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IMPULSIVE NABLA FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. This article deals with the study of impulsive nabla fractional difference equations. First, we present a first-order initial value problem (IVP) on impulsive nabla difference equations and write its equivalent sum equation. To illustrate the proposed procedure, we provide an example. In this line, we propose a particular class of IVPs for α^{th}-order $(0 < \alpha \leqslant 1)$ impulsive nabla difference equations in the Caputo sense and establish its equivalent sum equation. We furnish an eigenvalue problem to demonstrate the proposed method. Next, we introduce a special class of α^{th}-order $(0 < \alpha \leqslant 1)$ impulsive nabla boundary value problems (BVPs) and analyse its solutions, using fixed point theorems. Finally, we support this analysis through a few examples.

1. Introduction

Classical theory of differential equations with impulses is a prominent area of research due its applicability in modelling a variety of the real world problems. In line with its continuous counterpart, study of classical impulsive difference equations also gained momentum and as a result many research articles were reported. For the theory and applications of these equations, we refer [6, 7, 8, 12, 15, 16, 21].

Recently, many researchers have studied impulsive differential equations of fractional order rigorously and explored many aspects of these equations. An interesting remark on these equations is that fractional impulsive differential equations cannot be considered as a trivial generalization of classical impulsive differential equations due to the non-locality of fractional derivatives. Many important results on impulsive fractional differential equations are available in the literature. For example, see [3, 10, 19, 20, 22, 23].

On the other hand, the study of impulsive fractional difference equations was initiated very recently. In [22], the authors introduced the notion of impulsive delta fractional difference equations and discussed asymptotic stability and impulsive Mittag–Leffler stability for a particular class of impulsive delta fractional difference equations. But, articles on the theory of impulsive fractional difference equations in nabla perspective is not yet reported. Motivated by this discussion, we initiate the study of impulsive nabla fractional difference equations, by following the technique proposed in [10] and [22].

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We organize this article in the following manner: In section 2, we present preliminaries of discrete fractional calculus. In sections 3 and 4, we propose particular classes of IVPs for first-order and α^{th}-order $(0<\alpha\leqslant 1)$ impulsive nabla difference equations, respectively. Section 5 contains a study on a special class of α^{th}-order $(0<\alpha\leqslant 1)$ impulsive nabla fractional BVPs. We conclude this article with a few examples in Section 6.

2. Preliminaries

We shall utilize the following preliminaries [9, 11, 13] throughout the article. Denote by $\mathbb{N}_c=\{c,c+1,c+2,\ldots\}$ and $\mathbb{N}_c^d=\{c,c+1,c+2,\ldots,d\}$, for any real numbers c and d such that $d-c\in\mathbb{N}_1$. The backward jump operator $\rho:\mathbb{N}_{c+1}\to\mathbb{N}_c$ is defined by $\rho(t)=t-1$, for all $t\in\mathbb{N}_{c+1}$. The μ^{th}-order nabla fractional Taylor monomial is defined by

$$H_\mu(t,a)=\frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)}=\frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)},\quad \mu\in\mathbb{R}\setminus\{\ldots,-2,-1\},$$

provided the right-hand side exists. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

DEFINITION 1. (See [11]) Let $u:\mathbb{N}_{a+1}\to\mathbb{R}$ and $\nu>0$. The ν^{th}-order nabla sum of u based at a is given by

$$\left(\nabla_a^{-\nu}u\right)(t)=\sum_{s=a+1}^{t}H_{\nu-1}(t,\rho(s))u(s),\quad t\in\mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-\nu}u\right)(a)=0$.

We collect a few of the properties of nabla fractional Taylor monomials in the next lemma.

LEMMA 1. *The following hold provided the expressions in this lemma are well-defined.*

1. $\displaystyle\sum_{s=a+1}^{t}H_\mu(t,\rho(s))=H_{\mu+1}(t,a)$.
2. $\nabla_a^{-\nu}H_\mu(t,a)=H_{\mu+\nu}(t,a)$.
3. $\nabla H_\mu(t,a)=H_{\mu-1}(t,a)$.

DEFINITION 2. (See [4]) Assume $u:\mathbb{N}_a\to\mathbb{R}$ and $0<\nu\leqslant 1$. The ν^{th}-order Caputo nabla difference of $u:\mathbb{N}_a\to\mathbb{R}$ is given by

$$\left(\nabla_{a*}^{\nu}u\right)(t)=\left(\nabla_a^{-(1-\nu)}(\nabla u)\right)(t),\quad t\in\mathbb{N}_{a+1}.$$

THEOREM 1. (See [11]) *Assume* $0 < \nu \leqslant 1$. *Consider the IVP*

$$\begin{cases} \left(\nabla_{a*}^{\nu} u\right)(t) = h(t), \quad t \in \mathbb{N}_{a+1}, \\ u(a) = u_0, \end{cases} \tag{1}$$

where $u_0 \in \mathbb{R}$ *and* $h : \mathbb{N}_{a+1} \to \mathbb{R}$. *The unique solution to the IVP* (1) *is given by*

$$u(t) = u_0 + \left(\nabla_a^{-\nu} h\right)(t), \quad t \in \mathbb{N}_a. \tag{2}$$

Let λ, $u_0 \in \mathbb{R}$ and $0 < \nu \leqslant 1$. We study the following IVP

$$\begin{cases} \left(\nabla_{a*}^{\nu} u\right)(t) = \lambda u(t-1), \quad t \in \mathbb{N}_{a+1}, \\ u(a) = u_0. \end{cases} \tag{3}$$

THEOREM 2. *The solution of the IVP* (3) *is uniquely determined.*

Proof. We use the definition of the ν^{th}-order Caputo nabla difference of u to obtain the following iteration schema. Consider

$$\begin{aligned} \lambda u(t-1) &= \left(\nabla_{a*}^{\nu} u\right)(t) \\ &= \left(\nabla_a^{-(1-\nu)}\left(\nabla u\right)\right)(t) \quad \text{(By Definition 2)} \\ &= \sum_{s=a+1}^{t} H_{-\nu}(t,\rho(s))\left(\nabla u\right)(s) \quad \text{(By Definition 1)} \\ &= \sum_{s=a+1}^{t} H_{-\nu}(t,\rho(s))\left[u(s) - u(s-1)\right] \\ &= u(t) + \sum_{s=a+1}^{t-1} H_{-\nu}(t,\rho(s))u(s) - \sum_{s=a+1}^{t} H_{-\nu}(t,\rho(s))u(s-1), \end{aligned}$$

implying that

$$u(t) = \lambda u(t-1) - \sum_{s=a+1}^{t-1} H_{-\nu}(t,\rho(s))u(s) + \sum_{s=a+1}^{t} H_{-\nu}(t,\rho(s))u(s-1), \quad t \in \mathbb{N}_{a+1}.$$

This iteration schema ensures that the solution of the IVP (3) is uniquely determined. □

THEOREM 3. *The unique solution to the IVP*

$$\begin{cases} \left(\nabla_{0*}^{\nu} u\right)(t) = \lambda u(t-1), \quad t \in \mathbb{N}_1, \\ u(0) = u_0, \end{cases} \tag{4}$$

is given by

$$u(t) = u_0 \sum_{n=0}^{t} \lambda^n H_{n\nu}(t,\rho(n)), \quad t \in \mathbb{N}_0. \tag{5}$$

Proof. Denote by

$$v(t) = u_0 \tilde{E}_{\lambda,\nu}(t), \quad t \in \mathbb{N}_0,$$

where

$$\tilde{E}_{\lambda,\nu}(t) = \sum_{n=0}^{t} \lambda^n H_{n\nu}(t,\rho(n)), \quad t \in \mathbb{N}_0. \tag{6}$$

We show that v satisfies the IVP (4). That is,

$$\begin{cases} \left(\nabla^{\nu}_{0*} v\right)(t) = \lambda v(t-1), & t \in \mathbb{N}_1, \\ v(0) = u_0. \end{cases}$$

Since $\tilde{E}_{\lambda,\nu}(0) = 1$, v satisfies the initial condition. Consider

$$\begin{aligned} \nabla \tilde{E}_{\lambda,\nu}(t) &= \tilde{E}_{\lambda,\nu}(t) - \tilde{E}_{\lambda,\nu}(t-1) \\ &= \sum_{n=0}^{t} \lambda^n H_{n\nu}(t,\rho(n)) - \sum_{n=0}^{t-1} \lambda^n H_{n\nu}(t-1,\rho(n)) \\ &= \lambda^t + \sum_{n=0}^{t-1} \lambda^n \left[H_{n\nu}(t,\rho(n)) - H_{n\nu}(t-1,\rho(n))\right] \\ &= \lambda^t + \sum_{n=0}^{t-1} \lambda^n \nabla H_{n\nu}(t,\rho(n)) \\ &= \lambda^t + \sum_{n=1}^{t-1} \lambda^n H_{n\nu-1}(t,\rho(n)) \quad \text{(By Lemma 1)} \\ &= \sum_{n=1}^{t} \lambda^n H_{n\nu-1}(t,\rho(n)). \end{aligned} \tag{7}$$

Now, consider

$$\begin{aligned} \nabla^{\nu}_{0*}\left[\tilde{E}_{\lambda,\nu}(t)\right] &= \nabla_0^{-(1-\nu)} \nabla \left[\tilde{E}_{\lambda,\nu}(t)\right] \quad \text{(By Definition 2)} \\ &= \nabla_0^{-(1-\nu)} \left[\sum_{n=1}^{t} \lambda^n H_{n\nu-1}(t,\rho(n))\right] \quad \text{(By (7))} \\ &= \sum_{s=1}^{t} H_{-\nu}(t,\rho(s)) \left[\sum_{n=1}^{s} \lambda^n H_{n\nu-1}(s,\rho(n))\right] \quad \text{(By Definition 1)} \\ &= \sum_{n=1}^{t} \lambda^n \left[\sum_{s=n}^{t} H_{-\nu}(t,\rho(s)) H_{n\nu-1}(s,\rho(n))\right] \\ &= \sum_{n=1}^{t} \lambda^n \left[\nabla_{\rho(n)}^{-(1-\nu)} H_{n\nu-1}(t,\rho(n))\right] \quad \text{(By Definition 1)} \\ &= \sum_{n=1}^{t} \lambda^n H_{n\nu-\nu}(t,\rho(n)) \quad \text{(By Lemma 1)} \\ &= \lambda \sum_{n=0}^{t-1} \lambda^n H_{n\nu}(t-1,\rho(n)) = \lambda \tilde{E}_{\lambda,\nu}(t-1), \end{aligned}$$

implying that

$$\nabla_{0*}^{\nu}\left[u_0\tilde{E}_{\lambda,\nu}(t)\right] = \lambda u_0\tilde{E}_{\lambda,\nu}(t-1), \quad t\in\mathbb{N}_1.$$

That is,

$$\left(\nabla_{0*}^{\nu}v\right)(t) = \lambda v(t-1), \quad t\in\mathbb{N}_1.$$

Uniqueness of this solution follows from Theorem 2. □

3. Classical impulsive nabla difference equation

In this section, we introduce the notion of first-order impulsive nabla difference equations of the form

$$\left(\nabla u\right)(t) = f(t,u(t-1)), \quad t\in\mathbb{N}_1\setminus\{t_1,t_2,t_3,\cdots\}, \tag{8}$$

$$u(t_j) = u(\bar{t_j}) + c_j, \quad j\in\mathbb{N}_1, \tag{9}$$

$$u(0) = u_0. \tag{10}$$

Here f is a continuous real-valued function defined on $\mathbb{N}_1\times\mathbb{R}$, $\{c_1,c_2,c_3,\cdots\}$ are constants, and $\{t_1,t_2,t_3,\cdots\}\subseteq\mathbb{N}_2$ are fixed impulsive points with $0=t_0<t_1<\cdots<t_{j-1}<t_j<\cdots$ such that $t_j-t_{j-1}\in\mathbb{N}_2$, and

$$u(\bar{t_j}) = u(t_j-1) + f(t_j,u(t_j-1)), \quad j\in\mathbb{N}_1. \tag{11}$$

THEOREM 4. *The equivalent sum equation of* (8)–(11) *is given by*

$$u(t) = \begin{cases} u_0 + \sum_{s=1}^{t} f(s,u(s-1)), & t\in\mathbb{N}_0^{t_1-1}, \\ u_0 + \sum_{i=1}^{j} c_i + \sum_{s=1}^{t} f(s,u(s-1)), & t\in\mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j\in\mathbb{N}_1. \end{cases} \tag{12}$$

Proof. First, we consider the IVP

$$\begin{cases} \left(\nabla u\right)(t) = f(t,u(t-1)), & t\in\mathbb{N}_1^{t_1-1}, \\ u(0) = u_0. \end{cases} \tag{13}$$

The equivalent sum equation of (13) is given by

$$u(t) = u_0 + \sum_{s=1}^{t} f(s,u(s-1)), \quad t\in\mathbb{N}_0^{t_1-1}, \tag{14}$$

which is the first expression on the right-hand side of (12). From (9) and (11), we obtain

$$u(t_1) = u(\bar{t_1}) + c_1 = u_0 + c_1 + \sum_{s=1}^{t_1} f(s,u(s-1)) = u_1, \text{ say.} \tag{15}$$

Now, consider the IVP

$$\begin{cases} (\nabla u)(t) = f(t,u(t-1)), & t \in \mathbb{N}_{t_1+1}^{t_2-1}, \\ u(t_1) = u_1. \end{cases} \tag{16}$$

The equivalent sum equation of (16) is

$$u(t) = u_1 + \sum_{s=t_1+1}^{t} f(s,u(s-1)) = u_0 + c_1 + \sum_{s=1}^{t} f(s,u(s-1)), \quad t \in \mathbb{N}_{t_1}^{t_2-1}, \tag{17}$$

which is the second expression on the right-hand side of (12) for $j = 1$. Proceeding in a similar way, we achieve (12). The proof is complete. □

REMARK 1. A careful observation of (9) and (11) reveal that the expression for $u(\bar{t_j})$ is obtained by replacing $t_j - 1$ with t_j in the expression of $u(t_j - 1)$. We have

$$\begin{aligned} u(t_j - 1) &= u_0 + \sum_{i=1}^{j-1} c_i + \sum_{s=1}^{t_j-1} f(s,u(s-1)) \\ &= u_0 + \sum_{i=1}^{j-1} c_i + \left[\nabla^{-1} f(t,u(t-1))\right]_{t=t_j-1}, \ j \in \mathbb{N}_1. \end{aligned} \tag{18}$$

Then,

$$u(\bar{t_j}) = u_0 + \sum_{i=1}^{j-1} c_i + \left[\nabla^{-1} f(t,u(t-1))\right]_{t=t_j}, \ j \in \mathbb{N}_1. \tag{19}$$

Consequently, from (9), we have

$$u(t_j) = u_0 + \sum_{i=1}^{j} c_i + \left[\nabla^{-1} f(t,u(t-1))\right]_{t=t_j}, \ j \in \mathbb{N}_1. \tag{20}$$

EXAMPLE 1. Consider (8)–(11) with

$$f(t,u(t-1)) = \lambda u(t-1), \quad t \in \mathbb{N}_1, \quad \lambda \in \mathbb{R}.$$

Then, the unique solution of (8)–(11) is

$$u(t) = \begin{cases} u_0(1+\lambda)^t, & t \in \mathbb{N}_0^{t_1-1}, \\ u_0(1+\lambda)^t + \sum_{i=1}^{j} c_i (1+\lambda)^{t-t_i}, & t \in \mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j \in \mathbb{N}_1. \end{cases} \tag{21}$$

Proof. First, we consider the IVP

$$\begin{cases} (\nabla u)(t) = \lambda u(t-1), & t \in \mathbb{N}_1^{t_1-1}, \\ u(0) = u_0. \end{cases} \tag{22}$$

The unique solution of (22) is given by

$$u(t) = u_0 (1+\lambda)^t, \quad t \in \mathbb{N}_0^{t_1-1}, \tag{23}$$

which is the first expression on the right-hand side of (21). From (9), (11) and Remark 1, we have

$$u(t_1) = u(\overline{t_1}) + c_1 = u_0 (1+\lambda)^{t_1} + c_1 = u_1, \text{ say}. \tag{24}$$

Now, consider the IVP

$$\begin{cases} (\nabla u)(t) = \lambda u(t-1), & t \in \mathbb{N}_{t_1+1}^{t_2-1}, \\ u(t_1) = u_1. \end{cases} \tag{25}$$

The unique solution of (25) is given by

$$u(t) = u_1 (1+\lambda)^{t-t_1} = u_0 (1+\lambda)^t + c_1 (1+\lambda)^{t-t_1}, \quad t \in \mathbb{N}_{t_1}^{t_2-1}, \tag{26}$$

which is the second expression on the right-hand side of (21) for $j = 1$. Proceeding in a similar way, we achieve (21). The proof is complete. □

4. Impulsive nabla fractional difference equation

Analogous to (8), (9), (19) and (10), in this section, we introduce the notion of impulsive nabla fractional difference equations of the form

$$\left(\nabla_{0*}^{\alpha} u\right)(t) = f(t, u(t-1)), \quad t \in \mathbb{N}_1 \setminus \{t_1, t_2, t_3, \cdots\}, \tag{27}$$

$$u(t_j) = u(\overline{t_j}) + c_j, \quad j \in \mathbb{N}_1, \tag{28}$$

$$u(0) = u_0, \tag{29}$$

where $0 < \alpha \leqslant 1$, $f : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$ is continuous,

$$\{c_1, c_2, c_3, \cdots\} \subseteq \mathbb{R},$$

and $\{t_1, t_2, t_3, \cdots\} \subseteq \mathbb{N}_2$ are fixed impulsive points with

$$0 = t_0 < t_1 < \cdots < t_{j-1} < t_j < \cdots$$

such that $t_j - t_{j-1} \in \mathbb{N}_2$, and

$$u(\overline{t_j}) = u_0 + \sum_{i=1}^{j-1} c_i + \left[\nabla_0^{-\alpha} f(t, u(t-1))\right]_{t=t_j}, \quad j \in \mathbb{N}_1. \tag{30}$$

THEOREM 5. *The equivalent sum equation of* (27)–(30) *is*

$$u(t) = \begin{cases} u_0 + \nabla_0^{-\alpha} f(t, u(t-1)), & t \in \mathbb{N}_0^{t_1-1}, \\ u_0 + \sum_{i=1}^{j} c_i + \nabla_0^{-\alpha} f(t, u(t-1)), & t \in \mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j \in \mathbb{N}_1. \end{cases} \tag{31}$$

Proof. First, we consider the IVP

$$\begin{cases} \left(\nabla_{0*}^{\alpha} u\right)(t) = f(t, u(t-1)), & t \in \mathbb{N}_1^{t_1-1}, \\ u(0) = u_0. \end{cases} \tag{32}$$

From Theorem 1, the equivalent sum equation of (32) is given by

$$u(t) = u_0 + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_0^{t_1-1}, \tag{33}$$

which is the first expression on the right-hand side of (31). From (28) and (30), we have

$$u(t_1) = u(\overline{t_1}) + c_1 = u_0 + c_1 + \left[\nabla_0^{-\alpha} f(t, u(t-1))\right]_{t=t_1} = u_1, \text{ say}. \tag{34}$$

Now, consider the IVP

$$\begin{cases} \left(\nabla_{0*}^{\alpha} u\right)(t) = f(t, u(t-1)), & t \in \mathbb{N}_{t_1+1}^{t_2-1}, \\ u(t_1) = u_1. \end{cases} \tag{35}$$

From Theorem 1, the equivalent sum equation of (35) is given by

$$\begin{aligned} u(t) &= u_0 + \nabla_0^{-\alpha} f(t, u(t-1)) \\ &= u_1 - \left[\nabla_0^{-\alpha} f(t, u(t-1))\right]_{t=t_1} + \nabla_0^{-\alpha} f(t, u(t-1)) \\ &= u_0 + c_1 + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_{t_1}^{t_2-1}, \end{aligned} \tag{36}$$

which is the second expression on the right-hand side of (31) for $j = 1$. Proceeding in a similar way, we achieve (31). The proof is complete. □

EXAMPLE 2. Consider (27)–(30) with

$$f(t, u(t)) = \lambda u(t-1), \quad t \in \mathbb{N}_1, \quad \lambda \in \mathbb{R}.$$

Then, the unique solution of (27)–(30) is

$$u(t) = \begin{cases} u_0 \tilde{E}_{\lambda,\alpha}(t), & t \in \mathbb{N}_0^{t_1-1}, \\ u_0 \tilde{E}_{\lambda,\alpha}(t) + \sum_{i=1}^{j} c_i \dfrac{\tilde{E}_{\lambda,\alpha}(t)}{\tilde{E}_{\lambda,\alpha}(t_i)}, & t \in \mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j \in \mathbb{N}_1. \end{cases} \tag{37}$$

Proof. First, we consider the IVP

$$\begin{cases} \left(\nabla_{0*}^{\alpha} u\right)(t) = \lambda u(t-1), & t \in \mathbb{N}_1^{t_1-1}, \\ u(0) = u_0. \end{cases} \tag{38}$$

From Theorem 3, the unique solution of (38) is given by

$$u(t) = u_0 \tilde{E}_{\lambda,\alpha}(t), \quad t \in \mathbb{N}_0^{t_1-1}, \tag{39}$$

which is the first expression on the right-hand side of (37). From (28) and (30), we have

$$u(t_1) = u(\overline{t_1}) + c_1 = u_0\tilde{E}_{\lambda,\alpha}(t_1) + c_1 = u_1, \text{ say.} \tag{40}$$

Now, consider the IVP

$$\begin{cases} (\nabla^{\alpha}_{0*}u)(t) = \lambda u(t-1), & t \in \mathbb{N}^{t_2-1}_{t_1+1}, \\ u(t_1) = u_1. \end{cases} \tag{41}$$

From Theorem 3, the unique solution of (41) is given by

$$\begin{aligned} u(t) &= u_0\tilde{E}_{\lambda,\alpha}(t) \\ &= \frac{u_1}{\tilde{E}_{\lambda,\alpha}(t_1)}\tilde{E}_{\lambda,\alpha}(t) \\ &= \frac{u_0\tilde{E}_{\lambda,\alpha}(t_1) + c_1}{\tilde{E}_{\lambda,\alpha}(t_1)}\tilde{E}_{\lambda,\alpha}(t) \\ &= u_0\tilde{E}_{\lambda,\alpha}(t) + c_1\frac{\tilde{E}_{\lambda,\alpha}(t)}{\tilde{E}_{\lambda,\alpha}(t_1)}, \quad t \in \mathbb{N}^{t_2-1}_{t_1}, \end{aligned} \tag{42}$$

which is the second expression in the right-hand side of (37) for $j = 1$. From (28) and (30), we have

$$u(t_2) = u(\overline{t_2}) + c_2 = u_0\tilde{E}_{\lambda,\alpha}(t_2) + c_1\frac{\tilde{E}_{\lambda,\alpha}(t_2)}{\tilde{E}_{\lambda,\alpha}(t_1)} + c_2 = u_2, \text{ say.} \tag{43}$$

Now, consider the IVP

$$\begin{cases} (\nabla^{\alpha}_{0*}u)(t) = \lambda u(t-1), & t \in \mathbb{N}^{t_3-1}_{t_2+1}, \\ u(t_2) = u_2. \end{cases} \tag{44}$$

From Theorem 3, the unique solution of (44) is given by

$$\begin{aligned} u(t) &= u_0\tilde{E}_{\lambda,\alpha}(t) \\ &= \frac{u_2}{\tilde{E}_{\lambda,\alpha}(t_2)}\tilde{E}_{\lambda,\alpha}(t) \\ &= \frac{u_0\tilde{E}_{\lambda,\alpha}(t_2) + c_1\frac{\tilde{E}_{\lambda,\alpha}(t_2)}{\tilde{E}_{\lambda,\alpha}(t_1)} + c_2}{\tilde{E}_{\lambda,\alpha}(t_2)}\tilde{E}_{\lambda,\alpha}(t) \\ &= u_0\tilde{E}_{\lambda,\alpha}(t) + c_1\frac{\tilde{E}_{\lambda,\alpha}(t)}{\tilde{E}_{\lambda,\alpha}(t_1)} + +c_2\frac{\tilde{E}_{\lambda,\alpha}(t)}{\tilde{E}_{\lambda,\alpha}(t_2)}, \quad t \in \mathbb{N}^{t_3-1}_{t_2}, \end{aligned} \tag{45}$$

which is the second expression in the right-hand side of (37) for $j = 2$. Proceeding in a similar way, we achieve (37). The proof is complete. □

5. Impulsive nabla fractional boundary value problem

Assume $0 < \alpha \leqslant 1$ and consider the impulsive nabla fractional BVP

$$\left(\nabla_{0*}^{\alpha} u\right)(t) = f(t, u(t-1)), \quad t \in \mathbb{N}_1^T \setminus \{t_1, t_2, t_3, \cdots, t_k\}, \tag{46}$$

$$u(t_j) = u(\bar{t_j}) + c_j, \quad j \in \mathbb{N}_1^k, \tag{47}$$

$$au(0) + bu(T) = c, \quad a + b \neq 0, \tag{48}$$

where $T \in \mathbb{N}_1$, a, b, $c \in \mathbb{R}$, $\{c_1, c_2, c_3, \cdots, c_k\} \subseteq \mathbb{R}$, $f : \mathbb{N}_1^T \times \mathbb{R} \to \mathbb{R}$ is continuous, $\{t_1, t_2, t_3, \cdots, t_k\} \subseteq \mathbb{N}_2$ are fixed impulsive points with $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T$, such that $t_{j+1} - t_j \in \mathbb{N}_2$, and

$$u(\bar{t_j}) = u_0 + \sum_{i=1}^{j-1} c_i + \left[\nabla_0^{-\alpha} f(t, u(t-1))\right]_{t=t_j}, \quad j \in \mathbb{N}_1^k. \tag{49}$$

THEOREM 6. *The equivalent sum equation of* (46)–(49) *is given by*

$$u(t) = \begin{cases} \frac{1}{a+b}\left[c - b\sum_{i=1}^{k} c_i - b\nabla_0^{-\alpha} f(T, u(T-1))\right] + \nabla_0^{-\alpha} f(t, u(t-1)), \ t \in \mathbb{N}_0^{t_1 - 1}, \\ \frac{1}{a+b}\left[c - b\sum_{i=1}^{k} c_i - b\nabla_0^{-\alpha} f(T, u(T-1))\right] \\ \qquad + \sum_{i=1}^{j} c_i + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j \in \mathbb{N}_1^{k-1}, \\ \frac{1}{a+b}\left[c - b\sum_{i=1}^{k} c_i - b\nabla_0^{-\alpha} f(T, u(T-1))\right] \\ \qquad + \sum_{i=1}^{k} c_i + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_{t_k}^{T}. \end{cases} \tag{50}$$

Proof. Consider (46), (47) and (49). From Theorem 5, we have

$$u(t) = \begin{cases} u(0) + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_0^{t_1 - 1}, \\ u(0) + \sum_{i=1}^{j} c_i + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j \in \mathbb{N}_1^{k-1}, \\ u(0) + \sum_{i=1}^{k} c_i + \nabla_0^{-\alpha} f(t, u(t-1)), \quad t \in \mathbb{N}_{t_k}^{T}. \end{cases} \tag{51}$$

We notice that there exist no impulsive points between $t_{k+1} - 1$ and t_{k+1}. So, the last expression in (51) is valid for $t \in \mathbb{N}_{t_k}^{t_{k+1}}$ instead of $t \in \mathbb{N}_{t_k}^{t_{k+1}-1}$. From the boundary condition $au(0) + bu(T) = c$, we have

$$u(0) = \frac{1}{a+b}\left[c - b\sum_{i=1}^{k} c_i - b\nabla_0^{-\alpha} f(T, u(T-1))\right]. \tag{52}$$

Substituting the expression for $u(0)$ from (52) in (51), we obtain (50). The proof is complete. □

Note that any solution $u:\mathbb{N}_0^T \to \mathbb{R}$ of (46)–(49) can be viewed as a real $(T+1)$-tuple vector. Consequently, $u \in \mathbb{R}^{T+1}$. We use the fact that $\mathbb{R}^{T+1}$ is a Banach space equipped with the maximum norm

$$\|u\| = \max_{t\in\mathbb{N}_0^T} |u(t)|,$$

for any $u \in \mathbb{R}^{T+1}$. Denote by

$$\mathscr{B}_r = \{u \in \mathbb{R}^{T+1} : \|u\| \leqslant r\}.$$

5.1. Existence and uniqueness of solutions

In this subsection, we establish sufficient conditions on existence and uniqueness of solutions of (46)–(49) using Banach fixed point theorem. First, we recall the statement of Banach fixed point theorem.

THEOREM 7. [2, 18] *(Banach fixed point theorem) Let S be a closed subset of a Banach space X. Assume $T:S\to S$ is a contraction mapping. That is, there exist a constant λ, $0<\lambda<1$, such that*

$$\|Tx - Ty\| \leqslant \lambda \|x-y\|,$$

for all x, y in S. Then, T has a unique fixed point z in S.

THEOREM 8. *Assume*

(C1) f satisfies Lipschitz condition with respect to the second variable on $\mathbb{N}_0^T\times\mathbb{R}$ with Lipschitz constant K.

If

$$KH_\alpha(T,0)\left(1+\left|\frac{b}{a+b}\right|\right) < 1, \tag{53}$$

then the boundary value problem (46)–(49) has a unique solution in $\mathbb{R}^{T+1}$.

Proof. Define $T:\mathbb{R}^{T+1}\to\mathbb{R}^{T+1}$ by

$$\big(Tu\big)(t) = \begin{cases} \dfrac{1}{a+b}\left[c - b\displaystyle\sum_{i=1}^{k} c_i - b\nabla_0^{-\alpha} f(T,u(T-1))\right] + \displaystyle\sum_{i=1}^{j} c_i \\ \qquad\qquad +\nabla_0^{-\alpha} f(t,u(t-1)), \quad t\in\mathbb{N}_{t_j}^{t_{j+1}-1}, \quad j\in\mathbb{N}_0^{k-1}, \\ \dfrac{1}{a+b}\left[c - b\displaystyle\sum_{i=1}^{k} c_i - b\nabla_0^{-\alpha} f(T,u(T-1))\right] + \displaystyle\sum_{i=1}^{k} c_i \\ \qquad\qquad +\nabla_0^{-\alpha} f(t,u(t-1)), \quad t\in\mathbb{N}_{t_k}^{T}. \end{cases} \tag{54}$$

We show that T is a contraction mapping on $\mathbb{R}^{T+1}$. For u, $v \in \mathbb{R}^{T+1}$, $t \in \mathbb{N}_{t_j}^{t_{j+1}-1}$, $j \in \mathbb{N}_0^{k-1}$, consider

$$\begin{aligned}
&|(Tu)(t) - (Tv)(t)| \\
&= \left|\nabla_0^{-\alpha} f(t,u(t-1)) - \nabla_0^{-\alpha} f(t,v(t-1))\right| \\
&\quad + \left|\frac{b}{a+b}\right| \left|\nabla_0^{-\alpha} f(T,u(T-1)) - \nabla_0^{-\alpha} f(T,v(T-1))\right| \\
&= \left|\sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) f(s,u(s-1)) - \sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) f(s,v(s-1))\right| \\
&\quad + \left|\frac{b}{a+b}\right| \left|\sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s)) f(s,u(s-1))\right. \\
&\quad \left. - \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s)) f(s,v(s-1))\right| \\
&\leqslant \sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) \left|f(s,u(s-1)) - f(s,v(s-1))\right| \\
&\quad + \left|\frac{b}{a+b}\right| \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s)) \left|f(s,u(s-1)) - f(s,v(s-1))\right| \\
&\leqslant K \sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) \left|u(s-1) - v(s-1)\right| \\
&\quad + K \left|\frac{b}{a+b}\right| \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s)) \left|u(s-1) - v(s-1)\right| \\
&\leqslant K\|u-v\| \sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) + K\left|\frac{b}{a+b}\right| \|u-v\| \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s)) \\
&= K\|u-v\| H_\alpha(t,0) + K\left|\frac{b}{a+b}\right| \|u-v\| H_\alpha(T,0) \\
&= K\left[H_\alpha(t,0) + \left|\frac{b}{a+b}\right| H_\alpha(T,0)\right] \|u-v\|.
\end{aligned}$$

Similarly, for u, $v \in \mathbb{R}^{T+1}$, $t \in \mathbb{N}_{t_k}^{T}$, we obtain

$$|(Tu)(t) - (Tv)(t)| \leqslant K\left[H_\alpha(t,0) + \left|\frac{b}{a+b}\right| H_\alpha(T,0)\right] \|u-v\|.$$

Since $H_\alpha(t,0) \leqslant H_\alpha(T,0)$, we have

$$\|Tu - Tv\| \leqslant K H_\alpha(T,0) \left(1 + \left|\frac{b}{a+b}\right|\right) \|u-v\|.$$

Thus, by (53), T is a contraction mapping on $\mathbb{R}^{T+1}$ and hence, T has a unique fixed point by Theorem 7. $\square$

THEOREM 9. *Assume*

(C2) f satisfies Lipschitz condition with respect to the second variable on $\mathbb{N}_0^T \times \mathscr{B}_r$ with Lipschitz constant L.

Set

$$m = \max\{|f(t,0)| : t \in \mathbb{N}_0^T\}. \tag{55}$$

If

$$LH_\alpha(T,0)\left(1+\left|\frac{b}{a+b}\right|\right) < 1, \tag{56}$$

and we choose

$$r \geqslant \frac{\left|\frac{c}{a+b}\right| + \left[mH_\alpha(T,0)+\sum_{i=1}^k |c_i|\right]\left(1+\left|\frac{b}{a+b}\right|\right)}{1-LH_\alpha(T,0)\left(1+\left|\frac{b}{a+b}\right|\right)}, \tag{57}$$

then the boundary value problem (46)–(49) has a unique solution in $\mathbb{B}_r$.

Proof. First, we show that $T : \mathscr{B}_r \to \mathscr{B}_r$. To see this, let $u \in \mathscr{B}_r$, $t \in \mathbb{N}_{t_j}^{t_{j+1}-1}$, $j \in \mathbb{N}_0^{k-1}$, consider

$$\begin{aligned}
|(Tu)(t)| &= \left|\frac{1}{a+b}\left[c - b\sum_{i=1}^k c_i - b\nabla_0^{-\alpha} f(T,u(T-1))\right] + \sum_{i=1}^j c_i + \nabla_0^{-\alpha} f(t,u(t-1))\right| \\
&= \left|\frac{c}{a+b} - \frac{b}{a+b}\sum_{i=1}^k c_i - \frac{b}{a+b}\sum_{s=1}^T H_{\alpha-1}(T,\rho(s))f(s,u(s-1))\right. \\
&\quad \left. + \sum_{i=1}^j c_i + \sum_{s=1}^t H_{\alpha-1}(t,\rho(s))f(s,u(s-1))\right| \\
&\leqslant \left|\frac{c}{a+b}\right| + \left|\frac{b}{a+b}\sum_{i=1}^k c_i\right| \\
&\quad + \left|\frac{b}{a+b}\right|\sum_{s=1}^T H_{\alpha-1}(T,\rho(s))\,|f(s,u(s-1)) - f(s,0)| \\
&\quad + \left|\frac{b}{a+b}\right|\sum_{s=1}^T H_{\alpha-1}(T,\rho(s))\,|f(s,0)| + \left|\sum_{i=1}^j c_i\right| \\
&\quad + \sum_{s=1}^t H_{\alpha-1}(t,\rho(s))\,|f(s,u(s-1)) - f(s,0)| + \sum_{s=1}^t H_{\alpha-1}(t,\rho(s))\,|f(s,0)|
\end{aligned}$$

$$\leqslant \left|\frac{c}{a+b}\right| + \left|\frac{b}{a+b}\right| \sum_{i=1}^{k} |c_i| + L\left|\frac{b}{a+b}\right| \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s))\,|u(s-1)|$$
$$+ m\left|\frac{b}{a+b}\right| \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s)) + L\sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s))\,|u(s-1)|$$
$$+ m\sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) + \sum_{i=1}^{j} |c_i|$$
$$\leqslant \left|\frac{c}{a+b}\right| + \left|\frac{b}{a+b}\right| \sum_{i=1}^{k} |c_i| + \left|\frac{b}{a+b}\right| [Lr+m] \sum_{s=1}^{T} H_{\alpha-1}(T,\rho(s))$$
$$+ [Lr+m] \sum_{s=1}^{t} H_{\alpha-1}(t,\rho(s)) + \sum_{i=1}^{j} |c_i|$$
$$= \left|\frac{c}{a+b}\right| + \left|\frac{b}{a+b}\right| \sum_{i=1}^{k} |c_i| + \left|\frac{b}{a+b}\right| [Lr+m]\, H_{\alpha}(T,0)$$
$$+ [Lr+m]\, H_{\alpha}(t,0) + \sum_{i=1}^{j} |c_i|.$$

Similarly, for $u \in \mathscr{B}_r$, $t \in \mathbb{N}_{t_k}^{T}$, we obtain

$$|(Tu)(t)| \leqslant \left|\frac{c}{a+b}\right| + \left|\frac{b}{a+b}\right| \sum_{i=1}^{k} |c_i| + \left|\frac{b}{a+b}\right| [Lr+m]\, H_{\alpha}(T,0)$$
$$+ [Lr+m]\, H_{\alpha}(t,0) + \sum_{i=1}^{k} |c_i|.$$

Since $H_{\alpha}(t,0) \leqslant H_{\alpha}(T,0)$ and

$$\sum_{i=1}^{j} |c_i| \leqslant \sum_{i=1}^{k} |c_i|, \quad j \in \mathbb{N}_1^k,$$

we write

$$\|Tu\| \leqslant \left|\frac{c}{a+b}\right| + \left[(Lr+m)H_{\alpha}(T,0) + \sum_{i=1}^{k} |c_i|\right] \left(1 + \left|\frac{b}{a+b}\right|\right).$$

Thus, by (56) and (57), we have

$$\|Tu\| \leqslant r,$$

implying that $T : \mathscr{B}_r \to \mathscr{B}_r$. It follows from the proof of Theorem 8 and (56) that T is a contraction mapping with contraction constant $LH_{\alpha}(T,0)\left(1 + \left|\frac{b}{a+b}\right|\right)$. Hence, T has a unique fixed point by Theorem 7. □

5.2. Existence of solutions

In this subsection, we establish sufficient conditions on existence of solutions of (46)–(49) using Brouwer fixed point theorem. First, we recall the statement of Brouwer fixed point theorem.

THEOREM 10. [2, 18] (Brouwer fixed point theorem) *Let $\mathscr{C}$ be a non-empty compact convex subset of $\mathbb{R}^n$, and $T:\mathscr{C}\to\mathscr{C}$ is a continuous mapping. Then, T has a fixed point in $\mathscr{C}$.*

THEOREM 11. *Set*

$$M=\max\{|f(t,x)|: t\in\mathbb{N}_0^T,\ x\in\mathscr{B}_r\}. \tag{58}$$

If we choose

$$r\geqslant\left|\frac{c}{a+b}\right|+\left[MH_\alpha(T,0)+\sum_{i=1}^{k}|c_i|\right]\left(1+\left|\frac{b}{a+b}\right|\right), \tag{59}$$

then the boundary value problem (46)–(49) has a solution in $\mathbb{B}_r$.

Proof. First, we show that $T:\mathbb{B}_r\to\mathbb{B}_r$. To see this, let $u\in\mathscr{B}_r$, $t\in\mathbb{N}_{t_j}^{t_{j+1}-1}$, $j\in\mathbb{N}_0^{k-1}$, consider

$$\begin{aligned}
|(Tu)(t)|&=\left|\frac{1}{a+b}\left[c-b\sum_{i=1}^{k}c_i-b\nabla_0^{-\alpha}f(T,u(T-1))\right]+\sum_{i=1}^{j}c_i+\nabla_0^{-\alpha}f(t,u(t-1))\right|\\
&=\left|\frac{c}{a+b}-\frac{b}{a+b}\sum_{i=1}^{k}c_i-\frac{b}{a+b}\sum_{s=1}^{T}H_{\alpha-1}(T,\rho(s))f(s,u(s-1))\right.\\
&\quad\left.+\sum_{i=1}^{j}c_i+\sum_{s=1}^{t}H_{\alpha-1}(t,\rho(s))f(s,u(s-1))\right|\\
&\leqslant\left|\frac{c}{a+b}\right|+\left|\frac{b}{a+b}\sum_{i=1}^{k}c_i\right|+\left|\frac{b}{a+b}\right|\sum_{s=1}^{T}H_{\alpha-1}(T,\rho(s))\,|f(s,u(s-1))|\\
&\quad+\sum_{s=1}^{t}H_{\alpha-1}(t,\rho(s))\,|f(s,u(s-1))|+\left|\sum_{i=1}^{j}c_i\right|\\
&\leqslant\left|\frac{c}{a+b}\right|+\left|\frac{b}{a+b}\right|\sum_{i=1}^{k}|c_i|+M\left|\frac{b}{a+b}\right|\sum_{s=1}^{T}H_{\alpha-1}(T,\rho(s))\\
&\quad+M\sum_{s=1}^{t}H_{\alpha-1}(t,\rho(s))+\sum_{i=1}^{j}|c_i|\\
&=\left|\frac{c}{a+b}\right|+\left|\frac{b}{a+b}\right|\sum_{i=1}^{k}|c_i|+\left|\frac{b}{a+b}\right|MH_\alpha(T,0)+MH_\alpha(t,0)+\sum_{i=1}^{j}|c_i|.
\end{aligned}$$

Similarly, for $u \in \mathscr{B}_r$, $t \in \mathbb{N}_{t_k}^T$, we obtain

$$\big|(Tu)(t)\big| \leqslant \left|\frac{c}{a+b}\right| + \left|\frac{b}{a+b}\right| \sum_{i=1}^{k} |c_i| + \left|\frac{b}{a+b}\right| MH_\alpha(T,0) \tag{60}$$

$$+MH_\alpha(t,0) + \sum_{i=1}^{k} |c_i|. \tag{61}$$

Since $H_\alpha(t,0) \leqslant H_\alpha(T,0)$ and

$$\sum_{i=1}^{j} |c_i| \leqslant \sum_{i=1}^{k} |c_i|, \quad j \in \mathbb{N}_1^k,$$

we write

$$\|Tu\| \leqslant \left|\frac{c}{a+b}\right| + \left[MH_\alpha(T,0) + \sum_{i=1}^{k} |c_i|\right]\left(1 + \left|\frac{b}{a+b}\right|\right).$$

Thus, by (59), we have

$$\|Tu\| \leqslant r,$$

implying that $T : \mathscr{B}_r \to \mathscr{B}_r$. Clearly, T is continuous. Hence, T has a fixed point by Theorem 10. □

THEOREM 12. *If f is continuous and bounded on $\mathbb{N}_0^T \times \mathbb{R}$, then the boundary value problem* (46)–(49) *has a solution in $\mathbb{R}^{T+1}$.*

6. Examples

In this section, we provide two examples to illustrate the applicability of established results.

EXAMPLE 3. Consider the boundary value problem

$$\begin{cases} \big(\nabla_{0*}^{0.5} u\big)(t) = (0.1)\cos u(t) + t^3, & t \in \mathbb{N}_1^6 \setminus \{4\}, \\ u(4) = u(\overline{4}) + 1, \\ u(0) + u(6) = 0. \end{cases} \tag{62}$$

Here $\alpha = 0.5$, $T = 6$, $f(t,u) = (0.1)\cos u + t^3$ is continuous on $\mathbb{N}_1^6 \times \mathbb{R}$, and

$$u(\overline{4}) = u(0) + \left[\nabla_0^{-\alpha} f(t, u(t))\right]_{t=4}. \tag{63}$$

Clearly,

$$\|f(t,u) - f(t,v)\| \leqslant K\|u - v\|, \quad (t,u),(t,v) \in \mathbb{N}_1^6 \setminus \{4\} \times \mathbb{R},$$

$$KH_\alpha(T,0)\left(1 + \left|\frac{b}{a+b}\right|\right) = 0.4061 < 1,$$

with $K = 0.1$, implying that assumption (C1) and (53) hold. Thus, by Theorem 8, the boundary value problem (62)–(63) has a unique solution.

EXAMPLE 4. Consider the boundary value problem

$$\begin{cases} \left(\nabla_{0*}^{0.5}u\right)(t) = (0.01)\left(u^2(t)+1\right), & t \in \mathbb{N}_1^6 \setminus \{4\}, \\ u(4) = u(\bar{4}) + 1, \\ u(0) + u(6) = 0. \end{cases} \tag{64}$$

Here $\alpha = 0.5$, $T = 6$, $f(t,u) = (0.01)\left(u^2+1\right)$ is continuous on $\mathbb{N}_1^6 \times \mathbb{R}$, and

$$u(\bar{4}) = u(0) + \left[\nabla_0^{-\alpha} f(t,u(t))\right]_{t=4}. \tag{65}$$

We have

$$m = \max\{|f(t,0)| : t \in \mathbb{N}_0^6\} = \max\{(0.01) : t \in \mathbb{N}_0^6\} = (0.01).$$

Clearly,

$$\|f(t,u) - f(t,v)\| \leqslant L\|u-v\|, \quad (t,u),(t,v) \in \mathbb{N}_1^6 \setminus \{4\} \times \mathbb{R},$$

with $L = (0.02)r$. To apply Theorem 9, we must have

$$LH_\alpha(T,0)\left(1 + \left|\frac{b}{a+b}\right|\right) = (0.08121)r < 1,$$

and

$$r \geqslant \frac{\left|\frac{c}{a+b}\right| + \left[mH_\alpha(T,0) + \sum_{i=1}^{k}|c_i|\right]\left(1 + \left|\frac{b}{a+b}\right|\right)}{1 - LH_\alpha(T,0)\left(1 + \left|\frac{b}{a+b}\right|\right)} = \frac{(1.5406)}{1-(0.08121)r}.$$

Clearly, $r = 10$ satisfies the above two inequalities. Thus, by Theorem 9, the boundary value problem (46)–(49) has a unique solution in $\mathbb{B}_{10}$.

REFERENCES

[1] T. ABDELJAWAD AND F. M. ATICI, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal. **2012**, Art. ID 406757 (2012), 13 pp.

[2] R. P. AGARWAL, M. MEEHAN AND D. O'REGAN, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics **141**, Cambridge University Press, Cambridge, 2001.

[3] B. AHMAD AND S. SIVASUNDARAM, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Anal. Hybrid Syst. **3**, 3 (2009), 251–258.

[4] G. A. ANASTASSIOU, *Nabla discrete fractional calculus and nabla inequalities*, Math. Comput. Modelling **51**, 5–6 (2010), 562–571.

[5] F. M. ATICI AND P. W. ELOE, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl. **17**, 4 (2011), 445–456.

[6] D. BAĬNOV AND P. SIMEONOV, *Impulsive Differential Equations: Periodic Solutions and Applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics **66**, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993.

[7] M. BENCHOHRA, J. HENDERSON AND S. NTOUYAS, *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and Its Applications **2**, Hindawi Publishing Corporation, New York, 2006.

[8] H. BEREKETOGLU AND A. HUSEYNOV, *Boundary value problems for nonlinear second-order difference equations with impulse*, Appl. Math. Mech. (English Ed.) **30**, 8 (2009), 1045–1054.

[9] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser Boston, Inc., Boston, MA, 2001.

[10] M. FEČKAN, Y. ZHOU AND J. WANG, *On the concept and existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul. **17**, 7 (2012), 3050–3060.

[11] C. GOODRICH AND A. C. PETERSON, *Discrete Fractional Calculus*, Springer, Cham, 2015.

[12] Z. HE AND X. ZHANG, *Monotone iterative technique for first order impulsive difference equations with periodic boundary conditions*, Appl. Math. Comput. **156**, 3 (2004), 605–620.

[13] W. G. KELLEY AND A. C. PETERSON, *Difference Equations. An Introduction with Applications*, Second edition, Harcourt/Academic Press, San Diego, CA, 2001.

[14] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies **204**, Elsevier Science B.V., Amsterdam, 2006.

[15] V. LAKSHMIKANTHAM, D. BAĬNOV AND P. SIMEONOV, *Theory of impulsive Differential Equations*, Series in Modern Applied Mathematics **6**, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[16] J. LI AND J. SHEN, *Positive solutions for first order difference equations with impulses*, Int. J. Difference Equ. **1**, 2 (2006), 225–239.

[17] I. PODLUBNY, *Fractional Differential Equations*, Mathematics in Science and Engineering **198**, Academic Press, Inc., San Diego, CA, 1999.

[18] D. R. SMART, *Fixed Point Theorems*, Cambridge Tracts in Mathematics **66**, Cambridge University Press, London-New York, 1974.

[19] J. WANG, Y. ZHOU AND M. FEČKAN, *Nonlinear impulsive problems for fractional differential equations and Ulam stability*, Comput. Math. Appl. **64**, 10 (2012), 3389–3405.

[20] J. WANG, M. FEČKAN AND Y. ZHOU, *A survey on impulsive fractional differential equations*, Fract. Calc. Appl. Anal. **19**, 4 (2016), 806–831.

[21] P. WANG AND W. WANG, *Boundary value problems for first order impulsive difference equations*, Int. J. Difference Equ. **1**, 2 (2006), 249–259.

[22] G. WU AND D. BALEANU, *Stability analysis of impulsive fractional difference equations*, Fract. Calc. Appl. Anal. **21**, 2 (2018), 354–375.

[23] G. WU, D. ZENG AND D. BALEANU, *Fractional impulsive differential equations: exact solutions, integral equations and short memory case*, Fract. Calc. Appl. Anal. **22**, 1 (2019), 180–192.

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