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HERMITE–HADAMARD WEIGHTED INTEGRAL INEQUALITIES FOR (h,m)–CONVEX MODIFIED FUNCTIONS

BAHTIYAR BAYRAKTAR* AND JUAN E. NÁPOLES VALDÉS

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Abstract. In this paper, some new integral inequalities of the Hermite–Hadamard type are were obtained for (h,m)-convex modified functions. The results are obtained on the basis of the introduced definition of a generalized weighted integral operator by using the convexity property, the well-known Hölder's inequality and its modification. Some results existing in the literature are some special cases of our results.

1. Introduction

A function $\psi:[\rho,\sigma]\to\mathbb{R}$, is said to be convex if $\psi(\theta x_1+(1-\theta)x_2)\leqslant\theta\psi(x_1)+(1-\theta)\psi(x_2)$ holds $\forall x_1,x_2\in[\rho,\sigma]$ and $\theta\in[0,1]$. If the above inequality is reversed, then the function ψ will be concave on $[\rho,\sigma]$.

Convex functions have been summed up widely hypothetically ; these extensions incorporate the m-convex, r-concave, h-convex, s-convex, (h,m)-convex functions and numerous others. Readers interested in its multiple ramifications and extensions can consult [26], where a fairly complete overview of the development of the convex function concept was presented.

DEFINITION 1. [32] Let $\psi:[0,\sigma]\to\mathbb{R}$ and $m\in[0,1]$, If $\forall x_1,x_2\in[0,\sigma]$ and $\theta\in[0,1]$ the inequality

$$\psi(\theta x_1+(1-\theta)x_2)\leqslant\theta\psi(x_1)+m(1-\theta)\psi(x_2) \tag{1}$$

holds. Then ψ is said to be m-convex on $[0,\sigma]$.

If the above inequality is holds in reverse, then we say that the function ψ is m-concave.

The following definitions are the successive extensions of the concept of convex function and, as we will see later, they are the particular cases of our Definition.

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* Corresponding author.

DEFINITION 2. [9, 15] Let $\psi:[0,\sigma]\to\mathbb{R}$ and $s\in(0,1]$. If, $\forall x_1,x_2\in[0,\sigma]$ and $\theta\in[0,1]$ the inequality $\psi(\theta x_1+(1-\theta)x_2)\leqslant\theta^s\psi(x_1)+(1-\theta^s)\psi(x_2)$ is valid, then on $[0,\sigma]$ the function ψ is called the s-convex(in the first sense).

DEFINITION 3. [9, 15] Let $\psi:[0,\sigma]\to\mathbb{R}$ and $s\in(0,1]$. If $\forall x_1,x_2\in[0,\sigma]$ and $\theta\in[0,1]$ the inequality $\psi(\theta x_1+(1-\theta)x_2)\leqslant\theta^s\psi(x_1)+(1-\theta)^s\psi(x_2)$ is valid, then on $[0,\sigma]$ the function ψ is called the s-convex(in the second sense).

DEFINITION 4. [34] Let $\psi:[0,\sigma]\to\mathbb{R}$ and $s\in[-1,1]$. If $\forall x_1,x_2\in[0,\sigma]$ and $\theta\in(0,1)$ the inequality $\psi(\theta x_1+(1-\theta)x_2)\leqslant\theta^s\psi(x_1)+(1-\theta)^s\psi(x_2)$ is valid, then on $[0,\sigma]$ the function ψ is called the extended s-convex.

In [21], the present class of (a,m)-convex functions as follows.

DEFINITION 5. Let $\psi:[0,\sigma]\to\mathbb{R}$ and $\alpha,m\in[0,1]$. If $\forall x_1,x_2\in[0,\sigma]$ and $\theta\in[0,1]$ and the inequality $\psi(\theta x_1+m(1-\theta)x_2)\leqslant\theta^a\psi(x_1)+m(1-\theta^a)\psi(x_2)$ is valid, then on $[0,\sigma]$ the function ψ is called the (a,m)-convex.

In [20], the following definition is introduced.

DEFINITION 6. Let $h:[0,1]\to\mathbb{R}$ and $\psi:[0,\sigma]\to[0,+\infty)$ are non-negative functions and the function h is not identically zero. If $\forall x_1,x_2\in[0,\sigma]$ and $\theta\in[0,1]$ the inequality

$$\psi\left(\theta x_1+m(1-\theta)x_2\right)\leqslant h(\theta)\psi(x_1)+mh(1-\theta)\psi(x_2) \tag{2}$$

is fulfilled for $m\in[0,1]$, then on $[0,\sigma]$ the function ψ is called the (h,m)-convex.

If the above inequality is reversed, then ψ is said to be (h,m)-concave. Note that in Definition 6, if we take $h(\theta)=\theta$, then we get the definition of an m-convex function, and if, additionally, we take $m=1$, then we get the definition of classical convexity.

In [25], the authors presented the class of $s-(a,m)$-convex functions as follows ("redefined" in [35]).

DEFINITION 7. Let $\psi:[0,+\infty)\to[0,+\infty)$ and $(a,m)\in[0,1]^2$, and $s\in(0,1]$. If, $\forall\rho,\sigma\in[0,+\infty)$ and $\theta\in[0,1]$ the inequality

$$\psi(\theta\rho+m(1-\theta)\sigma)\leqslant\theta^{as}\psi(\rho)+m(1-\theta^{as})\psi(\sigma)$$

is valid, then the function ψ is called the $s-(a,m)$-convex in the first sense.

DEFINITION 8. Let $\psi:[0,+\infty)\to[0,+\infty)$ and $a,m\in[0,1]$, and $s\in(0,1]$. If, $\forall\rho,\sigma\in[0,+\infty)$ and $\theta\in[0,1]$ the inequality

$$\psi(\theta\rho+m(1-\theta)\sigma)\leqslant(\theta^a)^s\psi(\rho)+m(1-\theta^a)^s\psi(\sigma),$$

is valid, then the function ψ is called the $s-(a,m)$-convex in the second sense.

On the basis of these definitions, we will present the classes of functions that will be the basis of our work ([3]).

DEFINITION 9. Let $h:[0,1]\to[0,1]$ and $\psi:[0,\sigma]\to[0,+\infty)$ be non-negative functions and the function h is not identically zero. If, $\forall x_1,x_2\in[0,+\infty)$ and $\theta\in[0,1]$ the inequality

$$\psi(\theta x_1+m(1-\theta)x_2)\leqslant h^s(\theta)\psi(x_1)+m(1-h^s(\theta))\psi(x_2) \tag{3}$$

is valid for $m\in[0,1]$ and $s\in[-1,1]$, then on $[0,+\infty)$ the function ψ is called the (h,m)-convex modified of the first type.

DEFINITION 10. Let $h:[0,1]\to[0,1]$ and $\psi:[0,\sigma]\to[0,+\infty)$ be non-negative functions and the function h is not identically zero. If, $\forall x_1,x_2\in[0,+\infty)$ and $\theta\in[0,1]$ the inequality

$$\psi(\theta x_1+m(1-\theta)x_2)\leqslant h^s(\theta)\psi(x_1)+m(1-h(\theta))^s\psi(x_2) \tag{4}$$

is valid for $m\in[0,1]$ and $s\in[-1,1]$, then on $[0,+\infty)$ the function ψ is called the (h,m)-convex modified of the second type.

REMARK 1. From Definitions 9 and 10, we have

1. If $h(\theta)=\theta^a$ with $a\in(0,1]$, then ψ is a $s-(a,m)$-convex function on $[0,+\infty)$.
2. If $s=1$, then ψ is an (h,m)-convex function on $[0,+\infty)$.
3. If $h(\theta)=\theta$, $s\in(0,1]$ and $m=1$, then ψ is a s-convex function on $[0,+\infty)$.
4. If $h(\theta)=\theta$, $s\in[-1,1]$ and $m=1$, then ψ is an extended s-convex function on $[0,+\infty)$.
5. If $h(\theta)=\theta$ and $s=m=1$, then ψ is a convex function on $[0,+\infty)$.

One of the most important inequalities, that has attracted the attention many inequality experts in the last few decades, is the famous Hermite–Hadamard inequality:

$$\psi\left(\frac{\rho+\sigma}{2}\right)\leqslant\frac{1}{\sigma-\rho}\int_\rho^\sigma\psi(\theta)\,d\theta\leqslant\frac{\psi(\rho)+\psi(\sigma)}{2} \tag{5}$$

which holds for any function ψ convex on the interval $[\rho,\sigma]$. This inequality was proved by the French mathematicians Charles Hermite in 1883 and independently, 10 years later, by Jacques Hadamard. The importance of this inequality is that it makes it possible to estimate the mean value of a convex function, moreover, it provides a refinement of the well-known Jensen inequality. Several results can be consulted in [1, 8, 11, 4, 7, 12, 13, 14, 16, 18, 20, 23, 22, 27] as well as the references therein for more information and other extensions of the Hermite–Hadamard inequality.

To promote the understanding of the subject, we firstly presented the definition of the classical Riemann-Liouville fractional integrals (with $0\leqslant\rho<\theta<\sigma\leqslant\infty$).

DEFINITION 11. Let $\psi \in L_1[\rho,\sigma]$. Then, the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):

$$ {}^{\alpha}I_{\rho+}\psi(x_1) = \frac{1}{\Gamma(\alpha)}\int_{\rho}^{x_1}(x_1-\theta)^{\alpha-1}\psi(\theta)\,d\theta, \quad x_1 > \rho, $$
$$ {}^{\alpha}I_{\sigma-}\psi(x_1) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{\sigma}(\theta-x_1)^{\alpha-1}\psi(\theta)\,d\theta, \quad x_1 < \sigma, $$

where Γ is the Euler gamma function.

The basis of our work is the next definition of the weighted integral operators:

DEFINITION 12. Let $\psi \in L_1[\rho,\sigma]$ and $w : [0,1] \to \mathbb{R}^+$, $w \in C([0,1])$ having the first and second–order derivatives piecewise continuous on $[0,1]$, $w(0) = 0$ and $w(1) = 0$. Then the weighted fractional integrals are defined by (right and left respectively):

$$ I_{\rho+}\psi(x_1) = \int_{\rho}^{x_1} w''\left(\frac{x_1-\theta}{\frac{\rho+\sigma}{2}-\rho}\right)\psi(\theta)\,d\theta, \quad x_1 > \rho $$
$$ I_{\sigma-}\psi(x_1) = \int_{x_1}^{\sigma} w''\left(\frac{\theta-x_1}{\sigma-\frac{\rho+\sigma}{2}}\right)\psi(\theta)\,d\theta, \quad x_1 < \sigma. $$

REMARK 2. If $w''(\theta) = \frac{\left(\frac{\sigma-\rho}{2}\right)^{\alpha-1}\theta^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the Riemann-Liouville fractional integral, right and left.

REMARK 3. By putting $w''(\theta) \equiv 1$, we obtain the classical Riemann integral.

In this paper, within the framework of generalized integral operators (Definition 12), we presented some variants of the inequality (5) for (h,m)-convex modified functions.

2. Hermite–Hadamard type inequalities for (h,m)-convex modified functions

To establish our results, we need the following Lemma.

LEMMA 1. *Let* $\psi : I = [\rho,\sigma] \to \mathbb{R}$ *and* $\psi \in C^2(I^{\circ})$. *If* $\psi'' \in L_1(I)$, *then*

$$ \begin{aligned} &-\frac{1}{2}\left[w'(1)\psi(\rho) - w'(0)\psi\left(\frac{\rho+\sigma}{2}\right) + w'(1)\psi\left(\frac{\rho+\sigma}{2}\right) - w'(0)\psi(\sigma)\right] \\ &+\frac{1}{\sigma-\rho}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right) + I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right) \\ =&\frac{(\sigma-\rho)^2}{8}\int_0^1 w(\theta)\left[\psi''\left(\theta\rho+(1-\theta)\frac{\rho+\sigma}{2}\right)+\psi''\left(\theta\frac{\rho+\sigma}{2}+(1-\theta)\sigma\right)\right]d\theta. \end{aligned} \tag{6} $$

Proof. Denoting

$$I = \int_0^1 w(\theta)\left[\psi''\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right) + \psi''\left(\theta\frac{\rho+\sigma}{2} + (1-\theta)\sigma\right)\right]d\theta$$
$$= I_1 + I_2.$$

If we integrate by parts and we take into account the definition of $w(\theta)$, we obtain

$$\begin{aligned} I_1 &= \int_0^1 w(\theta)\,\psi''\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right)d\theta \\ &= \frac{2}{\sigma-\rho}\int_0^1 w'(\theta)\,\psi'\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right)d\theta \\ &= \frac{2}{\sigma-\rho}\left[\frac{2w'(\theta)}{\rho-\sigma}\psi\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right)\Bigg|_0^1\right. \\ &\quad \left. - \frac{2}{\rho-\sigma}\int_0^1 w''(\theta)\,\psi\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right)d\theta\right] \\ &= \frac{-4}{(\sigma-\rho)^2}\left[\left(w'(1)\psi(\rho) - w'(0)\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right. \\ &\quad \left. - \int_0^1 w''(\theta)\;\psi\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right)d\theta\right] \end{aligned}$$

By using the change of the variable $z = \theta\rho + (1-\theta)\frac{\rho+\sigma}{2}$ for $\theta \in [0,1]$, which gives

$$I_1 = \frac{-4}{(\sigma-\rho)^2}\left[w'(1)\psi(\rho) - w'(0)\psi\left(\frac{\rho+\sigma}{2}\right)\right] + \frac{8}{(\sigma-\rho)^3}I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right).$$

Similarly, we obtain

$$I_2 = \frac{-4}{(\sigma-\rho)^2}\left[w'(1)\psi\left(\frac{\rho+\sigma}{2}\right) - w'(0)\psi(\sigma)\right] + \frac{8}{(\sigma-\rho)^3}I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right).$$

Then,

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{-4}{(\sigma-\rho)^2}\left[w'(1)\psi(\rho) - w'(0)\psi\left(\frac{\rho+\sigma}{2}\right) + w'(1)\psi\left(\frac{\rho+\sigma}{2}\right) - w'(0)\psi(\sigma)\right] \\ &\quad + \frac{8}{(\sigma-\rho)^3}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right) + I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right). \end{aligned}$$

From here, it is very easy to obtain the required equality. □

REMARK 4. If we use

$$w(\theta)=\begin{cases}\theta^{\alpha}, & \theta\in[0,\frac{1}{2})\\ (1-\theta)^{\alpha}, & \theta\in[\frac{1}{2},1]\end{cases},$$

then

1. For $\alpha>1$, from the lemma above with only $\psi(\theta\rho+m(1-\theta)\sigma)$, the Lemma 2.1 of [5] is obtained;
2. If we choose $\alpha=1$ and, from the lemma above with only $\psi(\theta\rho+m(1-\theta)\sigma)$, the Lemma 2.1 of [6] is obtained;
3. If we choose $\alpha=2$ and consider the functions $\psi(\theta\rho+(1-\theta)\sigma)$ and $\psi(\theta\sigma+(1-\theta)\rho)$, then, from (6), we obtain Lemma 2 from [31].

REMARK 5. If we consider the functions $\psi(\theta\rho+(1-\theta)\sigma)$ and $\psi(\theta\sigma+(1-\theta)\rho)$ use $w(\theta)=\theta\,(1-\theta)^{\alpha}$, then, from (6), we obtain Lemma 3.1 from [5].

REMARK 6. Under the same requirements of the previous Remark, and if we consider $w(\theta)=\frac{1-\,(1-\theta)^{\alpha+1}-\,\theta^{\alpha+1}}{\alpha+1}$, from the Lemma above with only $\psi(\theta\rho+(1-\theta)\sigma)$, the Lemma 2.1 of [33] is obtained.

REMARK 7. If we put $w(\theta)=\theta(1-\theta)$, then from (6), we obtain Lemma 2.1 of [35]. However, if we put $w(\theta)=\frac{\theta^{(\alpha+1)}}{\alpha(\alpha+1)\Gamma(\alpha)}$, we obtain the following new equality for Fractional Integral of Riemann-Liouville:

$$\begin{aligned}&-\frac{1}{2\Gamma(\alpha+1)}\left[\psi(\rho)+\psi\left(\frac{\rho+\sigma}{2}\right)\right]-\frac{\sigma-\rho}{4\Gamma(\alpha+2)}\left[\psi'(\rho)+\psi'\left(\frac{\rho+\sigma}{2}\right)\right] \qquad (7)\\ &+\frac{2^{\alpha-1}}{(\sigma-\rho)^{\alpha}}\left[{}^{\alpha}I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+\,{}^{\alpha}I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right]\\ &=\frac{(\sigma-\rho)^2}{8\Gamma(\alpha+2)}\int_0^1\theta^{\alpha+1}\left[\psi''\left(\theta\rho+(1-\theta)\frac{\rho+\sigma}{2}\right)+\psi''\left(\theta\frac{\rho+\sigma}{2}+(1-\theta)\sigma\right)\right]d\theta.\end{aligned}$$

THEOREM 1. *Let $\psi:I=[\rho,\sigma]\longrightarrow\mathbb{R}$ and $\psi\in C^2(I^{\circ})$ with $\psi''\in L_1(I)$. If $|\,\psi''|$ is a convex function ($h(\theta)=\theta$ and $s=m=1$), then the inequality*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}}{(\sigma-\rho)^{\alpha}}\left[{}^{\alpha}I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+\,{}^{\alpha}I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right]-\frac{\psi(\rho)+\psi\left(\frac{\rho+\sigma}{2}\right)}{2\Gamma(\alpha+1)}\right.\\ &\left.-\frac{(\sigma-\rho)\left[\psi'(\rho)+\psi'\left(\frac{\rho+\sigma}{2}\right)\right]}{4\Gamma(\alpha+2)}\right| \qquad (8)\\ &\leqslant\frac{(\sigma-\rho)^2}{8\Gamma(\alpha+2)}\left[\frac{|\psi''(\rho)|-|\psi''(\sigma)|}{\alpha+3}+\frac{\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|+|\psi''(\sigma)|}{\alpha+2}\right].\end{aligned}$$

holds $\forall\alpha>0$.

Proof. It is easy to see that for the right-hand side of the identity (7), by using the properties of the modulus and taking into account the convexity of the function ψ, we obtain:

$$\begin{aligned}&\left|\int_0^1 \theta^{\alpha+1}\left[\psi''\left(\theta\rho+(1-\theta)\frac{\rho+\sigma}{2}\right)+\psi''\left(\theta\frac{\rho+\sigma}{2}+(1-\theta)\sigma\right)\right]d\theta\right|\\ &\leqslant \frac{|\psi''(\rho)|-|\psi''(\sigma)|}{\alpha+3}+\frac{\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|+|\psi''(\sigma)|}{\alpha+2}.\end{aligned}$$

This inequality shows the validity of (8). □

Now we get some new inequalities of the Hermite-Hadamard type.

THEOREM 2. *Let* $\psi : I=\left[\rho,\frac{\sigma}{m}\right]\longrightarrow\mathbb{R}$ *and* $\psi\in C^2(I^\circ)$ *with* $\psi''\in L_1(I)$. *If* $|\psi''|$ *is* (h,m)*-convex modified of the second sense on* I*, then*

$$\begin{aligned}&\left|R_w+\frac{1}{\sigma-\rho}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right| \\ &\leqslant\frac{(\sigma-\rho)^2}{8}\left[\left(|\psi''(\rho)|+\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|\right)\mathbb{A}+m\left(\left|\psi''\left(\frac{\rho+\sigma}{2m}\right)\right|+\left|\psi''\left(\frac{\sigma}{m}\right)\right|\right)\mathbb{B}\right],\end{aligned}\tag{9}$$

with

$$R_w=-\frac{1}{2}\left[w'(1)\psi(\rho)-w't(0)\psi\left(\frac{\rho+\sigma}{2}\right)+w'(1)\psi\left(\frac{\rho+\sigma}{2}\right)-w'(0)\psi(\sigma)\right],$$

$$\mathbb{A}=\int_0^1|w(\theta)|h^s(\theta)d\theta \quad and \quad \mathbb{B}=\int_0^1|w(\theta)|(1-h(\theta))^s d\theta.$$

Proof. From Lemma 1, it follows that

$$\begin{aligned}&\left|R_w+\frac{1}{(\sigma-\rho)}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right|\\ &\leqslant\frac{(\sigma-\rho)^2}{8}(I_1+I_2).\end{aligned}$$

with

$$\begin{aligned}I_1&=\int_0^1|w(\theta)|\left|\psi''\left(\theta\rho+(1-\theta)\frac{\rho+\sigma}{2}\right)\right|d\theta\\ &=\int_0^1|w(\theta)|\left|\psi''\left(\theta\rho+m(1-\theta)\frac{\rho+\sigma}{2m}\right)\right|d\theta\end{aligned}$$

and

$$\begin{aligned}I_2&=\int_0^1|w(\theta)|\left|\psi''\left(\theta\frac{\rho+\sigma}{2}+(1-\theta)\sigma\right)\right|d\theta\\ &=\int_0^1|w(\theta)|\left|\psi''\left(\theta\frac{\rho+\sigma}{2}+m(1-\theta)\frac{\sigma}{m}\right)\right|d\theta.\end{aligned}$$

To obtain the inequality (9), just use the (h,m)-convex modified of the second sense of $|\psi''|$, in I_1 and I_2, apply the elementary properties of the module and group like terms. □

REMARK 8. For $h(\theta)=\theta$, that is, if ψ is a (s,m)-convex function, then

1. By taking into account Remark 4.1, it is not difficult to obtain Theorem 2.1 of [5];
2. By taking into account Remark 4.2, it is easy to obtain Theorem 2.1 of [6];
3. By taking into account Remark 5, we obtain Theorem 3.1 of [5].

REMARK 9. By taking into account Remark 4.3, it is easy to obtain Theorem 3 of [31] by considering that ψ that is a convex function that is, if $h(\theta)=\theta$ and $s=1$.

THEOREM 3. *Let* $\psi : I=\left[\rho,\frac{\sigma}{m}\right] \longrightarrow \mathbb{R}$ *and* $\psi \in C^2(I^\circ)$ *such that* $\psi'' \in L_1(I)$. *If* $|\psi|^q$ *is* (h,m)*-convex modified of the second sense on* I, $q \geqslant 1$, *then*

$$\begin{aligned}
&\left|R_w+\frac{1}{(\sigma-\rho)}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right| \\
&\leqslant \frac{(\sigma-\rho)^2}{8}\left(\int_0^1 |w(\theta)|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left\{\left(|\psi''(\rho)|^q\mathbb{H}_1+m\left|\psi''\left(\frac{\rho+\sigma}{2m}\right)\right|^q\mathbb{H}_2\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|^q\mathbb{H}_1+m\left|\psi''\left(\frac{\sigma}{m}\right)\right|^q\mathbb{H}_2\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

with

$$R_w=-\frac{1}{2}\left[w'(1)\psi(\rho)-w'(0)\psi\left(\frac{\rho+\sigma}{2}\right)+w'(1)\psi\left(\frac{\rho+\sigma}{2}\right)-w'(0)\psi(\sigma)\right],$$

$$\mathbb{H}_1=\int_0^1 h^s(\theta)dt \quad \text{and} \quad \mathbb{H}_2=\int_0^1 (1-h(\theta))^s dt.$$

Proof. From Lemma 1 and using well–known Hölder's integral inequality, we get

$$\begin{aligned}
&\left|R_w+\frac{1}{(\sigma-\rho)}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right| \\
&\leqslant \frac{(\sigma-\rho)^2}{8}\left(\int_0^1 |w(\theta)|^p d\theta\right)^{\frac{1}{p}}\left\{I_3^{\frac{1}{q}}+I_4^{\frac{1}{q}}\right\},
\end{aligned} \tag{10}$$

with $I_3=\int_0^1\left|\psi''\left(\theta\rho+(1-\theta)\frac{\rho+\sigma}{2}\right)\right|^q d\theta$, $I_4=\int_0^1\left|\psi''\left(\theta\frac{\rho+\sigma}{2}+(1-\theta)\sigma\right)\right|^q d\theta$.

Since $|\psi''|^q$ is (h,m)-convex modified of the second sense on $\left[\rho,\frac{\sigma}{m}\right]$, we know that for $\theta\in[0,1]$

$$\left|\psi''\left(\theta\rho+m(1-\theta)\frac{\rho+\sigma}{2m}\right)\right|^q\leqslant h^s(\theta)\left|\psi''(\rho)\right|^q+m(1-h(\theta))^s\left|\psi''\left(\frac{\rho+\sigma}{2m}\right)\right|^q, \tag{11}$$

$$\left|\psi''\left(\theta\frac{\rho+\sigma}{2}+m(1-\theta)\frac{\sigma}{m}\right)\right|^q\leqslant h^s(\theta)\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|^q+m(1-h(\theta))^s\left|\psi''\left(\frac{\sigma}{m}\right)\right|^q \tag{12}$$

Hence, by substituting (11) and (12) in (10), we get

$$\begin{aligned}
&\left(\int_0^1|w(\theta)|^p d\theta\right)^{\frac{1}{p}}\left\{I_3^{\frac{1}{q}}+I_4^{\frac{1}{q}}\right\}\leqslant\left(\int_0^1|w(\theta)|^p d\theta\right)^{\frac{1}{p}}\\
&\times\left\{\left(\int_0^1\left[h^s(\theta)\left|\psi''(\rho)\right|^q+m(1-h(\theta))^s\left|\psi''\left(\frac{\rho+\sigma}{2m}\right)\right|^q\right]d\theta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left[h^s(\theta)\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|^q+m(1-h(\theta))^s\left|\psi''\left(\frac{\sigma}{m}\right)\right|^q\right]d\theta\right)^{\frac{1}{q}}\right\}\\
=&\left(\int_0^1|w(\theta)|^p d\theta\right)^{\frac{1}{p}}\left\{\left(\left|\psi''(\rho)\right|^q\mathbb{H}_1+m\left|\psi''\left(\frac{\rho+\sigma}{2m}\right)\right|^q\mathbb{H}_2\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|^q\mathbb{H}_1+m\left|\psi''\left(\frac{\sigma}{m}\right)\right|^q\mathbb{H}_2\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

REMARK 10. By considering $w(\theta)=\theta(1-\theta)$ and taking ψ'' as a function (α,m)-convex, we obtain Theorem 3.1 of [35].

A variant of the previous result is given in the following theorem.

THEOREM 4. *Let $\psi:I=\left[\rho,\frac{\sigma}{m}\right]\longrightarrow\mathbb{R}$ and $\psi\in C^2(I^\circ)$ such that $\psi''\in L_1(I)$. If $|\psi|^q$ is (h,m)-convex modified of the second sense on I, $q>1$, then*

$$\begin{aligned}
&\left|R_w+\frac{1}{(\sigma-\rho)}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right)+I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right| \qquad (13)\\
&\leqslant\frac{(\sigma-\rho)^2}{8}\left(\int_0^1|w(\theta)|^p d\theta\right)^{\frac{1}{p}}\\
&\times\left\{\left(\left|\psi''(\rho)\right|^q\mathbb{H}_1+m\left|\psi''\left(\frac{\rho+\sigma}{2m}\right)\right|^q\mathbb{H}_2\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|^q\mathbb{H}_1+m\left|\psi''\left(\frac{\sigma}{m}\right)\right|^q\mathbb{H}_2\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

with

$$R_w = -\frac{1}{2}\left[w'(1)\psi(\rho) - w'(0)\psi\left(\frac{\rho+\sigma}{2}\right) + w'(1)\psi\left(\frac{\rho+\sigma}{2}\right) - w'(0)\psi(\sigma)\right],$$

$$\mathbb{H}_1 = \int_0^1 |h^s(\theta)d\theta \quad \textit{and} \quad \mathbb{H}_2 = \int_0^1 (1-h(\theta))^s d\theta.$$

Proof. From Lemma 1 and by using well–known power mean inequality, we have

$$\left|R_w + \frac{1}{(\sigma-\rho)}\left(I_{\rho+}\psi\left(\frac{\rho+\sigma}{2}\right) + I_{\sigma-}\psi\left(\frac{\rho+\sigma}{2}\right)\right)\right| \tag{14}$$
$$\leqslant \frac{(\sigma-\rho)^2}{8}\left(\int_0^1 |w(\theta)|d\theta\right)^{1-\frac{1}{q}}\left\{I_5^{\frac{1}{q}} + I_6^{\frac{1}{q}}\right\},$$

with

$$I_5 = \left(\int_0^1 |w(\theta)|\left|\psi''\left(\theta\rho + (1-\theta)\frac{\rho+\sigma}{2}\right)\right|^q d\theta\right)$$

and

$$I_6 = \left(\int_0^1 |w(\theta)|\left|\psi''\left(\theta\frac{\rho+\sigma}{2} + (1-\theta)\sigma\right)\right|^q d\theta\right).$$

Since $|\psi''|^q$ is (h,m)-convex modified of the second sense on $\left[\rho,\frac{\sigma}{m}\right]$, we have (11) and (12).

By substituting these values in (14), grouping the terms and proceeding as the previous Theorem, we obtain the inequality (13).

Theh, this completes the proof. □

REMARK 11. By taking into account Remark 10, it is easy to obtain the Theorem 3.2 of [35] from the above result.

3. Conclusions

Although we have pointed out throughout the article that our results have special cases, some of which are found in the literature, we wanted to point out one more detail.

If we consider the function

$$w(\theta) = \begin{cases} \theta^{1+\alpha}, & \theta \in \left[0,\frac{1}{2}\right) \\ (1-\theta)^{1+\alpha}, & \theta \in \left[\frac{1}{2},1\right], \end{cases}$$

from Lemma 1, we will obtain the equality

$$-\frac{1+\alpha}{2^\alpha}\left[\psi\left(\frac{3\rho+\sigma}{4}\right) + \psi\left(\frac{\rho+3\sigma}{4}\right)\right] + \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(\sigma-\rho)^\alpha} \tag{15}$$
$$\times\left[I^\alpha_{\left(\frac{3\rho+\sigma}{4}\right)^+}\psi\left(\frac{\rho+\sigma}{2}\right) + I^\alpha_{\left(\frac{\rho+3\sigma}{4}\right)^+}\psi(\sigma) + I^\alpha_{\left(\frac{3\rho+\sigma}{4}\right)^-}\psi(\rho) + I^\alpha_{\left(\frac{\rho+3\sigma}{2}\right)^-}\psi\left(\frac{\rho+\sigma}{2}\right)\right]$$

$$= \frac{(\sigma-\rho)^2}{8}\left\{\int_0^{1/2} t^{1+\alpha}\left[\psi''\left(ta+(1-t)\frac{\rho+\sigma}{2}\right)+\psi''\left(t\frac{\rho+\sigma}{2}+(1-t)\,\sigma\right)\right]dt\right.$$
$$\left.+\int_{1/2}^1 (1-\theta)^{1+\alpha}\left[\psi''\left(ta+(1-t)\frac{\rho+\sigma}{2}\right)+\psi''\left(t\frac{\rho+\sigma}{2}+(1-t)\sigma\right)\right]dt\right\}.$$

This result is better than the one offered in Remark 7 and it is not reported in the literature.

In this way, we have the following inequality:

THEOREM 5. *Let* $\psi : I=[\rho,\sigma]\longrightarrow\mathbb{R}$ *and* $\psi\in C^2(I^\circ)$ *with* $\psi''\in L_1(I)$. *If* $|\psi''|$ *is a convex function (* $h(\theta)=\theta$ *and* $s=m=1$ *), then the inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(\sigma-\rho)^{\alpha}}\left[I^{\alpha}_{\left(\frac{3\rho+\sigma}{4}\right)^+}\psi\left(\frac{\rho+\sigma}{2}\right)+I^{\alpha}_{\left(\frac{\rho+3\sigma}{4}\right)^+}\psi(\sigma)\right.\right. \tag{16}$$
$$\left.\left.+I^{\alpha}_{\left(\frac{3\rho+\sigma}{4}\right)^-}\psi(\rho)+I^{\alpha}_{\left(\frac{\rho+3\sigma}{2}\right)^-}\psi\left(\frac{\rho+\sigma}{2}\right)\right]-\frac{1+\alpha}{2^{\alpha}}\left[\psi\left(\frac{3\rho+\sigma}{4}\right)+\psi\left(\frac{\rho+3\sigma}{4}\right)\right]\right|$$
$$\leqslant \frac{(\sigma-\rho)^2}{2^{\alpha+5}(\alpha+2)}\left[\frac{(|\psi''(\rho)|-|\psi''(\sigma)|)\left(\alpha^2+6\alpha+10\right)}{2(\alpha+3)}\right.$$
$$\left.+(\alpha+3)\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|+|\psi''(\sigma)|\right)\right].$$

holds $\forall\alpha>0$.

Proof. From (15) and the module properties, we can write

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(\sigma-\rho)^{\alpha}}\right.$$
$$\times\left[I^{\alpha}_{\left(\frac{3\rho+\sigma}{4}\right)^+}\psi\left(\frac{\rho+\sigma}{2}\right)+I^{\alpha}_{\left(\frac{\rho+3\sigma}{4}\right)^+}\psi(\sigma)+I^{\alpha}_{\left(\frac{3\rho+\sigma}{4}\right)^-}\psi(\rho)+I^{\alpha}_{\left(\frac{\rho+3\sigma}{4}\right)^-}\psi\left(\frac{\rho+\sigma}{2}\right)\right]$$
$$\left.-\frac{1+\alpha}{2^{\alpha}}\left[\psi\left(\frac{3\rho+\sigma}{4}\right)+\psi\left(\frac{\rho+3\sigma}{4}\right)\right]\right|$$
$$\leqslant \frac{(\sigma-\rho)^2}{8}\left(\int_0^{1/2} t^{1+\alpha}\left|\psi''\left(ta+(1-t)\frac{\rho+\sigma}{2}\right)+\psi''\left(t\frac{\rho+\sigma}{2}+(1-t)\sigma\right)\right|dt\right.$$
$$\left.+\int_{1/2}^1 (1-\theta)^{1+\alpha}\left|\psi''\left(ta+(1-t)\frac{\rho+\sigma}{2}\right)+\psi''\left(t\frac{\rho+\sigma}{2}+(1-t)\sigma\right)\right|dt\right).$$

Since ψ is a convex for the integrals, we get:

$$|I_1| = \int_0^{1/2} t^{1+\alpha}\left|\psi''\left(ta+(1-t)\frac{\rho+\sigma}{2}\right)+\psi''\left(t\frac{\rho+\sigma}{2}+(1-t)\,\sigma\right)\right|dt$$
$$\leqslant \left(|\psi''(\rho)|-|\psi''(\sigma)|\right)\int_0^{1/2}\theta^{\alpha+2}d\theta+\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|+|\psi''(\sigma)|\right)\int_0^{1/2}\theta^{\alpha+1}d\theta$$

$$= \frac{|\psi''(\rho)| - |\psi''(\sigma)|}{2^{\alpha+3}} + \frac{\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + |\psi''(\sigma)|}{2^{\alpha+2}},$$

and

$$\begin{aligned}
|I_2| &= \int_{1/2}^{1} (1-\theta)^{1+\alpha} \left|\psi''\left(ta + (1-t)\frac{\rho+\sigma}{2}\right) + \psi''\left(t\frac{\rho+\sigma}{2} + (1-t)\sigma\right)\right| dt \\
&\leqslant \int_{1/2}^{1} (1-\theta)^{1+\alpha} \left[\theta\left|\psi''(\rho)\right| + (1-\theta)\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right|\right. \\
&\quad \left. + \theta\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + (1-\theta)\left|\psi''(\sigma)\right| d\theta\right] \\
&= \left(\left|\psi''(\rho)\right| - \left|\psi''(\sigma)\right|\right) \int_{1/2}^{1} (1-\theta)^{1+\alpha} \theta d\theta \\
&\quad + \left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + \left|\psi''(\sigma)\right|\right) \int_{1/2}^{1} (1-\theta)^{1+\alpha} d\theta \\
&= \frac{(\alpha+4)\left(|\psi''(\rho)| - |\psi''(\sigma)|\right)}{2^{\alpha+3}(\alpha+2)(\alpha+3)} + \frac{\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + |\psi''(\sigma)|}{2^{\alpha+2}(\alpha+2)}.
\end{aligned}$$

By summing the last inequality, we get:

$$\begin{aligned}
&|I_1| + |I_2| \\
&\leqslant \frac{|\psi''(\rho)| - |\psi''(\sigma)|}{2^{\alpha+3}} + \frac{\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + |\psi''(\sigma)|}{2^{\alpha+2}} \\
&\quad + \frac{(\alpha+4)\left(|\psi''(\rho)| - |\psi''(\sigma)|\right)}{2^{\alpha+3}(\alpha+2)(\alpha+3)} + \frac{\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + |\psi''(\sigma)|}{2^{\alpha+2}(\alpha+2)} \\
&= \frac{|\psi''(\rho)| - |\psi''(\sigma)|}{2^{\alpha+3}}\left(1 + \frac{\alpha+4}{(\alpha+2)(\alpha+3)}\right) + \frac{\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + |\psi''(\sigma)|\right)(\alpha+3)}{2^{\alpha+2}(\alpha+2)}.
\end{aligned}$$

Or, by multiplying both sides of the inequality by the expression $\frac{(\sigma-\rho)^2}{8}$, we get

$$\begin{aligned}
&\frac{(\sigma-\rho)^2}{8}\left(|I_1| + |I_2|\right) \\
&\leqslant \frac{(\sigma-\rho)^2}{8}\left[\frac{|\psi''(\rho)| - |\psi''(\sigma)|}{2^{\alpha+3}}\left(1 + \frac{\alpha+4}{(\alpha+2)(\alpha+3)}\right)\right. \\
&\quad \left. + \frac{\left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + |\psi''(\sigma)|\right)(\alpha+3)}{(\alpha+2)2^{\alpha+2}}\right] \\
&= \frac{(\sigma-\rho)^2}{(\alpha+2)2^{\alpha+5}}\left[\frac{\left(|\psi''(\rho)| - |\psi''(\sigma)|\right)\left(\alpha^2 + 6\alpha + 10\right)}{2(\alpha+3)}\right. \\
&\quad \left. + \left(\left|\psi''\left(\frac{\rho+\sigma}{2}\right)\right| + \left|\psi''(\sigma)\right|\right)(\alpha+3)\right]
\end{aligned}$$

The proof is completed. □

Finally, we must remember that the presented results contain generalized inequalities wich are valid for convex functions, m-convex functions, h-convex functions, and s-convex functions in the second sense, and defined in a closed interval of negative non-real numbers. It is clear that the problem of extending these results to the case of (h,m)-convex functions of the first type remains open.

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Bahtiyar Bayraktar
Bursa Uludağ University
Department of Mathematics and Science Education
Görükle Campus, 16059, Bursa, Turkey
e-mail: bbayraktar@uludag.edu.tr

Juan E. Nápoles Valdés
UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina
e-mail: jnapoles@frre.utn.edu.ar
and
UNNE, FaCENA
Ave. Libertad 5450, Corrientes 3400, Argentina
e-mail: jnapoles@exa.unne.edu.ar