Fractional Differential Calculus
Volume 12, Number 2 (2022), 197–208
doi:10.7153/fdc-2022-12-12

EXISTENCE AND UNIQUENESS RESULTS FOR GENERALIZED CAPUTO ITERATIVE FRACTIONAL BOUNDARY VALUE PROBLEMS

ABDELKRIM SALIM* AND MOUFFAK BENCHOHRA

(*Communicated by N. Nymoradi*)

Abstract. In this paper, we present some results on existence and uniqueness for a class of boundary value problems for iterative fractional differential equations with generalized Caputo fractional derivative. For our proofs, we employ some suitable fixed point theorems. Finally, we provide an illustration for more clarity.

Due to its applications in the modeling and physical understanding of natural phenomena, the domain of fractional calculus is constantly expanding at the moment. Non-integer derivatives of fractional order have been effectively applied to the generalization of fundamental laws of nature particularly in the transport phenomenon. For more details, we suggest [30, 18, 15, 2, 1, 25, 26, 27, 21, 20, 14, 8, 6, 24, 3, 28], and the references therein.

In [22], some properties of Caputo-type modification of the Erdélyi-Kober fractional derivative are provided by the authors. More information are available in [5, 19, 29, 4].

Iterative differential equations are frequently used to describe a broad variety of natural phenomena, including disease transmission models in epidemiology, the two-body problem of classical electrodynamics, population, mechanical and physical models, and many more. For some studied concerning this type of equations, we refer the readers to the papers [9, 10, 11, 12].

In [10], Kaufmann studied the existence and uniqueness of solutions to the following second order iterative boundary-value problem:

$$\chi''(\tau) = \Psi\left(\tau, \chi(\tau), \chi^{[2]}(\tau)\right), \quad \kappa_1 \leqslant \tau \leqslant \kappa_2,$$

where $\chi^{[2]}(\tau) = \chi(\chi(\tau))$, with solutions satisfying one of the boundary conditions $\chi(\kappa_1) = \kappa_1$, $\chi(\kappa_2) = \kappa_2$ or $\chi(\kappa_1) = \kappa_2$, $\chi(\kappa_2) = \kappa_1$. The Schauder fixed point theorem is the primary method used to arrive at his conclusions.

Mathematics subject classification (2020): 26A33, 34K10, 34A08, 34K37.
Keywords and phrases: Iterative problems, boundary value problems, generalized Caputo fractional derivative, existence, uniqueness.

* Corresponding author.

In [7], the authors proved some existence and uniqueness of solutions for the following problem of Erdélyi-Kober functional fractional differential equations with Caputo-type modification:

$$\begin{cases} {}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta,\rho}\chi(\tau) = \Psi_1\left(\tau,\chi^{\tau},\xi^{\tau},{}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta,\rho}\chi(\tau),{}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta,\rho}\xi(\tau)\right), \\ {}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta,\rho}\xi(\tau) = \Psi_2\left(\tau,\chi^{\tau},\xi^{\tau},{}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta,\rho}\chi(\tau),{}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta,\rho}\xi(\tau)\right), \end{cases} \quad \tau \in [\upsilon,\kappa],$$

$$\begin{cases} (\chi(\tau),\xi(\tau)) = (\phi_1(\tau),\phi_2(\tau)), \tau \in [\upsilon - r,\upsilon], \quad r > 0, \\ (\chi(\tau),\xi(\tau)) = (\psi_1(\tau),\psi_2(\tau)), \tau \in [\kappa,\kappa+\beta], \quad \beta > 0, \end{cases}$$

where ${}^{C}\mathscr{D}_{\upsilon^+}^{\vartheta}$ is the Caputo type modification of the Erdelyi-Kober fractional derivative and $\Psi_i : I \times C([-r,\beta],\mathbb{R})^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a given function, $\phi_i \in C([\upsilon - r,\upsilon],\mathbb{R})$ with $\phi_i(\upsilon) = 0$ and $\psi_i \in C([\kappa,\kappa+\beta],\mathbb{R})$ with $\psi_i(\kappa) = 0$, $i = 1,2$. Their arguments are based upon the Banach contraction principle and Schauder's fixed point theorem.

Motivated by the works mentioned above, first, we establish some existence results to the boundary value problem of the following fractional iterative differential equation:

$${}^{C}\mathscr{D}_{\kappa_1^+}^{\vartheta,\rho}\chi(\tau) = \Psi\left(\tau,\chi(\tau),\chi^{[2]}(\tau),\ldots,\chi^{[m]}(\tau)\right), \quad \tau \in (\kappa_1,\kappa_2), \tag{1}$$

with the boundary condition:

$$\xi_1\chi(\kappa_1) + \xi_2\chi(\kappa_2) = \xi_3, \tag{2}$$

where $m \geqslant 2$, $\chi^{[2]}(\tau) = \chi(\chi(\tau))$ and for $j = 3,\ldots,m$, we have $\chi^{[j]}(\tau) = \chi\left(\chi^{[j-1]}(\tau)\right)$, ${}^{C}\mathscr{D}_{\kappa_1^+}^{\vartheta,\rho}$ is the generalized Caputo fractional derivative of order $0 < \vartheta \leqslant 1$, $\Psi : [\kappa_1,\kappa_2] \times \mathbb{R}^m \to \mathbb{R}$ is a given continuous function, $0 < \kappa_1 < \kappa_2 < \infty$ and $\xi_1,\xi_2,\xi_3 \in \mathbb{R}$ such that $\xi_1 + \xi_2 \neq 0$. We shall assume that $\kappa_1 < \chi(\tau) < \kappa_2$ for all $\tau \in [\kappa_1,\kappa_2]$ in order for the solutions to be well-defined.

The following is how the current paper is arranged. In Section 2, we present certain notations and review some preliminary notions concerning generalized fractional derivatives used throughout this manuscript. Section 3 presents existence and uniqueness results to our problem (1)–(2). In the last section, an illustrative example is provided in support of the obtained results.

1. Preliminaries

This part introduces the preliminary information that will be utilized throughout the study. Let $0 < \kappa_1 < \kappa_2 < \infty$ and $\Xi = [\kappa_1,\kappa_2]$.

By $C(\Xi,\mathbb{R})$ we denote the Banach space of all continuous functions from Ξ into $\mathbb{R}$ with the norm

$$\|\chi\| = \sup\{|\chi(\tau)| : \tau \in \Xi\}.$$

Let $C^n(\Xi,\mathbb{R})$ be the spaces of n-times absolutely continuous and n-times continuously differentiable functions on Ξ, respectively.

For $\kappa_1 < \eta_1 \leqslant \kappa_2$ and $0 < \eta_2$, we define the set

$$\Omega(\eta_1,\eta_2) = \{\chi \in C(\Xi,\mathbb{R}) : \kappa_1 \leqslant \chi(\tau) \leqslant \eta_1 \\ \text{and } |\chi(\tau_2) - \chi(\tau_1)| \leqslant \eta_2|\tau_2 - \tau_1|, \text{ for all } \tau_1,\tau_2 \in \Xi\}.$$

We note that $\Omega(\eta_1,\eta_2)$ is a bounded closed convex subset of $C(\Xi,\mathbb{R})$.

Consider the space $X_b^p(\kappa_1,\kappa_2)$, $(b \in \mathbb{R},\ 1 \leqslant p \leqslant \infty)$ of those complex-valued Lebesgue measurable functions f on $[\kappa_1,\kappa_2]$ for which $\|f\|_{X_b^p} < \infty$, where the norm is defined by:

$$\|f\|_{X_b^p} = \left(\int_{\kappa_1}^{\kappa_2} |\tau^b f(\tau)|^p \frac{d\tau}{\tau}\right)^{\frac{1}{p}}, \quad (1 \leqslant p < \infty, b \in \mathbb{R}).$$

DEFINITION 1. (Generalized Riemann-Liouville integral [17, 19]) Let $\vartheta \in \mathbb{R}$, $b \in \mathbb{R}$ and $\Phi \in X_b^p(\kappa_1,\kappa_2)$, the generalized RL fractional integral of order ϑ is defined by:

$$({}^\rho \mathscr{I}_{a^+}^{\vartheta}\Phi)(\tau) = \frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)} \int_a^\tau s^{\rho-1}\left(\tau^\rho - s^\rho\right)^{\vartheta-1}\Phi(s)ds, \qquad \tau > a, \rho > 0. \tag{3}$$

DEFINITION 2. ([16]) Let $0 \leqslant a < \tau$. The generalized fractional derivative, corresponding to the fractional integral (3), is defined by:

$$\begin{aligned} {}^\rho D_{a^+}^{\vartheta}\Phi(\tau) &= \frac{\rho^{1-n+\vartheta}}{\Gamma(n-\vartheta)}\left(\tau^{1-\rho}\frac{d}{d\tau}\right)^n \int_a^\tau \frac{s^{\rho-1}}{(\tau^\rho - s^\rho)^{1-n+\vartheta}}\Phi(s)ds \\ &= \delta_\rho^n({}^\rho \mathscr{I}_{a^+}^{n-\vartheta}\Phi)(\tau), \end{aligned} \tag{4}$$

where $\delta_\rho^n = \left(\tau^{1-\rho}\frac{d}{d\tau}\right)^n$.

DEFINITION 3. ([16, 22]) The Caputo-type generalized fractional derivative ${}^C\mathscr{D}_{a^+}^{\vartheta,\rho}$ is defined by

$$\left({}^C\mathscr{D}_{a^+}^{\vartheta,\rho}\Phi\right)(\tau) = \left({}^\rho D_{a^+}^{\vartheta}\left[\Phi(\tau) - \sum_{k=0}^{n-1}\frac{\Phi^{(k)}(a)}{k!}(s-a)^k\right]\right). \tag{5}$$

LEMMA 1. ([16]) *Let* $\vartheta,\rho \in \mathbb{R}^+$, *then*

$$({}^\rho \mathscr{I}_{a^+}^{\vartheta}\, {}^C\mathscr{D}_{a^+}^{\vartheta,\rho}\Phi)(\tau) = \Phi(\tau) - \sum_{k=0}^{n-1}\varepsilon_k\left(\frac{\tau^\rho - a^\rho}{\rho}\right)^k,$$

for some $\varepsilon_k \in \mathbb{R}$, $n = [\vartheta] + 1$.

LEMMA 2. ([16]) *Let* $\vartheta,\rho \in \mathbb{R}^+$, *then*

$${}^C\mathscr{D}_{a^+}^{\vartheta,\rho}\,({}^\rho \mathscr{I}_{a^+}^{\vartheta}\Phi)(\tau) = \Phi(\tau).$$

LEMMA 3. ([23]) *If* $\chi > n$*, then we have*

$$\left[{}^{\rho}\mathscr{J}_{a^+}^{\vartheta}\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\vartheta_2-1}\right](\chi)=\frac{\Gamma(\vartheta_2)}{\Gamma(\vartheta_2+\vartheta)}\left(\frac{\chi^{\rho}-a^{\rho}}{\rho}\right)^{\vartheta+\vartheta_2-1}. \tag{6}$$

In the sequel, we will provide some lemmas that are essential for our existence results.

LEMMA 4. ([9, 10]) *If* $\varpi_1,\varpi_2\in\Omega(\eta_1,\eta_2)$*, then*

$$\left|\varpi_1{}^{[m]}(\tau_1)-\varpi_1{}^{[m]}(\tau_2)\right|\leqslant\eta_2{}^m|\tau_1-\tau_2|;\;\; m=0,1,\dots$$

for all $\tau_1,\tau_2\in\Xi$ *and*

$$\left\|\varpi_1{}^{[m]}-\varpi_2{}^{[m]}\right\|\leqslant\sum_{i=0}^{m-1}\eta_2{}^i\|\varpi_1-\varpi_2\|;\;\; m=1,2,\dots$$

2. Existence results

Le us start this section by defining what we mean by a solution of the problem (1)–(2).

DEFINITION 4. A function $\chi\in C([\kappa_1,\kappa_2],\mathbb{R})$ is a solution of problem (1)–(2) if χ verifies the differential equation (1), the boundary condition (2) and $\kappa_1<\chi(\tau)<\kappa_2$ for all $\tau\in[\kappa_1,\kappa_2]$.

LEMMA 5. *The function* $\chi\in C([\kappa_1,\kappa_2],\mathbb{R})$ *is a solution of the problem* (1)–(2) *if and only if* χ *verifies the following integral equation:*

$$\begin{aligned}\chi(\tau)=&\frac{\xi_3}{\xi_1+\xi_2}\\&-\frac{\xi_2\rho^{1-\vartheta}}{(\xi_1+\xi_2)\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2}s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds\\&+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau}s^{\rho-1}(\tau^{\rho}-s^{\rho})^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds,\end{aligned} \tag{7}$$

and $\kappa_1<\chi(\tau)<\kappa_2$*, for all* $\tau\in[\kappa_1,\kappa_2]$.

Proof. Let $\chi\in C([\kappa_1,\kappa_2],\mathbb{R})$ be a solution of the problem (1)–(2). By applying the integral operator ${}^{\rho}\mathscr{J}_{\kappa_1+}^{\vartheta}(\cdot)$ on both sides of equation (1), by Lemma 1 we obtain

$$\chi(\tau)=\varepsilon_0+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau}s^{\rho-1}(\tau^{\rho}-s^{\rho})^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds. \tag{8}$$

Now, we apply the boundary condition (2) to get

$$\xi_3=(\xi_1+\xi_2)\varepsilon_0+\xi_2\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2}s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds.$$

Thus

$$\begin{aligned}\varepsilon_0=&\frac{\xi_3}{\xi_1+\xi_2}\\&-\frac{\xi_2\rho^{1-\vartheta}}{(\xi_1+\xi_2)\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2}s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds.\end{aligned}$$

Substituting the value of ε_0 in equation (8), we obtain

$$\begin{aligned}\chi(\tau)=&\frac{\xi_3}{\xi_1+\xi_2}\\&-\frac{\xi_2\rho^{1-\vartheta}}{(\xi_1+\xi_2)\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2}s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds\\&+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau}s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\dots,\chi^{[m]}(s)\right)ds.\end{aligned}$$

Thus, if χ verifies the problem (1)–(2) then it verifies the integral equation (7).

Reciprocally, it is easy to demonstrate by applying the operator ${}^{C}\mathscr{D}_{\kappa_1^+}^{\vartheta,\rho}(\cdot)$ on both sides of equation (7) and using Lemma 2 that if the function χ satisfies (7) then it verifies equation (1) as well as the boundary condition (2). $\square$

In the sequel, we present some necessary hypotheses. Suppose that the function $\Psi:[\kappa_1,\kappa_2]\times\mathbb{R}^m\to\mathbb{R}$ is continuous and verifies the requirements:

(Ax_1) There exist positive constants $\beta_1,\dots,\beta_m$ such that:

$$|\Psi(\tau,\wp_1,\dots,\wp_m)-\Psi(\tau,\bar{\wp}_1,\dots,\bar{\wp}_m)|\leqslant\sum_{i=1}^{m}\beta_i|\wp_i-\bar{\wp}_i|,$$

for any $\wp_i,\bar{\wp}_i\in\mathbb{R}$ and $\tau\in\Xi$.

(Ax_2) There exist positive constants $\overline{\beta},\underline{\beta}$ such that:

$$-\underline{\beta}\leqslant\Psi(\tau,\wp_1,\dots,\wp_m)\leqslant\overline{\beta},$$

for any $\tau,\wp_i\in\Xi$.

Set

$$\Psi^*=\sup_{\tau\in\Xi}|\Psi(\tau,0,0,\dots,0)|$$

and

$$\Lambda=\Psi^*+\eta_1\sum_{i=1}^{m}\beta_i\sum_{j=0}^{i-1}\eta_2{}^{j}.$$

Our next existence result is based on Schauder's fixed point theorem [13].

THEOREM 1. *Assume that* (Ax_1) *and* (Ax_2) *are verified. If*

$$\max\{\ell_1,\ell_2,\ell_3\} \leqslant 0, \tag{9}$$

where

$$\ell_1 = \left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\Lambda\rho^{-\vartheta}\left(\kappa_2^\rho-\kappa_1^\rho\right)^\vartheta}{\Gamma(\vartheta+1)}+\frac{|\xi_3|}{|\xi_1+\xi_2|}-\eta_1,$$

$$\ell_2 = \kappa_1+\frac{\xi_2\rho^{-\vartheta}\overline{\beta}\left(\kappa_2{}^\rho-\kappa_1{}^\rho\right)^\vartheta}{(\xi_1+\xi_2)\Gamma(\vartheta+1)}-\frac{\xi_3}{\xi_1+\xi_2}+\frac{\rho^{-\vartheta}\underline{\beta}\left(\kappa_2{}^\rho-\kappa_1{}^\rho\right)^\vartheta}{\Gamma(\vartheta+1)},$$

$$\ell_3 = \frac{\Lambda\rho^{1-\vartheta}\kappa^*\left(\kappa_2^\rho-\kappa_1^\rho\right)^{\vartheta-1}}{\Gamma(\vartheta)}-\eta_2, \text{ and } \kappa^*=\max\left\{\kappa_1^{\rho-1},\kappa_2^{\rho-1}\right\},$$

then problem (1)–(2) *has at least one solution* $\chi\in\Omega(\eta_1,\eta_2)$.

Proof. First, let us define the operator $\mathscr{H}:\Omega(\eta_1,\eta_2)\to C([\kappa_1,\kappa_2],\mathbb{R})$ by:

$$\begin{aligned}(\mathscr{H}\chi)(\tau)&\\ =-&\frac{\xi_2\rho^{1-\vartheta}}{(\xi_1+\xi_2)\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2}s^{\rho-1}\left(\kappa_2{}^\rho-s^\rho\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)ds\\ &+\frac{\xi_3}{\xi_1+\xi_2}+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau}s^{\rho-1}\left(\tau^\rho-s^\rho\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)ds,\end{aligned} \tag{10}$$

where $\kappa_1\leqslant(\mathscr{H}\chi)(\tau)\leqslant\kappa_2$ for all $\tau\in[\kappa_1,\kappa_2]$. By Lemma 5, we can say that the fixed points of the operator are solutions of the problem (1)–(2).

It is clear that $\Omega(\eta_1,\eta_2)$ is a uniformly bounded, closed and equicontinuous subset of $C([\kappa_1,\kappa_2],\mathbb{R})$. Consequently, $\Omega(\eta_1,\eta_2)$ is a compact subset. Thus, we need only to demonstrate that $\mathscr{H}$ is continuous and well defined. We will establish the proof in two steps.

Step 1. $\mathscr{H}$ is continuous.

Let $\{\chi_n\}$ be a sequence such that $\chi_n\longrightarrow\chi$ in $C([\kappa_1,\kappa_2],\mathbb{R})$. For $\tau\in\Xi$, we have

$$\begin{aligned}&|(\mathscr{H}\chi_n)(\tau)-(\mathscr{H}\chi)(\tau)|\\ &\leqslant\frac{|\xi_2|\rho^{1-\vartheta}}{|\xi_1+\xi_2|\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2}s^{\rho-1}\left(\kappa_2^\rho-s^\rho\right)^{\vartheta-1}\left|\Psi\left(s,\chi_n(s),\chi_n^{[2]}(s),\ldots,\chi_n^{[m]}(s)\right)\right.\\ &\quad\left.-\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds\\ &\quad+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau}s^{\rho-1}\left(\tau^\rho-s^\rho\right)^{\vartheta-1}\left|\Psi\left(s,\chi_n(s),\chi_n^{[2]}(s),\ldots,\chi_n^{[m]}(s)\right)\right.\\ &\quad\left.-\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds.\end{aligned}$$

By hypothesis (Ax_1), we obtain

$$\begin{aligned}
&|(\mathscr{H}\chi_n)(\tau)-(\mathscr{H}\chi)(\tau)| \\
&\leqslant \frac{|\xi_2|\rho^{1-\vartheta}}{|\xi_1+\xi_2|\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2} s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\sum_{i=1}^{m}\beta_i\left\|\chi_n^{[i]}-\chi^{[i]}\right\|ds \\
&\quad+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{\vartheta-1}\sum_{i=1}^{m}\beta_i\left\|\chi_n^{[i]}-\chi^{[i]}\right\|ds \\
&\leqslant\left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\rho^{-\vartheta}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\sum_{i=1}^{m}\beta_i\left\|\chi_n^{[i]}-\chi^{[i]}\right\|.
\end{aligned}$$

By Lemma 4, we get

$$\|\mathscr{H}\chi_n-\mathscr{H}\chi\|\leqslant\left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\rho^{-\vartheta}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\sum_{i=1}^{m}\beta_i\sum_{j=0}^{i-1}\eta_2{}^{j}\,\|\chi_n-\chi\|.$$

Since $\chi_n\longrightarrow\chi$, then

$$\|\mathscr{H}\chi_n-\mathscr{H}\chi\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Consequently, $\mathscr{H}$ is continuous.

Step 2. $\mathscr{H}(\Omega(\eta_1,\eta_2))\subset\Omega(\eta_1,\eta_2)$.
For $\tau\in\Xi$, we get

$$\begin{aligned}
&|\mathscr{H}\chi(\tau)| \\
&\leqslant \frac{|\xi_2|\rho^{1-\vartheta}}{|\xi_1+\xi_2|\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2} s^{\rho-1}\left({\kappa_2}^{\rho}-s^{\rho}\right)^{\vartheta-1}\left|\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds \\
&\quad+\frac{|\xi_3|}{|\xi_1+\xi_2|}+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{\vartheta-1}\left|\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds.
\end{aligned}\tag{11}$$

By the hypothesis (Ax_1) and Lemma 4, for $\tau\in\Xi$, we have

$$\begin{aligned}
&|\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)| \\
&\leqslant\left|\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)-\Psi(s,0,0,\ldots,0)\right|+|\Psi(s,0,0,\ldots,0)| \\
&\leqslant\Psi^{*}+\sum_{i=1}^{m}\beta_i\sum_{j=0}^{i-1}\eta_2{}^{j}\,\|\chi\| \\
&\leqslant\Lambda.
\end{aligned}$$

Thus for $\tau\in\Xi$, from (11) we obtain

$$\begin{aligned}
|\mathscr{H}\chi(\tau)|&\leqslant\left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\Lambda\rho^{-\vartheta}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{|\xi_3|}{|\xi_1+\xi_2|} \\
&\leqslant\eta_1.
\end{aligned}\tag{12}$$

For $\tau \in \Xi$ and by hypothesis (Ax_2), we have

$$\begin{aligned}
&(\mathscr{H}\chi)(\tau)\\
&\quad = -\frac{\xi_2\rho^{1-\vartheta}}{(\xi_1+\xi_2)\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2} s^{\rho-1}\left(\kappa_2{}^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)ds\\
&\qquad + \frac{\xi_3}{\xi_1+\xi_2}+\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)ds\\
&\quad \geqslant -\frac{\xi_2\rho^{-\vartheta}\overline{\beta}\left(\kappa_2{}^{\rho}-\kappa_1{}^{\rho}\right)^{\vartheta}}{(\xi_1+\xi_2)\Gamma(\vartheta+1)}+\frac{\xi_3}{\xi_1+\xi_2}-\frac{\rho^{-\vartheta}\underline{\beta}\left(\kappa_2{}^{\rho}-\kappa_1{}^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\\
&\quad \geqslant \kappa_1. \qquad (13)
\end{aligned}$$

By (12) and (13) we have

$$\kappa_1 \leqslant (\mathscr{H}\chi)(\tau) \leqslant |(\mathscr{H}\chi)(\tau)| \leqslant \eta_1.$$

Let $\tau_1,\tau_2 \in \Xi$ such that $\tau_1 < \tau_2$, then

$$\begin{aligned}
&|(\mathscr{H}\chi)(\tau_2)-(\mathscr{H}\chi)(\tau_1)|\\
&\leqslant \left|\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau_2} s^{\rho-1}\left(\tau_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right.\\
&\quad \left. -\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau_1} s^{\rho-1}\left(\tau_1^{\rho}-s^{\rho}\right)^{\vartheta-1}\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds\\
&\leqslant \frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau_1} s^{\rho-1}\left|\left(\tau_2^{\rho}-s^{\rho}\right)^{\vartheta-1}-\left(\tau_1^{\rho}-s^{\rho}\right)^{\vartheta-1}\right|\left|\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds\\
&\quad +\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\tau_1}^{\tau_2} s^{\rho-1}\left(\tau_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\left|\Psi\left(s,\chi(s),\chi^{[2]}(s),\ldots,\chi^{[m]}(s)\right)\right|ds.
\end{aligned}$$

By condition (Ax_1), we obtain

$$\begin{aligned}
|(\mathscr{H}\chi)(\tau_2)-(\mathscr{H}\chi)(\tau_1)| &\leqslant \frac{\Lambda\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau_1} s^{\rho-1}\left|\left(\tau_2^{\rho}-s^{\rho}\right)^{\vartheta-1}-\left(\tau_1^{\rho}-s^{\rho}\right)^{\vartheta-1}\right|ds\\
&\quad +\frac{\Lambda\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\tau_1}^{\tau_2} s^{\rho-1}\left(\tau_2^{\rho}-s^{\rho}\right)^{\vartheta-1}ds\\
&\leqslant \frac{\Lambda\rho^{-\vartheta}}{\Gamma(\vartheta+1)}\left[\left(\tau_2^{\rho}-\kappa_1{}^{\rho}\right)^{\vartheta}-\left(\tau_1^{\rho}-\kappa_1{}^{\rho}\right)^{\vartheta}\right]\\
&\leqslant \frac{\Lambda\rho^{1-\vartheta}\kappa^{*}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta-1}}{\Gamma(\vartheta)}|\tau_2-\tau_1|.
\end{aligned}$$

Thus,

$$|(\mathscr{H}\chi)(\tau_2)-(\mathscr{H}\chi)(\tau_1)| \leqslant \eta_2|\tau_2-\tau_1|.$$

Then, $\mathscr{H}$ is well defined and $\mathscr{H}(\Omega(\eta_1,\eta_2)) \subset \Omega(\eta_1,\eta_2)$. By Arzela-Ascoli theorem, we can conclude that $\mathscr{H}$ is continuous and completely continuous. From Schauder's theorem [13], we conclude that $\mathscr{H}$ has a fixed point which is a solution of the problem (1)–(2). □

THEOREM 2. *Suppose that all the requirements of Theorem 1 are verified. If*

$$\ell_4 := \left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\rho^{-\vartheta}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\sum_{i=1}^{m}\beta_i\sum_{j=0}^{i-1}\eta_2{}^{j}<1, \tag{14}$$

then problem (1)–(2) *has a unique solution in* $\Omega(\eta_1,\eta_2)$.

Proof. Following Theorem 1, we can say that the problem (1)–(2) has at least one solution. Now, for the uniqueness result it is only necessary to demonstrate that the operator $\mathscr{H}$ defined in (10) is a contraction in $\Omega(\eta_1,\eta_2)$. Let $\chi_1,\chi_2 \in \Omega(\eta_1,\eta_2)$. For $\tau \in \Xi$, we have

$$\begin{aligned}
&|(\mathscr{H}\chi_1)(\tau)-(\mathscr{H}\chi_2)(\tau)| \\
&\leqslant \frac{|\xi_2|\rho^{1-\vartheta}}{|\xi_1+\xi_2|\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2} s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\left|\Psi\left(s,\chi_1(s),\chi_1^{[2]}(s),\ldots,\chi_1^{[m]}(s)\right)\right. \\
&\quad \left.-\Psi\left(s,\chi_2(s),\chi_2^{[2]}(s),\ldots,\chi_2{}^{[m]}(s)\right)\right|ds \\
&\quad +\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{\vartheta-1}\left|\Psi\left(s,\chi_1(s),\chi_1^{[2]}(s),\ldots,\chi_1^{[m]}(s)\right)\right. \\
&\quad \left.-\Psi\left(s,\chi_2(s),\chi_2^{[2]}(s),\ldots,\chi_2{}^{[m]}(s)\right)\right|ds.
\end{aligned}$$

By hypothesis (Ax_1), we obtain

$$\begin{aligned}
&|(\mathscr{H}\chi_1)(\tau)-(\mathscr{H}\chi_2)(\tau)| \\
&\leqslant \frac{|\xi_2|\rho^{1-\vartheta}}{|\xi_1+\xi_2|\Gamma(\vartheta)}\int_{\kappa_1}^{\kappa_2} s^{\rho-1}\left(\kappa_2^{\rho}-s^{\rho}\right)^{\vartheta-1}\sum_{i=1}^{m}\beta_i\left\|\chi_1^{[i]}-\chi_2{}^{[i]}\right\|ds \\
&\quad +\frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)}\int_{\kappa_1}^{\tau} s^{\rho-1}\left(\tau^{\rho}-s^{\rho}\right)^{\vartheta-1}\sum_{i=1}^{m}\beta_i\left\|\chi_1^{[i]}-\chi_2{}^{[i]}\right\|ds \\
&\leqslant \left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\rho^{-\vartheta}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\sum_{i=1}^{m}\beta_i\left\|\chi_1^{[i]}-\chi_2{}^{[i]}\right\|.
\end{aligned}$$

By Lemma 4, we get

$$\begin{aligned}
\|\mathscr{H}\chi_n-\mathscr{H}\chi\| &\leqslant \left[1+\frac{|\xi_2|}{|\xi_1+\xi_2|}\right]\frac{\rho^{-\vartheta}\left(\kappa_2^{\rho}-\kappa_1^{\rho}\right)^{\vartheta}}{\Gamma(\vartheta+1)}\sum_{i=1}^{m}\beta_i\sum_{j=0}^{i-1}\eta_2{}^{j}\|\chi_1-\chi_2\| \\
&\leqslant \ell_4\|\chi_1-\chi_2\|.
\end{aligned}$$

Hence, by the Banach contraction principle [13], $\mathscr{H}$ has a unique fixed point which is a unique solution of the problem (1)–(2). □

3. An example

Consider the following problem which is an example of problem (1)–(2):

$$ {}^{C}\mathscr{D}_{0^+}^{\frac{1}{2},\rho}\chi(\tau)=\Psi\left(\tau,\chi(\tau),\chi^{[2]}(\tau),\chi^{[3]}(\tau)\right),\quad \tau\in(0,e), \tag{15} $$

$$ \chi(0)+\chi(e)=1, \tag{16} $$

where $m=3$, $\vartheta=\frac{1}{2}$, $\kappa_1=0$, $\kappa_2=e$, $\eta_1=\eta_2=2$ and $\xi_1=\xi_2=\xi_3=1$. Set

$$ \Psi\left(\tau,\chi(\tau),\chi^{[2]}(\tau),\chi^{[3]}(\tau)\right) = \frac{\cos(\tau)+1+\cos(\chi(\tau))+\cos^2\left(\chi^{[2]}(\tau)\right)+\cos^3\left(\chi^{[3]}(\tau)\right)}{251e^{\tau+11}\left(1+|\chi(\tau)|+\left|\chi^{[2]}(\tau)\right|+\left|\chi^{[3]}(\tau)\right|\right)}. $$

For each $\wp_1,\wp_2,\wp_3,\bar{\wp}_1,\bar{\wp}_2,\bar{\wp}_3\in\mathbb{R}$; and $\tau\in\Xi$, we have

$$ |\Psi(\tau,\wp_1,\wp_2,\wp_3)-\Psi(\tau,\bar{\wp}_1,\bar{\wp}_2,\bar{\wp}_3)|\leqslant\sum_{i=1}^{3}\frac{5}{251e^{11}}|\wp_i-\bar{\wp}_i|. $$

Therefore, (Ax_1) is verified with

$$ \beta_1=\beta_2=\beta_3=\frac{5}{251e^{11}}. $$

Also, hypothesis (Ax_2) is satisfied with $\overline{\beta}=\frac{5}{251e^{11}}$ and $\underline{\beta}=\frac{3}{251e^{11}}$. Since

$$ \ell_1=\frac{690\sqrt{e}}{502e^{11}\sqrt{\pi}}-\frac{3}{2}\approx-1.49997864604234, $$

$$ \ell_2=\frac{11\sqrt{e}}{251e^{11}\sqrt{\pi}}-\frac{1}{2}\approx-0.499999319149176, $$

$$ \ell_3=\frac{115}{251e^{11}\sqrt{e\pi}}-2\approx-1.99999738143933, $$

then all the conditions of Theorem 1 are satisfied, therefore, the problem (15)–(16) has at least one solution in $\Omega(2,2)$. Moreover, we have

$$ \ell_4=\frac{330\sqrt{e}}{502e^{11}\sqrt{\pi}}\approx 1.02127623586372\cdot 10^{-5}<1. $$

Thus by Theorem 2, we can deduce that the solution is unique.

Conclusion

Under certain conditions, we demonstrated the existence and uniqueness of a solution for an iterative fractional differential equation with generalized Caputo fractional derivative. The fixed point theorem of Schauder is the key instrument used in this study. We believe that these results will have an influence on the related literature and we anticipate that more generalized studies will be done using other methods and with different conditions.

Declarations

Conflict of interest. The authors state that they have no conflicting interests.

List of abbreviations. Not Applicable.

Availability of data and materials. Not Applicable.

Funding. Not Applicable.

Authors' contributions. Each of the authors contributed equally to each part of this work.

Acknowledgement. Not Applicable.

REFERENCES

[1] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.

[2] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2014.

[3] M. I. ABBAS, M. GHADERI, SH. REZAPOUR, S. T. M. THABET, *On a Coupled System of Fractional Differential Equations via the Generalized Proportional Fractional Derivatives*, Journal of Function Spaces **2022** (2022), 1–10, https://doi.org/10.1155/2022/4779213.

[4] R. ALMEIDA, A. B. MALINOWSKA AND T. ODZIJEWICZ, *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, J. Comput. Nonlinear Dynam. **11** (6): 11 pages, 2016.

[5] B. AL-SAQABI, V. S. KIRYAKOVA, *Explicit solutions of fractional integral and differential equations involving Erdélyi-Kober operators*, Appl. Math. Comput. **95** (1998), 1–13.

[6] M. BENCHOHRA, F. BOUAZZAOUI, E. KARAPINAR AND A. SALIM, *Controllability of second order functional random differential equations with delay*, Mathematics. **10** (2022), 16 pp, https://doi.org/10.3390/math10071120.

[7] M. BOUMAAZA, M. BENCHOHRA AND JUAN J. NIETO, *Caputo type modification of the Erdélyi-Kober coupled implicit fractional differential systems with retardation and anticipation*, Differ. Equ. Appl. **2** (2021) 101–114.

[8] C. DERBAZI, H. HAMMOUCHE, A. SALIM AND M. BENCHOHRA, *Measure of noncompactness and fractional Hybrid differential equations with Hybrid conditions*, Differ. Equ. Appl. **14** (2022), 145–161, http://dx.doi.org/10.7153/dea-2022-14-09.

[9] E. R. KAUFMANN, *A fourth-order iterative boundary value problem with Lidstone boundary conditions*, Differ. Equ. Appl. **14** (2022) 305–312.

[10] E. R. KAUFMANN, *Existence and uniqueness of solutions for a second-order iterative boundary-value problem*, Electron. J. Differential Equations. **2018** (2018) 1–6.

[11] A. GUERFI AND A. ARDJOUNI, *Existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative fractional differential equation*, CUBO, A Mathematical Journal. **24** (2022) 83–94.

[12] A. GUERFI AND A. ARDJOUNI, *Existence of nonnegative solutions for a hybrid nonlinear differential equation with iterative terms*, Acta Math. Univ. Comenianae. **2** (2022) 141–148.

[13] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[14] A. HERIS, A. SALIM, M. BENCHOHRA AND E. KARAPINAR, *Fractional partial random differential equations with infinite delay*, Results in Physics. (2022), https://doi.org/10.1016/j.rinp.2022.105557.

[15] R. HILFER, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[16] U. N. KATUGAMPOLA, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (4) (2014), 1–15.

[17] U. N. KATUGAMPOLA, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (3) (2011), 860–865.

[18] A. A. KILBAS, HARI M. SRIVASTAVA, AND JUAN J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[19] V. KIRYAKOVA, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Math. 301, Longman, Harlow – J. Wiley, New York, 1994.

[20] N. LALEDJ, A. SALIM, J. E. LAZREG, S. ABBAS, B. AHMAD AND M. BENCHOHRA, *On implicit fractional q-difference equations: Analysis and stability*, Math Meth Appl Sci. **45** (2022), 1–23, https://doi.org/10.1002/mma.8417.

[21] J. E. LAZREG, M. BENCHOHRA AND A. SALIM, *Existence and Ulam stability of k-Generalized ψ-Hilfer Fractional Problem*, J. Innov. Appl. Math. Comput. Sci. **2** (2022), 01–13.

[22] Y. LUCHKO AND J. J. TRUJILLO, *Caputo-type modification of the Erdélyi-Kober fractional derivative*, Fract. Calc. Appl. Anal. **10** (3) (2007), 249–267.

[23] D. S. OLIVEIRA, E. CAPELAS DE OLIVEIRA, *Hilfer-Katugampola Fractional derivative*, Comput. Appl. Math. **37** (2018), 3672–3690.

[24] SH. REZAPOUR, S. T. M. THABET, M. M. MATAR, J. ALZABUT, S. ETEMAD, *Some Existence and Stability Criteria to a Generalized FBVP Having Fractional Composite p-Laplacian Operator*, Journal of Function Spaces. **2021** (2021), 1–10, https://doi.org/10.1155/2021/9554076.

[25] A. SALIM, S. ABBAS, M. BENCHOHRA AND E. KARAPINAR, *Global stability results for Volterra-Hadamard random partial fractional integral equations*, Rend. Circ. Mat. Palermo (2) (2022), 1–13, https://doi.org/10.1007/s12215-022-00770-7.

[26] A. SALIM, M. BENCHOHRA, J. R. GRAEF AND J. E. LAZREG, *Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations*, J. Fixed Point Theory Appl. **24** (2022), 14 pp, https://doi.org/10.1007/s11784-021-00920-x.

[27] A. SALIM, J. E. LAZREG, B. AHMAD, M. BENCHOHRA AND J. J. NIETO, *A Study on k-Generalized ψ-Hilfer Derivative Operator*, Vietnam J. Math. (2022), https://doi.org/10.1007/s10013-022-00561-8.

[28] S. T. M. THABET, M. B. DHAKNE, *On abstract fractional integro-differential equations via measure of noncompactness*, Adv. Fixed Point Theory **6** (2016), 175–193.

[29] S. B. YAKUBOVICH, Y. F. LUCHKO, *The Hypergeometric Approach to Integral Transforms and Convolutions*, Mathematics and its Appl. 287, Kluwer Acad. Publ., Dordrecht-Boston-London, 1994.

[30] Y. ZHOU, *Basic Theory of Fractional Differential Equations*, World scientific, 2014.

(Received August 20, 2022)

Abdelkrim Salim
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89 Sidi Bel Abbes 22000, Algeria
Faculty of Technology
Hassiba Benbouali University of Chlef
P.O. Box 151 Chlef 02000, Algeria,
e-mail: salim.abdelkrim@yahoo.com
a.salim@univ-chlef.dz

Mouffak Benchohra
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89 Sidi Bel Abbes 22000, Algeria
e-mail: benchohra@yahoo.com