we remind some properties of Bessel functions [31]: $J_0'(x) = -J_1(x)$; $2J_1'(x) = J_0(x) - J_2(x)$ and asymptotic formula $|J_\nu(\lambda_k x)| \leqslant \frac{2A}{\sqrt{\lambda_k x}}$, $\nu > -1/2$, $A = const$.

It is not difficult to see that the series representation of $u_{xx}(x,t)$ is bigger than $u_x(x,t)$ hence it is enough to show the uniform convergence of $u_{xx}(x,t)$. By using these properties we have

$$|u_{xx}(x,t)| \leqslant \sum_{k=1}^{\infty}|u_k(t)|\left|\frac{d^2}{dx^2}J_0(\lambda_k x)\right| = \sum_{k=1}^{\infty}|u_k(t)|\frac{\lambda_k^2}{2}\left|J_2(\lambda_k x) - J_0(\lambda_k x)\right|$$

And as a similar way of $u(x,t)$ we can show that

$$|u_{xx}(x,t)| \leqslant \sum_{k=1}^{\infty}\left(\frac{p^{\alpha_1}}{p^{\alpha_1} + \lambda_k^2|t^{p\alpha_1}|}\frac{M_2}{|\Delta_k|\lambda_k^{7/2}} + \frac{M_1 T^{\alpha_1 p}}{\lambda_k^{7/2}}\right)\frac{2A}{\sqrt{\lambda_k x}}.$$

From the last inequality we can see that the series representation of $u_{xx}(x,t)$ is bounded by convergent series. According to Weierstrass M-test, the series of $u_{xx}(x,t)$ converges uniformly in Ω_1.

The uniform convergence of $^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha u(x,t)$ which is defined as

$$^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha u(x,t) = u_{xx}(x,t) - \frac{1}{x}u_x(x,t) + f(x,t)$$

is similar to the way of showing convergence of the series representation of $u_{xx}(x,t)$.

In Ω_2 domain it is enough to show the uniform convergence of $u_{xx}(x,t)$ which is bigger than other series. Hence the convergence of the series of $u(x,t)$, $u_x(x,t)$, $D_{0-}^{(\alpha_2,\beta_2)\mu}u(x,t)$ can be derived from the uniform convergence of $u_{xx}(x,t)$.

From Theorem 1 and Lemma 3, in Ω_2 we can have

$$
\begin{aligned}
|u_{xx}(x,t)| &\leqslant \sum_{k=1}^{\infty} |u_k(t)| \frac{\lambda_k^2}{2} \left| J_2(\lambda_k x) - J_0(\lambda_k x) \right| \\
&\leqslant \sum_{k=1}^{\infty} \lambda_k^2 |\varphi_k| |(-t)^{\gamma_2-2}| |E_{\delta_2,\gamma_2-1}(-\lambda_k^2(-t)^{\delta_2})| \\
&\quad + \sum_{k=1}^{\infty} \lambda_k^2 |\psi_k| |(-t)^{\gamma_2-1}| |E_{\delta_2,\gamma_2}(-\lambda_k^2(-t)^{\delta_2})| \\
&\quad + \sum_{k=1}^{+\infty} \lambda_k^2 \int_t^0 |z-t|^{\delta_2-1} |E_{\delta_2,\delta_2}[-\lambda_k^2(z-t)^{\delta_2}]| |f_k(z)| dz \\
&\leqslant \sum_{k=1}^{\infty} \left[\frac{\lambda_k^2 M_2}{|\Delta_k|\lambda_k^{7/2}} \frac{|(-t)^{\gamma_2-2}|M^*}{1+\lambda_k^2|(-t)^{\delta_2}|} + \frac{\lambda_k^2 M_2}{|\Delta_k|\lambda_k^{3/2}p^{\alpha_1}\Gamma(\alpha_1)} \frac{|(-t)^{\gamma_2-1}|M^*}{1+\lambda_k^2|(-t)^{\delta_2}|} \right] \\
&\quad + \sum_{k=1}^{+\infty} \lambda_k^2 \int_t^0 |z-t|^{\delta_2-1} \frac{M^*}{1+\lambda_k^2|(z-t)^{\delta_2}|} \frac{M}{\lambda_k^{7/2}} dz \\
&\leqslant \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{3/2}} \left[\frac{M_2 M^* T^{\gamma_2-2}}{|\Delta_k|(1+\lambda_k^2 T^{\delta_2})} + \frac{M_2 M^* T^{\gamma_2-\delta_2-1}}{|\Delta_k|} + \frac{MM^* \ln(1+\lambda_k^2 T^{\delta_2})}{\delta_2\lambda_k^2} \right] \\
&\leqslant \sum_{k=1}^{\infty} \frac{M_3}{\lambda_k^{3/2}}
\end{aligned}
$$

where $\lim\limits_{\lambda_k\to\infty} \frac{\ln(1+\lambda_k^2 T^{\delta_2})}{\delta_2\lambda_k^2} < \infty$ according to l'Hopital's rule. It can be seen that the series representation of $u_{xx}(x,t)$ is bounded by convergent numerical series and due to Weierstrass M-test, the series of $u_{xx}(x,t)$ converges uniformly in Ω_2

Using (30), (31) and Theorem 1, Lemma 3, we can show the uniform convergence of $u(x,t)$, $u_x(x,t)$, and $D_{0-}^{(\alpha_2,\beta_2)\mu}u(x,t)$ in a similar method used for $u_{xx}(x,t)$.

4. Main results

Finally, we have proved the uniqueness and existence of the solution to the considered problem as stated in the following theorem.

THEOREM 2. *Let* $\Delta_k \neq 0$ *and* $\sum_{i=1}^{m} \dfrac{\zeta_i}{\Gamma(\alpha_1)p^{\alpha_1}\Gamma(2-\delta_2)} > 0$, *and also the following conditions hold for* $f(x,t) \in C(\Omega)$ *such that*

- $f(0,t) = f'_x(0,t) = \ldots = f'''_x(0,t) = 0$;
- $f(1,t) = f'_x(1,t) = f''_x(1,t) = 0$;
- $\dfrac{\partial^4}{\partial x^4} f(x,t)$ *is bounded*;

then, there exist the unique solution of the considered problem which is represented by

$$u(x,t) = \sum_{k=1}^{\infty} U_k(t)J_0(\lambda_k x)$$

where

$$U_k(t) = \begin{cases} \tau_k E_{\alpha_1,1}\left(-\dfrac{\lambda_k^2}{p^{\alpha_1}}t^{p\alpha_1}\right) + G_k(t), \ t > 0, \\ \varphi_k(-t)^{\gamma_2-2}E_{\delta_2,\gamma_2-1}[-\lambda_k^2(-t)^{\delta_2}] - \psi_k(-t)^{\gamma_2-1}E_{\delta_2,\gamma_2}[-\lambda_k^2(-t)^{\delta_2}]J_0(\lambda_k x) \\ \qquad + \displaystyle\int_t^0 (z-t)^{\delta_2-1}E_{\delta_2,\delta_2}[-\lambda_k^2(z-t)^{\delta_2}]f_k(z)dz, \ t < 0 \end{cases}$$

here $p = 1 - \theta$ *and*

$$G_k(t) = \frac{1}{p^{\alpha_1}\Gamma(\alpha_1)}\int_0^t (t^p - \tau^p)^{\alpha_1-1}f_k(\tau)d(\tau^p)$$
$$- \frac{\lambda_k^2}{p^{2\alpha_1}}\int_0^t (t^p - \tau^p)^{2\alpha_1-1}E_{\alpha_1,2\alpha_1}\left[-\frac{\lambda_k^2}{p^{\alpha_1}}(t^p - \tau^p)^{\alpha_1}\right]f_k(\tau)d(\tau^p),$$

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