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A COMPARATIVE STUDY ON SOME SEMI–ANALYTICAL METHODS FOR THE SOLUTIONS OF FRACTIONAL PARTIAL INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract. This work focuses on the semi-analytical methods for obtaining the solutions of time fractional partial integro-differential equations. Adomian decomposition method (ADM) and homotopy perturbation method (HPM) are successfully applied. Further, the modified version of homotopy perturbation method is applied which is comparatively more accurate than the other two methods. These methods are shown to be efficient and converge rapidly to the exact solution. Graphs are plotted and tabular data are recorded which represents the accuracy of the proposed techniques.

1. Model problem

This article studies the following time fractional partial integro-differential equation (PIDE):

$$\begin{cases} \mathrm{D}_t^{\varsigma} u(y,t) = \mathrm{g}(y,t) + \displaystyle\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u(y,\eta))d\eta, \\ (y,t) \in \Omega := ([c,d] \times (0,T)), \\ u(y,0) = \rho_0(y),\ \forall y \in [c,d], \end{cases} \tag{1}$$

where the fractional order derivative is taken into account in Caputo sense and $0 < \varsigma \leqslant 1$. Here $\mathscr{K}$ stands for the kernel function while $\mathrm{g}(y,t)$ is a smooth function defined $\forall (y,t)$. $\mathrm{F}(u(y,\eta))$ is a nonlinear function continuous on the domain Ω and $\rho_0(y)$ is the initial condition.

The substantial boost in the study of fractional differential equations (FDEs) is due to its widespread application in pure and applied sciences, such as in fluid dynamics [14, 20], chemical sciences and medicine [2, 19]. But, it is usually intractable to compute the analytical solutions of FDEs because of their complicated nature [6, 18, 21]. Therefore, the study on various numerical and approximated methods to find the solutions of differential equations involving fractional derivatives and integral operators

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have become extremely crucial over the years. Several approximation and numerical approaches including the variational iteration method [14], finite difference method [12, 13, 17], differential transform method [8] and Adams Bashforth Moulton method [4] are developed to find the solutions of FDEs. Gradually, considerable efforts are also made towards the study of FDEs involving partial derivatives. Contributions are made in the past for obtaining the solutions of fractional PIDEs involving weakly singular kernel using the pseudo operational matrices [6], predictor-corrector scheme [4] and also the semi-analytical approaches. But, less efforts have been made for obtaining the solutions of (1) using the mesh less methods which are very efficient and also consumes less computational time. Hemeda [10] solved the fractional differential equations using the HPM and also established the existence and convergence properties. The structural stability and uniqueness of the solutions of nonlinear FDEs were established by Diethelm and Ford [5]. Elbeleze *et al.,* [7] applied HPM to generate the solutions of fractional order PDEs (FPDEs). They obtained a series solution and proved the reliability of their method by computing the maximum absolute truncated error of the solution. Further, distributed-order FPDEs were solved by using generalized fractional order Taylor wavelet method proposed by Yuttanan *et al.,* [22]. Various integral equations, system of linear equations and FDEs have been solved by modified HPM. One may refer [9, 15] for further details. This work deals with finding the solutions of such model problems using ADM, HPM and modified HPM with greater accuracy and less computational cost. The proposed methods provide rapid convergence of the solution. Moreover, the modified HPM gives a one term series solution of the governing model equation which makes it more easy to implement.

The structure of this article is as follows: Section 2 presents some of the important definitions of fractional derivatives and integrals. Section 3 discusses all the three proposed techniques. Numerical experiments are performed in Section 4 to support the theoretical results . Finally concluding remarks are given in Section 5.

2. Notations and preliminaries

This section comprises of some of the basic definitions and properties on the fractional order derivatives and integrals.

DEFINITION 1. [18] Consider a real valued function $u(y,t)$ defined on $\Omega \subset \mathbb{R}^2$. u is said to be in space $\mathbf{C}_\nu$, $\nu \in \mathbb{R}$, if $\exists$ a real number $q > \nu$, such that $u(y,t) = t^q u_1(y,t)$ where $u_1(y,t) \in \mathbf{C}(\Omega)$, and it is called to be in space $\mathbf{C}_\nu^{\mathrm{m}}$, $\mathrm{m} \in \mathbb{N}$ iff $\dfrac{\partial^{\mathrm{m}} u}{\partial t^{\mathrm{m}}} \in \mathbf{C}_\nu$, $\mathrm{m} \in \mathbb{N} \cup \{0\}$.

DEFINITION 2. [6] The Riemann-Liouville (R-L) time fractional partial integral denoted as $\mathfrak{I}_t^\varsigma u(y,t)$ of a function $u \in \mathbf{C}_\nu$, $\nu \geqslant -1$ is defined as:

$$\mathfrak{I}_t^\varsigma u(y,t) = \frac{1}{\Gamma(\varsigma)} \int_0^t (t-\eta)^{\varsigma-1} u(y,\eta)\, d\eta, \ \ t > 0,$$

where $0 < \varsigma \leqslant 1$ is the fractional order.

DEFINITION 3. [16] Let the order of derivative $0 < \varsigma \leqslant 1$ then for $t > 0$,

$$\mathscr{D}_t^{\varsigma} u(y,t) = \frac{\partial}{\partial t} \int_0^t \frac{(t-\eta)^{-\varsigma}}{\Gamma(1-\varsigma)} u(y,\eta) d\eta.$$

is called the R-L time fractional partial derivative for $u(y,t)$.

DEFINITION 4. [16] Let $0 < \varsigma \leqslant 1$ and $t > 0$, then

$$\begin{cases} \mathrm{D}_t^{\varsigma} u(y,t) = \dfrac{1}{\Gamma(1-\varsigma)} \displaystyle\int_0^t \frac{\partial u(y,\eta)}{\partial \eta} (t-\eta)^{-\varsigma} d\eta, \\ \mathrm{D}_t^{\varsigma} u(y,t) = \dfrac{\partial^{\mathrm{m}} u(y,t)}{\partial t^{\mathrm{m}}}, \quad \varsigma = \mathrm{m} \in \mathbb{N}. \end{cases}$$

is called Caputo time fractional partial derivative for $u(y,t)$.

3. Proposed techniques ADM, HPM, and MHPM

This section provides a detailed discussion on all the proposed semi-analytical methods and its application on the considered model problem.

3.1. Adomian decomposition method

ADM is a semi-analytical approach developed by G. Adomian [1] in 1984, This method is successfully applied in obtaining the solutions of linear and nonlinear ODEs and PDEs, integral equations, FDEs, FIDEs, and several others. The main advantage of this method lies in the fact that the method is free from discretizations and perturbations. The approach is less cumbersome and has a faster rate of convergence. ADM deals in finding the solution of (1) using an infinite series of the form

$$u(y,t) = \sum_{\mathrm{n}=0}^{\infty} u_{\mathrm{n}}(y,t). \tag{2}$$

Now, we solve the model problem (1) using the proposed technique. Applying the R-L integral $\mathfrak{I}_t^{\varsigma}$ on both sides of (1), we get

$$u(y,t) = u(y,0) + \mathfrak{I}_t^{\varsigma} \left(\mathrm{g}(y,t) + \int_0^t \mathscr{K}(y,t-\eta) \mathrm{F}(u(y,\eta)) d\eta \right).$$

It is possible to deconstruct the nonlinear function F as follows: $\mathrm{F} = \sum_{\mathrm{n}=0}^{\infty} \mathrm{A}_{\mathrm{n}}$, where A_n are the Adomian polynomials [16] given by:

$$\mathrm{A}_n = \frac{1}{\mathrm{n}!} \left[\frac{d^{\mathrm{n}}}{d\upsilon^{\mathrm{n}}} \mathrm{F}\left(\sum_{k=0}^{\mathrm{n}} \upsilon^k u_k \right) \right]_{\upsilon=0}. \tag{3}$$

The first few terms of A_n are listed below. These polynomials are the most significant ones and are useful in the approximation of nonlinear functions in most of the numerical algorithms.

$$\begin{aligned}
\mathrm{A}_0 &= \mathrm{F}_1(u_0),\\
\mathrm{A}_1 &= u_1\mathrm{F}_1'(u_0),\\
\mathrm{A}_2 &= u_2\mathrm{F}_1'(u_0)+\frac{1}{2!}u_1^2\mathrm{F}_1''(u_0),\\
\mathrm{A}_3 &= u_3\mathrm{F}_1'(u_0)+u_1u_2\mathrm{F}_1''(u_0)+\frac{1}{3!}u_1^3\mathrm{F}_1'''(u_0).
\end{aligned}$$

The following iterations are used to acquire the unknown solutions $u_0, u_1, u_2, \dots u_{\mathrm{n}}$:

$$\begin{cases} u_0 = u(y,0)\dfrac{t^0}{1} + \mathfrak{I}_t^{\varsigma}\mathrm{g}(y,t), \\ u_{\mathrm{n}+1} = \mathfrak{I}_t^{\varsigma}\left(\displaystyle\int_0^t \mathscr{K}(y,t-\eta)\mathrm{A}_{\mathrm{n}}d\eta\right) \quad \mathrm{n}=0,1,2,\dots \end{cases} \tag{4}$$

The solution is provided by $u(y,t) = \lim\limits_{\mathrm{n}\to\infty}\sum\limits_{i=0}^{\mathrm{n}} u_i(y,t)$. By truncating (2) upto first N terms (for finite N) the desired numerical solution can be obtained. The numerical approximation in N terms is defined as: $\overline{\Theta}_{\mathrm{N}} = \sum\limits_{\mathrm{n}=0}^{\mathrm{N}-1} u_{\mathrm{n}}(y,t)$. The initial condition and the function $\mathrm{g}(y,t)$, as stated above, define the zeroth component u_0 in this approach. The remaining components $u_1, u_2, \dots u_{\mathrm{n}}$ are generated recursively by using (4).

3.2. Homotopy perturbation method

HPM is a perturbation based approach used to generate the solution in the form of series for several mathematical model problems involving derivative and integral operator in both the linear and nonlinear cases. Homotopy $\mathfrak{V} : \tilde{\Omega}\times[0,1]\to\mathbb{R}$ for (1) is constructed as:

$$\begin{aligned}
\overline{h}(\mathfrak{V},\widetilde{p}) &= (1-\widetilde{p})(\mathrm{D}_t^{\varsigma}u(y,t)-\mathrm{g}(y,t))\\
&\quad +\widetilde{p}\left(\mathrm{D}_t^{\varsigma}u(y,t)-\mathrm{g}(y,t)-\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u(y,\eta))d\eta\right),
\end{aligned}$$

which implies

$$\overline{h}(\mathfrak{V},\widetilde{p}) = \mathrm{D}_t^{\varsigma}u(y,t)-\mathrm{g}(y,t)-\widetilde{p}\left(\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u(y,\eta))d\eta\right), \tag{5}$$

where $\widetilde{p}\in[0,1]$ is the embedding parameter. When $\widetilde{p}=0$, (5) becomes a linear equation, and when $\widetilde{p}=1$, then (5) reduces to (1). In addition the solution of (1) is represented as a series of the form:

$$u(y,t) = \sum_{\mathrm{n}=0}^{\infty} u_{\mathrm{n}}(y,t)\widetilde{p}^{\mathrm{n}}. \tag{6}$$

The approximated solution of (1) is found by taking $\widetilde{p}=1$ in (6)

$$u(y,t)=\sum_{\mathrm{n}=0}^{\infty}u_{\mathrm{n}}(y,t). \tag{7}$$

The convergence of series (7) has previously been demonstrated in [16]. The nonlinear function is operated using the Adomian polynomials (3). By inserting (7) into (5), the equivalent power of $\widetilde{p}$ is equalized, having obtained the following set of linear equations:

$$\begin{aligned}
\widetilde{p}^{0} &: \mathrm{D}_t^{\varsigma}u_0=\mathrm{g}(y,t),\\
\widetilde{p}^{1} &: \mathrm{D}_t^{\varsigma}u_1=\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u_0(y,\eta))d\eta,\\
\widetilde{p}^{2} &: \mathrm{D}_t^{\varsigma}u_2=\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u_1(y,\eta))d\eta,\\
&\vdots
\end{aligned} \tag{8}$$

By applying $\mathfrak{I}_t^{\varsigma}$ to the IVPs in (8), the approximated solution can be computed. Further, suitably selecting the initial conditions that are crucial in constructing the solution, the recurrence relations become simple to solve. Finally, the following series truncated upto N terms is used to approximate the series solution:

$$\widehat{\Theta}_N=\sum_{n=0}^{\mathrm{N}-1}u_{\mathrm{n}}(y,t).$$

The RHS of (8) results in Adomian polynomials. In addition, because ADM anticipates a series solution for (1) given by $u(y,t)=\lim_{\mathrm{n}\to\infty}\sum_{i=0}^{\mathrm{n}}u_i(y,t)$, using the Taylor series expansion in HPM reduces the complexity of the method and aligns it with ADM.

3.3. Modified homotopy perturbation technique

This subsection comprises of the modified version of HPM which is used to solve the linear and nonlinear FPIDEs in an efficient way. Consider (1) such that the construction of homotopy is done by following the usual way as in [11].

$$\begin{aligned}
\overline{h}(\mathfrak{V},\widetilde{p})=(1-\widetilde{p})&\left(\mathrm{D}_t^{\varsigma}u(y,t;\widetilde{p})-\mathrm{g}(y,t)\right)\\
&+\widetilde{p}\left(\mathrm{D}_t^{\varsigma}u(y,t;\widetilde{p})-\mathrm{g}(y,t)-\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u(y,\eta))d\eta\right),
\end{aligned}$$

where $\widetilde{p}\in[0,1]$ and $\mathfrak{V}(y,t,0)=u_0(y,t)$, $\mathfrak{V}(y,t,1)=u(y,t)$. In the view of modified HPM, $\mathrm{g}(y,t)$ is decomposed as $\mathrm{g}_0(y,t)+\mathrm{g}_1(y,t)$, so, the following homotopy is constructed as:

$$\mathrm{D}_t^{\varsigma}u-\mathrm{g}_0(y,t)=\widetilde{p}\left[\int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u(y,\eta))d\eta+\mathrm{g}_1(y,t)\right]. \tag{9}$$

The solution of FPIDE (1) can be presented as a power series in $\widetilde{p}$, given as,

$$u(y,t) = u_0(y,t) + \widetilde{p}u_1(y,t) + \widetilde{p}^2 u_2(y,t) + \dots. \tag{10}$$

Plugging the power series (10) into (9) yields the following iterations:

$$\begin{aligned}
&\widetilde{p}^0 : \mathrm{D}_t^{\varsigma} u_0(y,t) = \mathrm{g}_0(y,t), \ \ u_0(y,0) = \mathrm{g}(y,0),\\
&\widetilde{p}^1 : \mathrm{D}_t^{\varsigma} u_1(y,t) = g_1(y,t) + \int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u_0(y,\eta))d\eta, \ \ u_1(y,0) = 0,\\
&\widetilde{p}^2 : \mathrm{D}_t^{\varsigma} u_2(y,t) = \int_0^t \mathscr{K}(y,t-\eta)\mathrm{F}(u_1(y,\eta))d\eta, \ \ u_2(y,0) = 0.\\
&\vdots
\end{aligned} \tag{11}$$

Applying the R-L integral $\mathfrak{I}_t^{\varsigma}$ on both sides of (11), values of $u_0(y,t), u_1(y,t), u_2(y,t), \dots$ are obtained. The proper selection of g_0 and g_1 leads to the successful attainment of the exact solution in only one iteration which means that the modified HPM is a powerful method compared to ADM and HPM [16]. Finally the N^{th} term truncated series is obtained as:

$$\widetilde{\Theta}_{\mathrm{N}}(y,t) = \sum_{\mathrm{n}=0}^{\mathrm{N}-1} u_{\mathrm{n}}(y,t).$$

3.3.1. Uniqueness and convergence

ASSUMPTION 1. *A nonlinear function $F(u)$ satisfies the Lipschitz constant L such that* $||\mathrm{F}(u_1) - \mathrm{F}(u_2)|| \leqslant L||u_1 - u_2||$.

ASSUMPTION 2. *The kernel $\mathscr{K}$ defined in* (1) *is continuous and bounded on* (Ω). *Then there exists a* $\mathrm{M} > 0$ *such that* $||\mathscr{K}|| \leqslant \mathrm{M}$.

THEOREM 1. *Considering that assumptions* (1) *and* (2) *hold true. The considered model problem* (1) *has a unique* $u(y,t)$ *if* $L < \dfrac{\Gamma(\varsigma+1)}{\mathrm{M}T^{\varsigma}}$.

Proof. The detailed proof is available in [18]. □

THEOREM 2. *Under assumptions* (1) *and* (2) *and* $||u_1|| < \infty$, *the series solution obtained*

$$u(y,t) = \sum_{i=0}^{\infty} u_i(y,t),$$

converges to the exact solution.

Proof. The proof is done in [16]. One may refer to see the details. □

4. Numerical results

This section compiles some test examples to validate the theoretical estimates and give a more accurate idea of the implementation of proposed methods.

EXAMPLE 1. Consider the test case:

$$\begin{cases} \mathrm{D}_t^{\varsigma} u(y,t) + \int_0^t \mathscr{K}(y,t-\eta)u(y,\eta)d\eta = \mathrm{g}(y,t), & (y,t) \in [0,1] \times (0,T) \\ u(y,0) = 0. \end{cases}$$

The exact solution to Example 1 is $u(y,t) = yt^{\varsigma}$ with $g(y,t) = y\Gamma(\varsigma+1) + ye^{y}\dfrac{t^{\varsigma+1}}{\varsigma+1}$ and $\mathscr{K}(y,t) = e^{y}$.

According to HPM and ADM in [16], the two terms solution is given as:

$$\begin{aligned} u(y,t) = yt^{\varsigma} - \frac{ye^{y}\Gamma(\varsigma+2)t^{2\varsigma+1}}{(1+\varsigma)\Gamma(2\varsigma+2)} + \frac{e^{y}\Gamma(\varsigma+2)t^{2\varsigma+1}}{(\varsigma+1)\Gamma(2\varsigma+2)} \\ - \frac{e^{y}\Gamma(\varsigma+2)\Gamma(2\varsigma+3)t^{\varsigma+4}}{(1+\varsigma)(2\varsigma+2)\Gamma(2\varsigma+2)\Gamma(3\varsigma+3)} \end{aligned}$$

According to MHPM, the homotopy can be constructed as:

$$\mathrm{D}_t^{\varsigma} u(y,t) - y\Gamma(\varsigma+1) = \widetilde{p}\left[ye^{y}\frac{t^{\varsigma}+1}{1+\varsigma} - \int_0^t \mathscr{K}(y,t-\eta)u(y,\eta)d\eta\right].$$

Here $g_0 = -y\Gamma(\varsigma+1)$ and $g_1 = ye^{y}\dfrac{t^{\varsigma}+1}{1+\varsigma}$. Substituting (11) and equating the identical powers of $\widetilde{p}$, we obtain the following set of equations:

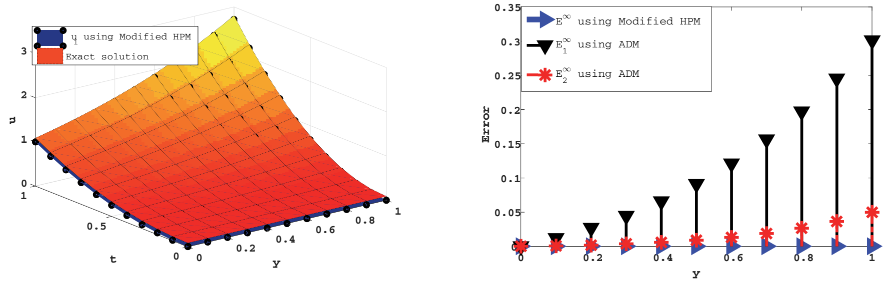
$$\begin{aligned} &\widetilde{p}^{0} : \mathrm{D}_t^{\varsigma} u_0 = ye^{y}\frac{t^{\varsigma}+1}{1+\varsigma}, \;\; u_0(y,0) = 0, \\ &\widetilde{p}^{1} : \mathrm{D}_t^{\varsigma} u_1 = ye^{y}\frac{t^{\varsigma}+1}{1+\varsigma} + \int_0^t \mathscr{K}(y,t-\eta)u_0(y,\eta)d\eta, \;\; u_1(y,0) = 0, \\ &\widetilde{p}^{2} : \mathrm{D}_t^{\varsigma} u_2 = \int_0^t \mathscr{K}(y,t-\eta)u_1(y,\eta)d\eta, \;\; u_2(y,0) = 0. \\ &\vdots \end{aligned}$$

Computing the first few terms of the modified homotopy perturbation solution for the above system gives $u_0(y,t) = yt^{\varsigma}$ and $u_k(y,t) = 0$ for $k \geqslant 1$. Thus, the exact solution follows immediately which is more faster in comparison of using the ADM and HPM.

EXAMPLE 2. Consider the test case

$$\begin{cases} \mathrm{D}_t^{\varsigma} u(y,t) - \int_0^t \eta e^{y}u^2(y,\eta)d\eta = \dfrac{2t^{2-\varsigma}e^{y}}{\Gamma(3-\varsigma)} - \dfrac{t^6e^{3y}}{6}, & (y,t) \in [0,1] \times (0,T)] \\ u(y,0) = 0. \end{cases}$$

The exact solution to Example 2 is $u(y,t) = t^2e^{y}$.



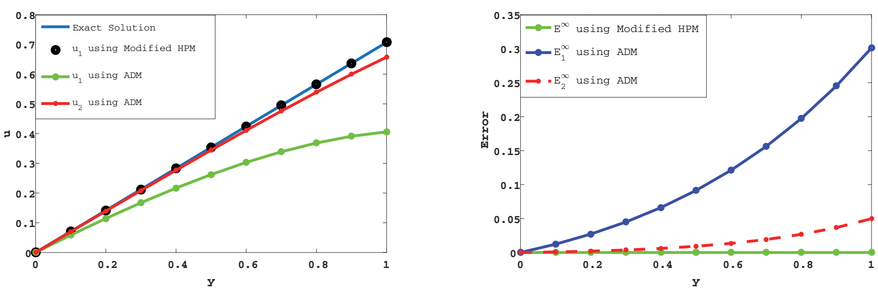
(a) $\varsigma = 0.25$. (b) $\varsigma = 0.8, \quad t \in (0,1]$.

Figure 1: *Solution plot and error plot for Example 1*

The absolute point-wise error is defined by:

$$\mathscr{E}_N^\infty = \left| u(y,t) - \sum_{\mathrm{n}=0}^{\mathrm{N}-1} u_{\mathrm{n}}(y,t) \right|.$$

Figure 1(a) depicts the 3D solution plot using modified HPM and the exact solution with one term approximation for Example 1. Figure 2(a) shows the solution plot at $t = 0.8$ and $y \in [0,1]$ for Example 2. The error plots for Example 1 and 2 are shown using Figure 1(b) and 2(b) respectively for different values of ς. The tabular data are recorded to represent the accuracy of the methods. Since, we have shown that ADM and HPM are equivalent techniques, so, Table 1 shows the solution obtained using ADM and modified HPM for Example 1. Similarly, errors obtained by three term approximation using ADM and HPM are shown using Table 2 for Example 2 where the error decreases as the number of terms in the series increases and Table 3 represents the solution obtained with the application of modified HPM and HPM for Example 2. It is evident from the tables that modified HPM converges very rapidly to the exact solution in comparison to ADM and HPM. Also it is observed that the methods are efficient and bear less computational cost making them more easier to implement.



(a) $\varsigma = 0.25$. (b) $\varsigma = 0.75, \quad t \in (0,1]$.

Figure 2: *Solution plot and error plot for Example 2.*

Table 1: *Solution obtained using ADM and modified HPM at* $\varsigma = 0.99$ *for Example* 1.

t	Exact solution	ADM I-term solution	ADM II-term solution	Modified HPM I-term solution
0.2	0.0406	0.0403	0.0406	0.0406
0.4	0.0807	0.0780	0.0807	0.0404
0.6	0.1206	0.1115	0.1204	0.1206
0.8	0.1604	0.1390	0.1595	0.1604
1.0	0.2000	0.1584	0.1974	0.2000
1.2	0.2396	0.1680	0.2331	0.2396
1.4	0.2791	0.1658	0.2652	0.2791
1.6	0.3185	0.1498	0.2916	0.3185
1.8	0.3579	0.1183	0.3095	0.3579
2.0	0.3972	0.0692	0.3156	0.3972

Table 2: $\mathscr{E}_N^\infty$ *for Example* 2.

	ADM with $\varsigma = 0.75$			HPM with $\varsigma = 0.5$		
y	$\mathscr{E}_1^\infty$	$\mathscr{E}_2^\infty$	$\mathscr{E}_3^\infty$	$\mathscr{E}_1^\infty$	$\mathscr{E}_2^\infty$	$\mathscr{E}_3^\infty$
0.1	4.7695e-3	3.1554e-5	1.5870e-5	8.5207e-3	1.1816e-4	5.9837e-5
0.2	6.4381e-3	5.1996e-5	2.6185e-5	1.1502e-2	1.9461e-4	9.8828e-5
0.3	8.6905e-3	8.5672e-5	4.3213e-5	1.5526e-2	3.2044e-4	1.6329e-4
0.4	1.1731e-2	1.4114e-4	7.1328e-5	2.0957e-2	5.2751e-4	2.6990e-4
0.5	1.5835e-2	2.3248e-4	1.1776e-4	2.8290e-2	8.6806e-4	4.4637e-4
0.6	2.1375e-2	3.8284e-4	1.9448e-4	3.8187e-2	1.4279e-3	7.3868e-4
0.7	2.8854e-2	6.3029e-4	3.2130e-4	5.1547e-2	2.3475e-3	1.2233e-3
0.8	3.8948e-2	1.0374e-3	5.3103e-4	6.9581e-2	3.8569e-3	2.0276e-3
0.9	5.2575e-2	1.7066e-3	8.7811e-4	9.3925e-2	6.3319e-3	3.3640e-3
1.0	7.0968e-2	2.8064e-3	1.4529e-3	1.2679e-1	1.0385e-2	5.5874e-3

Table 3: *Solution obtained using HPM and MHPM at* $t = 0.9$ *and* $\varsigma = 0.75$ *for Example* 2.

y	Exact solution	HPM I-term solution	HPM II-term solution	HPM III-term solution	MHPM I-term solution
0	0.9604	0.9262	9.5946	0.9599	0.9604
0.1	1.0614	1.0152	1.0600	1.0608	1.0614
0.2	1.1730	1.1107	1.1710	1.1722	1.1730
0.3	1.2964	1.2123	1.2933	1.2954	1.2964
0.4	1.4327	1.3192	1.4282	1.4316	1.4327
0.5	1.5834	1.4301	1.5767	1.5822	1.5834
0.6	1.7500	1.5430	1.7400	1.7490	1.7500
0.7	1.9340	1.6547	1.9195	1.9338	1.9340
0.8	2.1374	1.7604	2.1164	2.1391	2.1374
0.9	2.3622	1.8533	2.3321	2.3678	2.3622
1.0	2.6106	1.9237	2.5681	2.6235	2.6106

5. Concluding remarks

This work aims in obtaining the approximated solutions of fractional PIDEs using the semi-analytical methods such as ADM, HPM and modified HPM. Comparison among all the three techniques is represented with the tabular data and plots for different values of fractional order derivative ς. It can be clearly observed in this experiment that modified HPM gives better accuracy than ADM and HPM. Moreover, one can conclude that, modified HPM is more efficient and the computational cost is also very less making it easier to implement for a wide class of problems, which marks the novelty of the work.

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