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ON THE UNIQUENESS OF SOLUTIONS OF TWO INVERSE PROBLEMS FOR THE SUBDIFFUSION EQUATION

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Abstract. Consider an arbitrary positive self-adjoint operator A defined in a separable Hilbert space H. In the nonlocal boundary value problem $D_t^\rho u(t)+Au(t)=f(t)$ ($0<\rho<1$, $0<t\leqslant T$), $u(\xi_0)=\alpha u(0)+\varphi$ (α is a constant and $0<\xi_0\leqslant T$), where D_t is the Caputo derivative, assume that the right-hand side of the equation or the function φ is unknown. In this paper, we study the inverse problems of determining these unknown functions. For both inverse problems, $u(\xi_1)=V$ is taken as the over-determination condition. The main attention is paid to the study of the influence of the constant α on the existence and uniqueness of the solution to the problems. An interesting effect was discovered: when solving the forward problem, the uniqueness of the solution $u(t)$ was violated, while when solving the inverse problem for the same values of α, the solution $u(t)$ became unique.

1. Introduction

Consider an arbitrary positive self-adjoint operator A defined in a separable Hilbert space H with the scalar product $(\cdot,\cdot)$ and the norm $||\cdot||$. Let A have a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\lambda_k:\ 0<\lambda_1\leqslant\lambda_2\cdots\to+\infty$. We will also assume that the sequence $\{\lambda_k\}$ has no finite limit points. For a vector-functions (or simply functions) $h:\mathbb{R}_+\to H$, we define the Caputo fractional derivative of order $0<\rho<1$ as (see, e.g. [1])

$$D_t^\rho h(t)=\frac{1}{\Gamma(1-\rho)}\int_0^t\frac{h'(\xi)}{(t-\xi)^\rho}d\xi,\quad t>0,$$

where $\Gamma(\sigma)$ is Euler's gamma function. Denote by $C((a,b);H)$ the set of continuous functions $u(t)$ of $t\in(a,b)$ with values in H.

Let $f(t)\in C((0,T];H)$ and $\varphi\in H$. The main object studied in this work is the following non-local boundary value problem:

$$\begin{cases} D_t^\rho u(t)+Au(t)=f(t),\quad 0<t\leqslant T;\\ u(\xi_0)=\alpha u(0)+\varphi,\quad 0<\xi_0\leqslant T,\end{cases}\tag{1}$$

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where α is a constant and ξ_0 is a fixed point. Usually problem (1) is called *forward problem*.

In the case when $\xi_0 = T$ and parameter α is equal to zero: $\alpha = 0$, this problem is called *the backward problem* and it is well studied in the works [2]–[4] and [5]. And if $\alpha = 0$ and $\rho = 1$, then we get a classical problem called the *retrospective inverse problem*, which has been studied in detail by various specialists (see, e.g. Chapter 8.2 of [6] and literature therein).

It is well known that in most models described by differential (and pseudodifferential, see e.g., [7]) equations, an initial condition is used to select a single solution. However, there are also processes where we have to use non-local conditions, for example, the integral over time intervals (see, e.g. [8] for reaction diffusion equations or [9] for fractional equations), or connection of solution values at different times, for example, at the initial and final times (see, e.g. [10]–[11]). It should be noted that non-local conditions model some details of natural phenomena more accurately, since they take into account additional information in the initial conditions.

The non-local problem (1) in case of the diffusion equation, namely the following problem

$$\begin{cases} u'(t) + Au(t) = f(t), \quad 0 < t \leqslant T; \\ u(\xi_0) = u(0) + \varphi, \quad 0 < \xi \leqslant T, \end{cases} \tag{2}$$

actively studied by many researchers (see, for example, Ashiraliev A.O. et al. [10]–[11]). As the authors of these papers showed, in contrast to the retrospective inverse problem, problem (2) is solvable coercively in some spaces of smooth functions.

Let us return to the non-local problem (1). The authors of this paper in their previous work [12] studied in detail the influence of parameter $\alpha \neq 0$ on the correctness of problem (1). It turned out that the critical values of parameter α are in the interval $(0,1)$. In order to formulate the main result of work [12], we recall the definition of the Mittag-Leffler function $E_{\rho,\mu}(z)$ with two parameters (see, e.g. [13], Chapter 1):

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)},$$

where μ is an arbitrary complex number and if $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$. Recall (see, e.g. [12]), $E_\rho(-t)$ decreases strictly monotonically as $t > 0$ and, moreover, has the following estimate

$$0 < E_\rho(-t) < 1, \quad t > 0. \tag{3}$$

In work [12] it is proved that if $\alpha \in (0,1)$ and $E_\rho(-\lambda_k \xi_0^\rho) \neq \alpha$ for all k, then the solution of problem (1) exists and is unique. But it may turn out that for some eigenvalue λ_{k_0} of operator A, with multiplicity p_0 (obviously, p_0 is a finite number), equality

$$E_\rho(-\lambda_{k_0} \xi_0^\rho) = \alpha \tag{4}$$

will hold. Then, as proved in [12], in order for a solution to exist, one should require orthogonality conditions of the following form

$$(\varphi, v_k) = 0,\ (f(t), v_k) = 0, \text{ for all } t > 0,\ k \in K_0;\ K_0 = \{k_0, k_0+1, \ldots, k_0+p_0-1\}. \tag{5}$$

It should be emphasized that if the equality (4) holds, the solution will not be unique [12].

The paper [12] also studies two inverse problems of determining the function φ from the non-local condition (1) and the source function f, i.e. the right-hand side of the equation in (1) (in the latter case, it is assumed that f does not depend on t). It is proved that if $\alpha \notin (0,1)$, then the solutions of both inverse problems exist and are unique. The main goal of this paper is to study these inverse problems for critical values of parameter $\alpha \in (0,1)$.

PROBLEM 1. Let $\alpha \in (0,1)$. Find a pair $\{u(t), f\}$ of function $u(t) \in C([0,T];H)$ and $f \in H$ such that $D_t^\rho u(t), Au(t) \in C((0,T];H)$ and satisfying the non-local problem (1) (note, f does not depend on t) and the over-determination condition

$$u(\xi_1) = V, \quad 0 < \xi_1 < \xi_0, \tag{6}$$

where V is a given element of H.

In the case of $\xi_1 = \xi_0$ in (6), the non-local condition in (1) is the same as the Cauchy condition $u(0) = \varphi_1$ (note the $\alpha \neq 0$). The resulting inverse problem is studied in [14]. If the reverse inequality $\xi_1 > \xi_0$ holds, then it will be shown that the solution may not be unique.

PROBLEM 2. Let $\alpha \in (0,1)$. Find a pair $\{u(t), \varphi\}$ of function $u(t) \in C([0,T];H)$ and $\varphi \in H$ such that $D_t^\rho u(t)$, $Au(t) \in C((0,T];H)$ and satisfying the non-local problem (1) and the over-determination condition

$$u(\xi_2) = W, \quad 0 < \xi_2 \leqslant T, \quad \xi_2 \neq \xi_0, \tag{7}$$

where W is a given element of H.

If $\xi_2 = \xi_0$, then the non-local condition $u(\xi) = \alpha u(0) + \varphi$ is the same as the Cauchy condition $u(0) = \varphi_1$ (note $\alpha \neq 0$) and we have the backward problem, considered in [2]–[4].

Everywhere below, for the vector - function $h(t) \in H$ (which may or may not depend on t) by the symbol $h_k(t)$ we will denote the Fourier coefficients with respect to the system of eigenfunctions $\{v_k\}$: $h_k(t) = (h(t), v_k)$.

THEOREM 1. *Let $\varphi, V \in D(A)$ and let the orthogonality conditions (5) be satisfied. Then Problem 1 has only one solution $\{u(t), f\}$ and this solution has the form*

$$f = \sum_{k \notin K_0} \Bigg[\frac{\alpha - E_\rho(-\lambda_k \xi_0^\rho)}{E_\rho(-\lambda_k \xi_1^\rho)\xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho) + \xi_1^\rho E_{\rho,\rho+1}(-\lambda_k \xi_1^\rho)[\alpha - E_\rho(-\lambda_k \xi_0^\rho)]} V_k$$
$$+ \frac{E_\rho(-\lambda_k \xi_1^\rho)}{E_\rho(-\lambda_k \xi_1^\rho)\xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho) + \xi_1^\rho E_{\rho,\rho+1}(-\lambda_k \xi_1^\rho)[\alpha - E_\rho(-\lambda_k \xi_0^\rho)]} \varphi_k \Bigg] v_k, \tag{8}$$

$$u(t) = \sum_{k\notin K_0}\left[\frac{E_\rho(-\lambda_k t^\rho)}{E_\rho(-\lambda_k\xi_0^\rho)-\alpha}\left[\varphi_k - f_k\xi_0^\rho E_{\rho,\rho+1}(-\lambda_k\xi_0^\rho)\right] + f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)\right] v_k$$
$$+\sum_{k\in K_0}\frac{E_\rho(-\lambda_k t^\rho)\,V_k}{E_\rho(-\lambda_k\xi_1^\rho)}\,v_k. \tag{9}$$

Note that, due to the orthogonality condition (5), all Fourier coefficients f_k vanish for $k\in K_0$. Obviously, K_0 can also be an empty set; in this case the sum $\sum_{k\notin K_0}$ is the same as $\sum_{k=1}^{\infty}$.

In order to formulate a result on Problem 2 we define for an arbitrary real number τ the power of operator A as

$$A^\tau h = \sum_{k=1}^{\infty}\lambda_k^\tau h_k v_k,$$

where h_k are the Fourier coefficients of $h\in H$. The domain of definition of this operator is determined from the condition $A^\tau h\in H$ and has the form

$$D(A^\tau) = \{h\in H : \sum_{k=1}^{\infty}\lambda_k^{2\tau}|h_k|^2<\infty\}.$$

In $D(A^\tau)$ one can define the norm

$$||h||_\tau^2 = \sum_{k=1}^{\infty}\lambda_k^{2\tau}|h_k|^2 = ||A^\tau h||^2,$$

which turns $D(A^\tau)$ into a Hilbert space.

THEOREM 2. *Let $W\in D(A)$, $f\in C([0,T];D(A^\varepsilon))$ for some $\varepsilon\in(0,1)$ and let the orthogonality conditions (5) be satisfied. Then Problem 2 has only one solution $\{u(t),\varphi\}$ and this solution has the form*

$$\varphi = \sum_{k\notin K_0}\left[\frac{E_\rho(-\lambda_k\xi_0^\rho)-\alpha}{E_\rho(-\lambda_k\xi_2^\rho)}\left[W_k-\omega_k(\xi_2)\right]+\omega_k(\xi_0)\right]v_k, \tag{10}$$

$$u(t) = \sum_{k\notin K_0}\left[\frac{\varphi_k-\omega_k(\xi_0)}{E_\rho(-\lambda_k\xi_0^\rho)-\alpha}E_\rho(-\lambda_k t^\rho)+\omega_k(t)\right]v_k \tag{11}$$
$$+\sum_{k\in K_0}\frac{E_\rho(-\lambda_k t^\rho)\,W_k}{E_\rho(-\lambda_k\xi_2^\rho)}\,v_k,$$

where

$$\omega_k(t) = \int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta.$$

Note that for $k\in K_0$ all Fourier coefficients φ_k are equal to zero since the orthogonality condition (5). When K_0 is an empty set, then the sum $\sum_{k\notin K_0}$ coincides with $\sum_{k=1}^{\infty}$.

REMARK 1. It should be specially noted that, as was proved in [12] and noted above, when equality (4) holds, the solution to the forward problem is not unique. But it turns out that both inverse problems have a unique solution even under condition (4).

As far as we know, the inverse problem of determining the function φ in a non-local condition was considered only in the paper [15]. The authors investigated this problem for a subdiffusion equation with a fractional Caputo derivative, the elliptic part of which is a differential expression with constant coefficients, defined in a two-dimensional domain. On the other hand, it is not difficult to simulate a real process in which we will face just such an inverse problem. For example, in the temperature distribution process, the initial and final temperatures are not specified, and they are not sought, but it is required to find the difference between the initial and final temperatures.

As for the inverse problems of determining the source function f with final time observation, it is well studied, both for classical partial differential equations and for equations of fractional order. Many theoretical studies have been published. Kabanikhin [6] and Prilepko et al. [16] should be mentioned as classical monographs for integer-order equations. As for fractional differential equations, it is possible to construct theories parallel to the works of [6], [16], and work in this direction is ongoing. In this note, we will pay attention to only some of them, referring interested readers to a review paper [17]. Also note the works [12], [18, 19, 20], where there is a review of recent work in this direction.

We note right away that no one has yet proposed a method for finding the right-hand side given in the abstract form $f(x,t)$. Known results concern the separated source term $f(x,t) = q(t)p(x)$. The correct choice of over-determination conditions depend on whether $q(t)$ or $p(x)$ is unknown.

Quite a lot of papers are devoted to the case considered in this article, namely $q(t) \equiv 1$ and the unknown is $p(x)$. Subdiffusion equations with $Au = u_{xx}$ are considered, for example, in [18, 21, 22, 23]. The authors of [19, 20], studied the inverse problem for multi-term subdiffusion equations in which A is either the Laplace operator or a second order differential operator. The inverse problem for an equation in (1) with the Cauchy condition was studied in [14]. In recent papers [24]–[25] the inverse problem for equations with the Riemann-Liouville derivative has been studied.

The authors of [26] considered as A a non-self-adjoint differential operator (with non-local boundary conditions), and the solution of the inverse problem was found using a biorthogonal series.

The work [27] is devoted to the study of the inverse problem of the simultaneous determination of the Riemann-Liouville derivative and the source function in the subdiffusion equation. To prove the correctness of this inverse problem, the authors used the classical Fourier method.

It should be especially noted that in all the papers cited above, the Cauchy condition in time is considered (the exception is [28], where the integral condition is given by the variable t). As far as we know, in the article [12], the inverse problem for the subdiffusion equation with a time-nonlocal condition is considered for the first time.

More complicated are the inverse problems, in which the unknown is the function $q(t)$ (see the review article [17] and [3]). To determine the function $q(t)$ in such problems, the relation $u(x_0,t) = u_0(t)$ is taken as the over-determination condition. The authors studied mainly the uniqueness of the inverse problem's solution. In this regard, we note the recent papers [29], [30] where the inverse problem for determining the right-hand side of the form $q(t)$ was studied for the Schrodinger equation. Taking over-determination conditions of a rather general form $Bu(\cdot,t)$, where $B : H \to R$ is a linear bounded functional, the authors proved both the existence and uniqueness of a solution to the inverse problem.

The papers [31]–[32] consider the inverse problem of determining of the fractional derivative's order in the subdiffusion and wave equations, respectively.

2. Inverse Problem 1

2.1. Existence

Assume that all the conditions of Theorem 1 are satisfied, i.e. $\varphi, V \in D(A)$ and let the orthogonality conditions (5) be satisfied. Let us first prove the existence of a solution and that the solution has the form (8) and (9). The fact that these series converge in the norm H and in (9) the summation and operators D_t^ρ and A can be interchanged was proved in the work of the authors [12]. Therefore, it suffices to show that the series (8) and (9) formally satisfy the equation and the initial condition (1), and the over-determination condition (6). In order to do this, we rewrite the series (8) and (9) in the form $f = \sum f_k v_k$ and $u(t) = \sum u_k(t) v_k$. Now, according to the Fourier method, it suffices to show that the unknown coefficients f_k and $u_k(t)$ satisfy equation

$$D_t^\rho u_k(t) + \lambda_k u_k(t) = f_k, \tag{12}$$

the non-local condition

$$u_k(\xi_0) = \alpha u_k(0) + \varphi_k, \tag{13}$$

and finally the over-determination condition

$$u_k(\xi_1) = V_k, \tag{14}$$

for all $k \geqslant 1$.

Let us show that $u_k(t)$ and f_k satisfy equation (12). Let $k \notin K_0$. We have $u_k(t) = u_k^1(t) + u_k^2(t)$, where

$$u_k^1(t) = \frac{E_\rho(-\lambda_k t^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} [\varphi_k - f_k \xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho)]$$

and

$$u_k^2(t) = f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho).$$

It is known (see, e.g. [33], p. 174) that $u_k^1(t)$ is a solution to the homogeneous equation (12) with the initial condition

$$u_k^1(0) = \frac{\varphi_k - f_k \xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha}.$$

It is also known (see ibid.) that the function

$$\omega_k(t) = \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta$$

from Theorem 2 is a solution to equation (12) with the right-hand side $f_k(t)$ and with the initial condition $\omega_k(0) = 0$. If in this formula $f_k(t)$ does not depend on t, then the integral can be rewritten in the form (see e.g. [33], formula (4.4.4))

$$f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) d\eta = f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho).$$

Therefore, the function $u_k^2(t)$ is a solution to the inhomogeneous equation (12) with the initial condition $u_k^2(0) = 0$.

Now suppose that $k \in K_0$. Then the function

$$u_k(t) = \frac{E_\rho(-\lambda_k t^\rho) V_k}{E_\rho(-\lambda_k \xi_1^\rho)}$$

is a solution of homogeneous equation (12) with the initial data

$$u_k(0) = \frac{V_k}{E_\rho(-\lambda_k \xi_1^\rho)}.$$

Thus, it is proved that the functions (8) and (9) really satisfy equation (12).

It remains to verify the fulfillment of the non-local condition (13) and the over-determination condition (14).

Let $k \notin K_0$. Since we have calculated $u_k(0) = u_k^1(0) + u_k^2(0)$, we can write

$$\alpha u_k(0) + \varphi_k = \frac{\varphi_k E_\rho(-\lambda_k \xi_0^\rho) - \alpha f_k \xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha}.$$

On the other hand, according to (9), $u_k(\xi_0)$ has exactly the same value:

$$u_k(\xi_0) = \frac{\varphi_k E_\rho(-\lambda_k \xi_0^\rho) - \alpha f_k \xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha}.$$

Let now $k \in K_0$. Then $\varphi_k = 0$ (see (5)) and $E_\rho(-\lambda_k \xi_0^\rho) = \alpha$. Therefore

$$\alpha u_k(0) + \varphi_k = \frac{\alpha V_k}{E_\rho(-\lambda_k \xi_1^\rho)},$$

and

$$u_k(\xi_0) = \frac{E_\rho(-\lambda_k \xi_0^\rho) V_k}{E_\rho(-\lambda_k \xi_1^\rho)} = \frac{\alpha V_k}{E_\rho(-\lambda_k \xi_1^\rho)}.$$

Thus, the Fourier coefficients of function $u(t)$, defined by formula (9), satisfy the non-local condition (13) for all $k \geqslant 1$.

Let us check the fulfillment of the over-determination condition (14). Consider again the case $k \notin K_0$. By virtue of condition (14) we obtain:

$$\frac{E_\rho(-\lambda_k \xi_1^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha}[\varphi_k - f_k \xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho)] + f_k \xi_1^\rho E_{\rho,\rho+1}(-\lambda_k \xi_1^\rho) = V_k.$$

After simple calculations, we get

$$f_k = \frac{\alpha - E_\rho(-\lambda_k \xi_0^\rho)}{E_\rho(-\lambda_k \xi_1^\rho)\xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho) + \xi_1^\rho E_{\rho,\rho+1}(-\lambda_k \xi_1^\rho)[\alpha - E_\rho(-\lambda_k \xi_0^\rho)]} V_k$$
$$+ \frac{E_\rho(-\lambda_k \xi_1^\rho)}{E_\rho(-\lambda_k \xi_1^\rho)\xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho) + \xi_1^\rho E_{\rho,\rho+1}(-\lambda_k \xi_1^\rho)[\alpha - E_\rho(-\lambda_k \xi_0^\rho)]} \varphi_k$$

and this coincides with the Fourier coefficients of the function (8).

If $k \in K_0$, then

$$u_k(\xi_1) = \frac{E_\rho(-\lambda_k \xi_1^\rho) V_k}{E_\rho(-\lambda_k \xi_1^\rho)} = V_k.$$

Thus the existence of a solution to Problem 1 is proved.

2.2. Uniqueness

Let us proceed to the proof of the uniqueness of the solution of Problem 1.

We proceed in the standard way: assuming the existence of two solutions, we obtain contradictions. In other words, let $\{u_1(t), f_1\}$ and $\{u_2(t), f_2\}$ be two different solutions. We show that $u(t) \equiv u_1(t) - u_2(t) \equiv 0$ and $f \equiv f_1 - f_2 = 0$, i.e. we will prove that the solution $\{u(t), f\}$ of the inverse problem:

$$D_t^\rho u(t) + Au(t) = f, \quad t > 0; \tag{15}$$

$$u(\xi_0) = \alpha u(0), \quad 0 < \xi_0 \leqslant T, \tag{16}$$

$$u(\xi_1) = 0, \quad 0 < \xi_1 < \xi_0, \tag{17}$$

where ξ_0 and ξ_1 are the fixed points, is identically zero.

Let $\{u(t), f\}$ be a solution to this problem and denote $u_k(t) = (u(t), v_k)$, $f_k = (f, v_k)$. Then, since A is the self-adjoint of operator, problem (15)–(17) becomes the following inverse problem with respect to $\{u_k(t), f_k\}$:

$$D_t^\rho u_k(t) + \lambda_k u_k(t) = f_k, \quad t > 0; \quad u_k(\xi_0) = \alpha u_k(0), \quad u_k(\xi_1) = 0. \tag{18}$$

Note that if $k \in K_0$ then $f_k = 0$.

Let first $k \notin K_0$. If f_k is known, then the non-local condition implies (see, e.g. [33], p.174)

$$u_k(t) = \frac{f_k \xi_0^\rho E_{\rho,\rho+1}(-\lambda_k \xi_0^\rho)}{\alpha - E_\rho(-\lambda_k \xi_0^\rho)} E_\rho(-\lambda_k t^\rho) + f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho).$$

Scince $u_k(\xi_1)=0$ we have

$$f_k[\xi_0^\rho E_{\rho,\rho+1}(-\lambda_k\xi_0^\rho)E_\rho(-\lambda_k\xi_1^\rho)+\xi_1^\rho E_{\rho,\rho+1}(-\lambda_k\xi_1^\rho)(\alpha-E_\rho(-\lambda_k\xi_0^\rho))]=0. \quad (19)$$

Let us show that for $\xi_1<\xi_0$ the square bracket is not equal to zero. To do this, we introduce the notations: $a(t)=t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)>0$ and $b(t)=E_\rho(-\lambda_k t^\rho)>0$. It is known (see, e.g. [12]) that the function $a(t)$ is increasing and the function $b(t)$ is decreasing. Now let us rewrite the square bracket as

$$c(\xi_0,\xi_1)=a(\xi_0)b(\xi_1)-a(\xi_1)b(\xi_0)+\alpha b(\xi_0).$$

Obviously, for $\xi_1<\xi_0$ this expression is strictly positive. Therefore for all $k\notin K_0$ one has $f_k=0$ (see (19)).

It should be noted that if the inverse inequality $\xi_1>\xi_0$ is satisfied, then the first term in the expression for $c(\xi_0,\xi_1)$ becomes less than the second one and, as a result, there is $\alpha\in(0,1)$ that turns $c(\xi_0,\xi_1)$ into zero. Therefore, in this case f_k may not vanish, i.e., the uniqueness f_k for these α and k is violated.

Let us now consider the case $k\in K_0$. Denote $u_k(0)=b_k$. The differential equation in (18) has a unique solution with this initial condition: $u_k(t)=b_kE_\rho(-\lambda_k t^\rho)$ (see, e.g. [33], p.174). Since $E_\rho(-\lambda_k\xi_0^\rho)=\alpha$ in the considering case, then the non-local condition is satisfied for an arbitrary b_k. But the over-determination condition $u_k(\xi_1)=0$ implies $b_k=0$ for $k\in K_0$.

Therefore, due to the completeness of the system $\{v_k\}$ in H we get $f=0$ and $u(t)\equiv 0$. Thus the uniqueness and hence Theorem 1 is completely proved.

3. Inverse Problem 2

3.1. Existence

Suppose that $W\in D(A)$ and $f\in C([0,T];D(A^\varepsilon))$ for some $\varepsilon\in(0,1)$ and let the orthogonality conditions (5) be satisfied. Let us first show that series (10) and (11) are indeed solutions to Problem 2. The fact that $u(t)\in C([0,T];H)$ and $\varphi\in H$ and have properties $D_t^\rho u(t), Au(t)\in C((0,T];H)$ was proved in our previous paper [12]. Therefore, it suffices to prove that (10) and (11) together are a formal solution to Problem 2. In turn, for this it suffices to show that the Fourier coefficients φ_k and $u_k(t)$ of functions (10) and (11) respectively, satisfy equation (12), the non-local condition (13) and the over-determination condition

$$u_k(\xi_2)=W_k. \quad (20)$$

It is easy to see that $u_k(t)$ is a solution of equation (12). Indeed, let first, $k\notin K_0$. We introduce the notation

$$u_k^1(t)=\frac{\varphi_k-\omega_k(\xi_0)}{E_\rho(-\lambda_k\xi_0^\rho)-\alpha}E_\rho(-\lambda_k t^\rho).$$

Then $u_k(t) = u_k^1(t) + \omega_k(t)$. Here $u_k^1(t)$ is the solution of the homogeneous equation (12) with the initial condition

$$u_k^1(0) = \frac{\varphi_k - \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha},$$

and $\omega_k(t)$ is the solution of equation (12) with zero initial condition (see, e.g. [33], p. 174).

If $k \in K_0$, then according to the orthogonality conditions $f_k = 0$ and the function

$$u_k(t) = \frac{E_\rho(-\lambda_k t^\rho) W_k}{E_\rho(-\lambda_k \xi_2^\rho)}$$

is a solution of the homogeneous equation (12) with the initial condition

$$u_k(0) = \frac{W_k}{E_\rho(-\lambda_k \xi_2^\rho)}.$$

Thus we have shown that $u_k(t)$ is a solution of equation (12).

Let us check the non-local condition (13). Consider first the case $k \notin K_0$. We have

$$\alpha u_k(0) + \varphi_k = \alpha \frac{\varphi_k - \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} + \varphi_k.$$

On the other hand,

$$\begin{aligned} u_k(\xi_0) &= \frac{\varphi_k - \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} E_\rho(-\lambda_k \xi_0^\rho) + \omega_k(\xi_0) \\ &= \frac{\varphi_k E_\rho(-\lambda_k \xi_0^\rho) - \alpha\, \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} = \frac{\varphi_k \big(E_\rho(-\lambda_k \xi_0^\rho) - \alpha\big) + \varphi_k \alpha - \alpha \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} \\ &= \alpha \frac{\varphi_k - \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} + \varphi_k. \end{aligned}$$

Now consider the case $k \in K_0$. Note in this case $E_\rho(-\lambda_k \xi_0^\rho) = \alpha$ and all Fourier coefficients φ_k are equal to zero since the orthogonality condition (5). Therefore,

$$\alpha u_k(0) + \varphi_k = \alpha \frac{W_k}{E_\rho(-\lambda_k \xi_2^\rho)} = E_\rho(-\lambda_k \xi_0^\rho) \frac{W_k}{E_\rho(-\lambda_k \xi_2^\rho)} = u_k(\xi_0).$$

Let us move on to checking the over-determination condition (20). Let $k \notin K_0$. Then

$$\frac{\varphi_k - \omega_k(\xi_0)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} E_\rho(-\lambda_k \xi_2^\rho) + \omega_k(\xi_2) = W_k,$$

or

$$\varphi_k = \frac{E_\rho(-\lambda_k \xi_0^\rho) - \alpha}{E_\rho(-\lambda_k \xi_2^\rho)} [W_k - \omega_k(\xi_2)] + \omega_k(\xi_0),$$

and this coincides with the Fourier coefficients of the function (10).

If $k \in K_0$, then

$$u_k(\xi_2) = \frac{E_\rho(-\lambda_k \xi_2^\rho) W_k}{E_\rho(-\lambda_k \xi_2^\rho)} = W_k.$$

Thus the existence of a solution to Problem 2 is proved.

3.2. Uniqueness

Obviously, to prove the uniqueness of the solution to Problem 2, it suffices to show that the solution $\{u(t), \varphi\}$ to the following inverse problem:

$$D_t^\rho u(t) + Au(t) = 0, \qquad t > 0;$$

$$u(\xi_0) = \alpha u(0) + \varphi, \quad 0 < \xi_0 \leqslant T,$$

$$u(\xi_2) = 0, \quad 0 < \xi_2 \leqslant T, \ \xi_2 \neq \xi_0,$$

is identically zero: $u(t) \equiv 0$ and $\varphi = 0$.

Let $\{u(t), \varphi\}$ be a solution of this problem and let $u_k(t) = (u(t), v_k)$ and $\varphi_k = (\varphi, v_k)$. Then

$$D_t^\rho u_k(t) + \lambda_k u_k(t) = 0, \quad t > 0; \quad u_k(\xi_0) = \alpha u_k(0) + \varphi_k, \ u_k(\xi_2) = 0. \tag{21}$$

Let $k \notin K_0$. Then it is not hard to verify that the following function

$$u_k(t) = \frac{E_\rho(-\lambda_k t^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} \, \varphi_k$$

is the only solution to the equation and non-local condition in (21). The over-determination condition in (21) implies

$$u_k(\xi_2) = \frac{E_\rho(-\lambda_k \xi_2^\rho)}{E_\rho(-\lambda_k \xi_0^\rho) - \alpha} \, \varphi_k = 0.$$

Since $E_\rho(-\lambda_k \xi_0^\rho) \neq \alpha$ and $E_\rho(-\lambda_k \xi_2^\rho) \neq 0$, then we have $\varphi_k = 0$ and therefore $u_k(t) \equiv 0$ for all $k \notin K_0$.

Now consider the case $k \in K_0$. Denote $u_k(0) = b_k$. The differential equation in (21) has a unique solution with this initial condition: $u_k(t) = b_k E_\rho(-\lambda_k t^\rho)$ (see, e.g. [33], p.174). Since $E_\rho(-\lambda_k \xi_0^\rho) = \alpha$ and $\varphi_k = 0$ in the considering case, then the non-local condition is satisfied for an arbitrary b_k. But the over-determination condition $u_k(\xi_2) = 0$ implies $b_k = 0$ and therefore $u_k(t) \equiv 0$ for $k \in K_0$.

Therefore, due to the completeness of the system $\{v_k\}$ in H we get $\varphi = 0$ and $u(t) \equiv 0$. Thus the uniqueness and hence Theorem 2 is completely proved.

4. Conclusion

In the previous paper of the authors [12] it is proved that for $\alpha \notin (0,1)$ the solutions of the forward and two inverse problems of determining f and φ exist and are unique. If $\alpha \in (0,1)$ and equality (4) holds for some $k \in K_0$, then to ensure the existence of the solution to the forward problem, it is necessary to require the orthogonality condition (5). However, in this case the solution is not unique and it is determined up to the term

$$\sum_{k\in K_0} b_k E_\rho(-\lambda_k t^\rho) v_k,$$

where b_k are arbitrary numbers.

In this paper, we consider the above two inverse problems for critical values of parameter $\alpha \in (0,1)$. An interesting effect arises here: when solving the forward problem the uniqueness of solution $u(t)$ was violated, while when solving the inverse problem for the same values of α, solution $u(t)$ became unique. What is the matter here? It turns out, as follows from the main results of this paper, the over-determination condition

$$u(\tau) = V$$

can be rewritten in the form of two groups of conditions with respect to the Fourier coefficients

$$u_k(\tau) = V_k,\ k \notin K_0,$$

and

$$u_k(\tau) = V_k,\ k \in K_0.$$

With the help of the first group, the unique solutions of inverse problems are singled out, and since the coefficients f_k and φ_k are equal to zero for $k \in K_0$, the conditions from the second group are not used in this case. And the conditions from the second group ensure the uniqueness of the solution $u(t)$, namely, they determine uniquely the above arbitrary coefficients b_k.

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