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GENERALIZED FRACTIONAL OSTROWSKI TYPE INEQUALITIES VIA $h-s$–CONVEX FUNCTION

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Abstract. We are introducing very first time a generalized class named it the class of $h-s$-convex functions. This class of functions contains many important classes including class of h-convex, Godunova-Levin s-convex, s-convex in the 2^{nd} kind and hence contains class of convex and MT-convex functions. It also contains class of P-convex functions, class of Godunova-Levin functions and the class of ordinary convex functions. Also, we would like to state the generalization of the classical Ostrowski inequality via fractional integrals with respect to another function, which is obtained for functions whose first derivative in absolute values is $h-s$-convex function. Moreover we establish some Ostrowski type inequalities via fractional integrals with respect to another function and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $h-s$-convex functions by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

1. Introduction

In this section, from literature, we recall and introduce some definitions for various convex functions.

DEFINITION 1. [3] A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex(concave), if

$$\eta\left(tx+(1-t)y\right) \leqslant (\geqslant) t\eta(x)+(1-t)\eta(y),$$

$\forall x,y \in I$, $t \in [0,1]$.

DEFINITION 2. [3] A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be MT-convex(concave), if η is a non-negative and

$$\eta\left(tx+(1-t)y\right) \leqslant (\geqslant) \frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y),$$

$\forall x,y \in I$, $t \in [0,1]$.

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DEFINITION 3. [17] We say that $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a P-convex(concave), function, if η is a non-negative and $\forall x, y \in I$ and $t \in [0,1]$ we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant) \eta(x) + \eta(y).$$

DEFINITION 4. [20] We say that $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin convex(concave), function, if η is non-negative and $\forall x, y \in I$ and $t \in (0,1)$ we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant) \frac{1}{t}\eta(x) + \frac{1}{1-t}\eta(y).$$

DEFINITION 5. [4] Let $s \in [0,1]$. A function $\eta : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex(concave), in the 2^{nd} kind, if

$$\eta(tx + (1-t)y) \leqslant (\geqslant) t^s \eta(x) + (1-t)^s \eta(y),$$

$\forall x, y \in I$, $t \in [0,1]$.

DEFINITION 6. [9] We say that the function $\eta : I \subset \mathbb{R} \to [0,\infty)$ is of Godunova-Levin s-convex(concave), function, with $s \in [0,1]$, if

$$\eta(tx + (1-t)y) \leqslant (\geqslant) \frac{1}{t^s}\eta(x) + \frac{1}{(1-t)^s}\eta(y),$$

$\forall t \in (0,1)$ and $x, y \in I$.

DEFINITION 7. [33] Let $h : J \subseteq \mathbb{R} \to [0,\infty)$ with h not identical to 0. We say that η is an h-convex(concave), function if $\forall x, y \in I$, we have

$$\eta(tx + (1-t)y) \leqslant (\geqslant) h(t)\eta(x) + h(1-t)\eta(y),$$

$\forall t \in [0,1]$.

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality.

THEOREM 1. [30] *Let $\varphi : [a,b] \to \mathbb{R}$ be differentiable on (a,b) with the property that $|\varphi'(t)| \leqslant M$ $\forall t \in (a,b)$. Then*

$$\left| \varphi(x) - \frac{1}{b-a} \int_a^b \varphi(t)dt \right| \leqslant (b-a)M \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right], \tag{1}$$

$\forall x \in (a,b)$.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [11]–[16] and [21]–[25].

DEFINITION 8. The Riemann-Liouville integral operator of order $\zeta > 0$ with $a \geqslant 0$ is defined as

$$
\begin{aligned}
J_a^{\zeta}\varphi(x) &= \frac{1}{\Gamma(\zeta)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\zeta}}dt,\\
J_a^{0}\varphi(x) &= \varphi(x).
\end{aligned}
\tag{2}
$$

Here $\Gamma(\zeta) = \int_0^\infty e^{-u}u^{\zeta-1}du$ is the Gamma function. In case of $\zeta = 1$, the fractional integral reduces to the classical integral.

DEFINITION 9. [31] The Riemann-Liouville integrals $I_{a^+}^{\zeta}\varphi$ and $I_{b^-}^{\zeta}\varphi$ of $\varphi \in L_1[a,b]$ having order $\zeta > 0$ with $a \geqslant 0$, $a < b$ are defined by

$$
I_{a^+}^{\zeta}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\zeta}}dt, \quad x > a,
$$

and

$$
I_{b^-}^{\zeta}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_x^b \frac{\varphi(t)}{(t-x)^{1-\zeta}}dt, \quad x < b,
$$

respectively. Note that $I_{a^+}^{0}\varphi(x) = I_{b^-}^{0}\varphi(x) = \varphi(x)$.

DEFINITION 10. [31] Let $g : [a,b] : \mathbb{R}$ be an increasing and positive function on $[a,b]$, having a continuous derivatives $g'(x)$ on (a,b). The fractional integrals $I_{a^+,g}^{\zeta}\varphi$ and $I_{b^-,g}^{\zeta}\varphi$ of φ with respect to the function g on $[a,b]$ of order $\zeta > 0$ are defined by

$$
I_{a^+,g}^{\zeta}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_a^x \frac{g'(t)\varphi(t)}{(g(x)-g(t))^{1-\zeta}}dt, \quad x > a,
$$

and

$$
I_{b^-,g}^{\zeta}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_x^b \frac{g'(t)\varphi(t)}{(g(t)-g(x))^{1-\zeta}}dt, \quad x < b,
$$

respectively.

REMARK 1. If we replace $g(t) = t$ the above fractional integrals reduce to the Riemann-Liouville fractional integrals.

THEOREM 2. [16] *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $a,b \in I$, $a<b$ $\varphi' \in L_1[a,b]$ and for $\zeta > 1$, Montgomery identity for fractional integrals holds:*

$$\varphi(x) = \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^{\zeta}\varphi(b) - J_a^{\zeta-1}(P_1(x,b)\varphi(b)) + J_a^{\zeta}(P_1(x,b)\varphi'(b)),$$

where $P_1(x,t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}\dfrac{\Gamma(\zeta)}{(b-x)^{\zeta-1}}, & \text{if} \quad t \in [a,x], \\ \\ \dfrac{t-b}{b-a}\dfrac{\Gamma(\zeta)}{(b-x)^{\zeta-1}}, & \text{if} \quad t \in (x,b]. \end{cases}$$

THEOREM 3. [16] *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $a,b \in I$, $a<b$ $\varphi' \in L_1[a,b]$ and for $\zeta > 1$, generalized Montgomery identity for fractional integrals holds:*

$$\begin{aligned}(1-\varepsilon)\varphi(x) = &\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^{\zeta}\varphi(b) - J_a^{\zeta-1}(P_2(x,b)\varphi(b)) \\ &-\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}} J_a^{0}\varphi(a) + J_a^{\zeta}(P_2(x,b)\varphi'(b)),\end{aligned} \tag{3}$$

where $P_2(x,t)$ is the fractional Peano Kernel defined by:

$$P_2(x,t) = \begin{cases} \dfrac{t-\mu}{b-a}\dfrac{\Gamma(\zeta)}{(b-x)^{\zeta-1}}, & \text{if} \quad t \in [a,x], \\ \\ \dfrac{t-\nu}{b-a}\dfrac{\Gamma(\zeta)}{(b-x)^{\zeta-1}}, & \text{if} \quad t \in (x,b]. \end{cases}$$

$\forall x \in [\mu,\nu]$ for $\mu = a + \varepsilon\frac{b-a}{2}$ and $\nu = b - \varepsilon\frac{b-a}{2}$.

Throughout this paper, we will assume that $g : [a,b] \to \mathbb{R}$ is an increasing and positive function on $[a,b]$, having a continuous derivative $g'(x)$ on (a,b). In order to prove our results, we need the following Lemma.

LEMMA 1. [27] *Let $\varphi : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $\varphi' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then the identity for fractional integrals holds with respect to another function*

$$\begin{aligned}&\varphi(x) - \Gamma(\zeta+1)\left[\frac{I^{\zeta}_{x^+,g}\varphi(b)}{2\,(g(b)-g(x))^{\zeta}} + \frac{I^{\zeta}_{x^-,g}\varphi(a)}{2\,(g(x)-g(a))^{\zeta}}\right] \\ &= \frac{x-a}{2\,(g(x)-g(a))^{\zeta}}\int_0^1 \frac{\varphi'(tx+(1-t)a)}{(g(tx+(1-t)a)-g(a))^{-\zeta}}dt \\ &\quad - \frac{b-x}{2\,(g(b)-g(x))^{\zeta}}\int_0^1 \frac{\varphi'(tx+(1-t)b)}{(g(b)-g(tx+(1-t)b))^{-\zeta}}dt.\end{aligned}$$

Throughout this paper, we denote

$$
{}_{g}^{\zeta}\kappa_{a}^{b}(x)=\left[\frac{(x-a)^{\zeta+1}}{2\,(g(x)-g(a))^{\zeta}}+\frac{(b-x)^{\zeta+1}}{2\,(g(b)-g(x))^{\zeta}}\right],
$$

$$
{}_{\varphi,g}^{\zeta}\theta_{a}^{b}(x)=\varphi(x)-\Gamma(\zeta+1)\left[\frac{I_{x^{+},g}^{\zeta}\varphi(b)}{2\,(g(b)-g(x))^{\zeta}}+\frac{I_{x^{-},g}^{\zeta}\varphi(a)}{2\,(g(x)-g(a))^{\zeta}}\right].
$$

We also make use of the Euler's beta function, which is for $x,y>0$ defined as

$$
B(x,y)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
$$

The main aim of our study is to present Ostrowski inequality for fractional integrals with respect to another function, which is generalization of the classical Ostrowski inequality (1) via $h-s$-convex, which is given in Section 2. Moreover we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $h-s$-convex functions by using different techniques including Hölder's inequality [35] and power mean inequality [34]. Also we give the special cases of our results and applications of midpoint inequalities in special means. In the last section gives us conclusion with some remarks and future ideas to generalize the results.

2. Generalization of Ostrowski inequality via fractional with respect to another function

Now, we are introducing the very first time the new type of convex(concave) functions, named as $h-s$-convexity.

DEFINITION 11. Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a real valued function and $s\in[0,1]$. We say that the $\eta:I\subseteq\mathbb{R}\to\mathbb{R}$ is a $h-s$-convex(concave), function on the interval I if $\forall x,y\in I$ we have

$$
\eta\left(tx+(1-t)y\right)\leqslant(\geqslant)\left(\frac{h(t)}{t}\right)^{-s}\eta(x)+\left(\frac{h(1-t)}{1-t}\right)^{-s}\eta(y), \tag{4}
$$

$\forall t\in(0,1)$.

REMARK 2. In Definition 11, one can see the following.

1. If we take $h(t)=t^{\frac{s}{s+1}}$ where $s\in[0,1]$ in (4), we get h-convex function.
2. If we take $h(t)=t^{2}$ and $s\in[0,1)$ in (4), then we get the class of Godunova-Levin s-convex(concave) functions.
3. If we put $h(t)=1$ and $s\in(0,1]$ in (4), then we get the concept of s-convex(concave) in 2^{nd} kind.

4. If we put $h(t)=t^2$ and $s=1$ in (4), then we get the concept of Godunova-Levin convex(concave) function.

5. If we put $s=0$ in (4), then we get the concept of P-convex(concave) function.

6. If we put $h(t)=s=1$ in (4), then we get the concept of ordinary convex(concave) function.

7. If we put $s=1$ and $h(t)=2\sqrt{t(1-t)}$ in (4), then we get the concept of MT-convex(concave) function.

THEOREM 4. *Suppose all the assumptions of Lemma 1 hold. Additionally, assume that $|\varphi'|$ is $h-s$-convex function on $[a,b]$ with $h(t)\neq t^2$ and $|\varphi'(x)|\leqslant M$, $|g'(x)|\leqslant L$, $x\in[a,b]$. Then for each $x\in(a,b)$ the following inequality holds:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant ML^{\zeta}\left(\int_0^1 t^{\zeta}\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt\right)\,{}^{\zeta}_{g}\kappa_a^b(x). \tag{5}$$

Proof. From the Lemma 1 we have

$$\begin{aligned}\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| &\leqslant \frac{x-a}{2\left(g(x)-g(a)\right)^{\zeta}}\int_0^1 \frac{|\varphi'(tx+(1-t)a)|}{\left(g(tx+(1-t)a)-g(a)\right)^{-\zeta}}dt\\ &\quad+\frac{b-x}{2\left(g(b)-g(x)\right)^{\zeta}}\int_0^1 \frac{|\varphi'(tx+(1-t)b)|}{\left(g(b)-g(tx+(1-t)b)\right)^{-\zeta}}dt.\end{aligned} \tag{6}$$

Since g is differentiable and $|g'(x)|\leqslant L$ on $[a,b]$, we get that g is Lipschizian function, i.e.

$$g\left(tx+(1-t)a\right)-g(a)\leqslant Lt(x-a), \tag{7}$$

$$g(b)-g\left(tx+(1-t)b\right)\leqslant Lt(b-x). \tag{8}$$

Using the inequalities (7) and (8) in (6), we get

$$\begin{aligned}\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| &\leqslant L^{\zeta}\frac{(x-a)^{\zeta+1}}{2\left(g(x)-g(a)\right)^{\zeta}}\int_0^1 t^{\zeta}\left|\varphi'\left(tx+(1-t)a\right)\right|dt\\ &\quad+L^{\zeta}\frac{(b-x)^{\zeta+1}}{2\left(g(b)-g(x)\right)^{\zeta}}\int_0^1 t^{\zeta}\left|\varphi'\left(tx+(1-t)b\right)\right|dt.\end{aligned} \tag{9}$$

Since $|\varphi'|$ is $h-s$-convex on $[a,b]$ and $|\varphi'(x)|\leqslant M$, we have

$$\begin{aligned}&\int_0^1 t^{\zeta}\left(\left[\frac{h(t)}{t}\right]^{-s}|\varphi'(x)|+\left[\frac{h(1-t)}{1-t}\right]^{-s}|\varphi'(a)|\right)dt\\ &\leqslant M\int_0^1 t^{\zeta}\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt\end{aligned} \tag{10}$$

and similarly

$$\int_0^1 t^\zeta \left(\left[\frac{h(t)}{t}\right]^{-s} |\varphi'(x)| + \left[\frac{h(1-t)}{1-t}\right]^{-s} |\varphi'(b)| \right) dt$$
$$\leqslant M \int_0^1 t^\zeta \left(\left[\frac{h(t)}{t}\right]^{-s} + \left[\frac{h(1-t)}{1-t}\right]^{-s} \right) dt. \tag{11}$$

By using inequalities (10) and (11) in (9), we get

$$\left| {}^{\zeta}_{\varphi,g}\theta_a^b(x) \right| \leqslant ML^\zeta \left(\int_0^1 t^\zeta \left(\left[\frac{h(t)}{t}\right]^{-s} + \left[\frac{h(1-t)}{1-t}\right]^{-s} \right) dt \right) {}^{\zeta}_{g}\kappa_a^b(x),$$

which completes the proof. □

COROLLARY 1. *In Theorem 4, one can see the following.*

1. *If one takes $h(t) = t^{\frac{s}{s+1}}$ where $s \in [0,1]$ in (5), then one has the fractional Ostrowski type inequality for h-convex functions:*

$$\left| {}^{\zeta}_{\varphi,g}\theta_a^b(x) \right| \leqslant ML^\zeta \left(\int_0^1 t^\zeta \left[h(t) + h(1-t)\right] dt \right) {}^{\zeta}_{g}\kappa_a^b(x).$$

2. *If one takes $h(t) = t^2$ and $s \in [0,1)$ in (5), then one has the Ostrowski inequality for Godunova-Levin s-convex functions:*

$$\left| {}^{\zeta}_{\varphi,g}\theta_a^b(x) \right| \leqslant ML^\zeta \left(\frac{1}{1+\zeta-s} + \frac{\Gamma(1+\zeta)\Gamma(1-s)}{\Gamma(2+\zeta-s)} \right) {}^{\zeta}_{g}\kappa_a^b(x).$$

3. *If one takes $h(t) = t^2$ and $s \in [0,1)$ in (5), then one has the Ostrowski inequality for Godunova-Levin s-convex functions:*

$$\left| {}^{\zeta}_{\varphi,g}\theta_a^b(x) \right| \leqslant ML^\zeta \left(\frac{1}{1+\zeta+s} + \frac{\Gamma(1+\zeta)\Gamma(1+s)}{\Gamma(2+\zeta+s)} \right) {}^{\zeta}_{g}\kappa_a^b(x).$$

4. *If one takes If one takes $g(t) = t$, $h(t) = 1$ and $s \in (0,1]$ in inequality (5), then one has the inequality (2.6) of Theorem 7 in [32].*

5. *If one takes $g(t) = t$, $\zeta = h(t) = 1$ and $s \in (0,1]$ in inequality (5), then one has the inequality (2.1) of Theorem 2 in [1].*

6. *If one takes $s = 0$ in inequality (5), then one has the Ostrowski inequality for P-convex functions via fractional integrals:*

$$\left| {}^{\zeta}_{\varphi,g}\theta_a^b(x) \right| \leqslant \frac{2ML^\zeta}{1+\zeta} {}^{\zeta}_{g}\kappa_a^b(x).$$

7. *If one takes* $h(t)=s=1$ *in inequality* (5), *then one has the Ostrowski inequality for convex functions via fractional integrals:*

$$\left| {}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| \leqslant \frac{ML^{\zeta}}{1+\zeta}\, {}^{\zeta}_{g}\kappa^{b}_{a}(x).$$

8. *If one takes* $g(t)=t,\ h(t)=s=1$ *in inequality* (5), *then one has the Corollary* 1 *in* [32].

9. *If one takes* $g(t)=t,\ h(t)=\zeta=s=1$ *in inequality* (5), *then one has inequality* (1.3) *of Theorem* 3 *in* [32].

10. *If one takes* $s=1$ *and* $h(t)=2\sqrt{t(1-t)}$ *in* (5), *then one has the fractional Ostrowski type inequality MT -convex functions:*

$$\left| {}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| \leqslant ML^{\zeta}\left(\frac{\sqrt{\pi}\,\Gamma\left[\frac{1}{2}+\zeta\right]}{2\,\Gamma[1+\zeta]}\right) {}^{\zeta}_{g}\kappa^{b}_{a}(x).$$

THEOREM 5. *Suppose all the assumptions of Lemma* 1 *hold. Additionally, assume that* $|\varphi'|^{q}$ *is* $h-s$*-convex function on* $[a,b]$, $q\geqslant 1$ *with* $h(t)\neq t^{2}$ *and* $|\varphi'(x)|\leqslant M,\ |g'(x)|\leqslant L,\ x\in[a,b]$. *Then for each* $x\in(a,b)$ *the following inequality holds:*

$$\left| {}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| \leqslant \frac{ML^{\zeta}}{(\zeta+1)^{1-\frac{1}{q}}}\left(\int_{0}^{1}\left[t^{\zeta}\left[\frac{h(t)}{t}\right]^{-s}+t^{\zeta}\left[\frac{h(1-t)}{1-t}\right]^{-s}\right]dt\right)^{\frac{1}{q}} {}^{\zeta}_{g}\kappa^{b}_{a}(x).$$

Proof. From the inequality (9) and using power mean inequality [34], we have

$$\begin{aligned}
\left| {}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| &\leqslant L^{\zeta}\frac{(x-a)^{\zeta+1}}{2\,(g(x)-g(a))^{\zeta}}\int_{0}^{1}t^{\zeta}\left|\varphi'\left(tx+(1-t)a\right)\right|^{q}dt\\
&\quad+L^{\zeta}\frac{(b-x)^{\zeta+1}}{2\,(g(b)-g(x))^{\zeta}}\int_{0}^{1}t^{\zeta}\left|\varphi'\left(tx+(1-t)b\right)\right|^{q}dt\\
&\leqslant L^{\zeta}\frac{(x-a)^{\zeta+1}}{2\,(g(x)-g(a))^{\zeta}}\left(\int_{0}^{1}t^{\zeta}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t^{\zeta}\left|\varphi'\left(tx+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&\quad+L^{\zeta}\frac{(b-x)^{\zeta+1}}{2\,(g(b)-g(x))^{\zeta}}\left(\int_{0}^{1}t^{\zeta}dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t^{\zeta}\left|\varphi'\left(tx+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{12}$$

Since $|\varphi'|^{q}$ is $h-s$-convex on $[a,b]$. and $|\varphi'(x)|\leqslant M$, we get

$$\int_{0}^{1}t^{\zeta}\left|\varphi'\left(tx+(1-t)a\right)\right|^{q}dt\leqslant M^{q}\int_{0}^{1}t^{\zeta}\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt \tag{13}$$

and

$$\int_0^1 t^\zeta \left|\varphi'(tx+(1-t)b)\right|^q dt \leqslant M^q \int_0^1 t^\zeta \left(\left[\frac{h(t)}{t}\right]^{-s} + \left[\frac{h(1-t)}{1-t}\right]^{-s}\right) dt. \quad (14)$$

Using the inequalities (12)–(14), we get

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^\zeta \, {}^{\zeta}_{g}\kappa_a^b(x)}{(\zeta+1)^{1-\frac{1}{q}}} \left(\int_0^1 t^\zeta \left(\left[\frac{h(t)}{t}\right]^{-s} + \left[\frac{h(1-t)}{1-t}\right]^{-s}\right) dt\right)^{\frac{1}{q}},$$

which completes the proof. □

COROLLARY 2. *In Theorem 5, one can see the following.*

1. *If one takes $q=1$, one has the Theorem 4.*
2. *If one takes $h(t)=t^{\frac{s}{s+1}}$ where $s\in[0,1]$ in (12), then one has the fractional Ostrowski type inequality for h-convex functions:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^\zeta}{(1+\zeta)^{1-\frac{1}{q}}} \left(\int_0^1 t^\zeta\,[h(t)+h(1-t)]dt\right)^{\frac{1}{q}} \, {}^{\zeta}_{g}\kappa_a^b(x).$$

3. *If one takes $h(t)=t^2$ and $s\in[0,1)$ in (12), then one has the Ostrowski inequality for Godunova-Levin s-convex functions:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^\zeta}{(1+\zeta)^{1-\frac{1}{q}}} \left(\frac{1}{1+\zeta-s} + \frac{\Gamma(1+\zeta)\Gamma(1-s)}{\Gamma(2+\zeta-s)}\right)^{\frac{1}{q}} \, {}^{\zeta}_{g}\kappa_a^b(x).$$

4. *If one takes $h(t)=1$ and $s\in(0,1]$ in inequality (12), then one has the fractional Ostrowski type inequality for s-convex functions in 2^{nd} kind:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^\zeta}{(\zeta+1)^{1-\frac{1}{q}}} \left(\frac{1}{1+\zeta+s} + \frac{\Gamma(1+\zeta)\Gamma(1+s)}{\Gamma(2+\zeta+s)}\right)^{\frac{1}{q}} \, {}^{\zeta}_{g}\kappa_a^b(x).$$

5. *If one takes $g(t)=t$, $h(t)=1$ and $s\in(0,1]$ in inequality (12), then one has the inequality (2.8) of Theorem 9 in [32].*
6. *If one takes $g(t)=t$, $h(t)=\zeta=1$ and $s\in(0,1]$ in inequality (12), then one has the inequality (2.3) of Theorem 4 in [1].*
7. *If one takes $s=0$ in inequality (12), then one has the Ostrowski inequality for P-convex functions via fractional integrals:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{2^{\frac{1}{q}}ML^\zeta}{(1+\zeta)^{1-\frac{1}{q}}} \, {}^{\zeta}_{g}\kappa_a^b(x).$$

8. *If one takes* $h(t)=s=1$ *in inequality* (12), *then one has the Ostrowski inequality for convex functions via fractional integrals:*

$$\left|{}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| \leqslant \frac{ML^{\zeta}}{(1+\zeta)^{1-\frac{1}{q}}}\,{}^{\zeta}_{g}\kappa^{b}_{a}(x).$$

9. *If one takes* $g(t)=t$, $h(t)=s=1$ *in inequality* (12), *then one has the inequality of Corollary* 3 *in* [32].

10. *If one takes* $g(t)=t$, $h(t)=\zeta=s=1$ *in inequality* (12), *then one has inequality* (1.5) *of Theorem* 5 *in* [32].

11. *If one takes* $s=1$ *and* $h(t)=2\sqrt{t(1-t)}$ *in* (12), *then one has the fractional Ostrowski type inequality MT -convex functions:*

$$\left|{}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| \leqslant \frac{ML^{\zeta}}{(1+\zeta)^{1-\frac{1}{q}}}\left(\frac{\sqrt{\pi}\,\Gamma\left[\frac{1}{2}+\zeta\right]}{2\,\Gamma[1+\zeta]}\right)^{\frac{1}{q}}\,{}^{\zeta}_{g}\kappa^{b}_{a}(x).$$

THEOREM 6. *Suppose all the assumptions of Lemma* 1 *hold. Additionally, assume that* $|\varphi'|^{q}$ *is* $h-s$*-convex function on* $[a,b], q>1$ *with* $h(t)\neq t^{2}$ *and* $|\varphi'(x)|\leqslant M$, $|g'(x)|\leqslant L$, $x\in[a,b]$. *Then for each* $x\in(a,b)$ *the following inequality holds:*

$$\left|{}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| \leqslant \frac{ML^{\zeta}\,{}^{\zeta}_{g}\kappa^{b}_{a}(x)}{(\zeta p+1)^{\frac{1}{p}}}\left(\int_{0}^{1}\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt\right)^{\frac{1}{q}}, \tag{15}$$

where $p^{-1}+q^{-1}=1$.

Proof. From the inequality (9) and using Hölder's inequality [35], we have

$$\begin{aligned}\left|{}^{\zeta}_{\varphi,g}\theta^{b}_{a}(x)\right| &\leqslant L^{\zeta}\frac{(x-a)^{\zeta+1}}{2\left(g(x)-g(a)\right)^{\zeta}}\left(\int_{0}^{1}t^{\zeta p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\varphi'\left(tx+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\ &\quad+L^{\zeta}\frac{(b-x)^{\zeta+1}}{2\left(g(b)-g(x)\right)^{\zeta}}\left(\int_{0}^{1}t^{\zeta p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\varphi'\left(tx+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}.\end{aligned} \tag{16}$$

Since $|\varphi'|^{q}$ is $h-s$-convex and $|\varphi'(x)|\leqslant M$, we have

$$\int_{0}^{1}\left|\varphi'\left(tx+(1-t)a\right)\right|^{q}dt \leqslant M^{q}\int_{0}^{1}\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt, \tag{17}$$

and

$$\int_{0}^{1}\left|\varphi'\left(tx+(1-t)b\right)\right|^{q}dt \leqslant M^{q}\int_{0}^{1}\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt. \tag{18}$$

Using inequalities (16)–(18), we get

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\zeta}\ {}^{\zeta}_{g}\kappa_a^b(x)}{(\zeta p+1)^{\frac{1}{p}}}\left(\int_0^1\left(\left[\frac{h(t)}{t}\right]^{-s}+\left[\frac{h(1-t)}{1-t}\right]^{-s}\right)dt\right)^{\frac{1}{q}},$$

which completes the proof. □

COROLLARY 3. *In Theorem 6, one can see the following.*

1. *If one takes* $h(t)=t^{\frac{s}{s+1}}$ *where* $s\in[0,1]$ *in* (15), *then one has the fractional Ostrowski type inequality for* h*-convex functions:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\left(\int_0^1[h(t)+h(1-t)]\,dt\right)^{\frac{1}{q}}\ {}^{\zeta}_{g}\kappa_a^b(x).$$

2. *If one takes* $h(t)=t^2$ *and* $s\in[0,1)$ *in* (15), *then one has the Ostrowski inequality for Godunova-Levin* s*-convex functions:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}}\ {}^{\zeta}_{g}\kappa_a^b(x).$$

3. *If one takes* $h(t)=1$ *and* $s\in(0,1]$ *in inequality* (15), *then one has the fractional Ostrowski type inequality for* s*-convex functions in* 2^{nd} *kind:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\left(\frac{2}{1+s}\right)^{\frac{1}{q}}\ {}^{\zeta}_{g}\kappa_a^b(x).$$

4. *If one takes* $g(t)=t$, $h(t)=1$ *and* $s\in(0,1]$ *in inequality* (15), *then one has the inequality* (2.7) *of Theorem* 8 *in* [32].

5. *If one takes* $g(t)=t$, $h(t)=\zeta=1$ *and* $s\in(0,1]$ *in inequality* (15), *then one has the inequality* (2.4) *of Theorem* 3 *in* [1].

6. *If one takes* $s=0$ *in inequality* (15), *then one has the Ostrowski inequality for* P*-convex functions via fractional integrals:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{2^{\frac{1}{q}}ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\ {}^{\zeta}_{g}\kappa_a^b(x).$$

7. *If one takes* $h(t)=s=1$ *in inequality* (15), *then one has the Ostrowski inequality for convex functions via fractional integrals:*

$$\left|{}^{\zeta}_{\varphi,g}\theta_a^b(x)\right| \leqslant \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\ {}^{\zeta}_{g}\kappa_a^b(x).$$

8. *If one takes* $g(t)=t$, $h(t)=s=1$ *in inequality* (15), *then one has Corollary* 2 *in* [32].

9. *If one takes* $g(t)=t$, $h(t)=\zeta=s=1$ *in inequality* (15), *then one has inequality* (1.4) *of Theorem* 4 *in* [32].

10. *If one takes* $s=1$ *and* $h(t)=2\sqrt{t(1-t)}$ *in* (15), *then one has the fractional Ostrowski type inequality MT -convex functions:*

$$\left|{}_{\varphi,g}^{\zeta}\theta_a^b(x)\right| \leqslant \frac{ML^{\zeta}}{(\zeta p+1)^{\frac{1}{p}}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\ {}_{g}^{\zeta}\kappa_a^b(x).$$

THEOREM 7. *Let* $\varphi:[a,b]\to\mathbb{R}$ *be differentiable on* (a,b), $\varphi':[a,b]\to\mathbb{R}$ *be integrable on* $[a,b]$ *and* $\eta: I\subset\mathbb{R}\to\mathbb{R}$, *be a* $h-s$*-convex(concave) function, then we have the inequalities*

$$\begin{aligned}
&\eta\left[(1-\varepsilon)\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^{\zeta}\varphi(b)\right.\\
&\left.+J_a^{\zeta-1}(P_2(x,b)\varphi(b))+\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}}J_a^0\varphi(a)\right]\\
\leqslant(\geqslant)\ &\frac{(b-x)^{1-\zeta}}{(x-a)^{\zeta}}\left[\frac{b-a}{x-a}h\left(\frac{x-a}{b-a}\right)\right]^{-s}\int_a^x\eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\\
&+\frac{1}{(b-x)^{\zeta}}\left[\frac{b-a}{b-x}h\left(\frac{b-x}{b-a}\right)\right]^{-s}\int_x^b\eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt,
\end{aligned}\tag{19}$$

$\forall x\in[\mu,\nu]$ *and* $\varepsilon\in[0,1]$.

Proof. Utilizing the generalized montgomery identity (3) for fractional, we get

$$\begin{aligned}
&(1-\varepsilon)\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^{\zeta}\varphi(b)\\
&+J_a^{\zeta-1}(P_2(x,b)\varphi(b))+\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}}J_a^0\varphi(a)\\
&=J_a^{\zeta}(P_2(x,b)\varphi'(b))\\
=&\frac{1}{\Gamma(\zeta)}\int_a^b P_2(x,t)\frac{\varphi'(t)}{(b-t)^{1-\zeta}}dt\\
=&\left(\frac{x-a}{b-a}\right)\left[\frac{(b-x)^{1-\zeta}}{x-a}\int_a^x\frac{\{t-\mu\}\varphi'(t)}{(b-t)^{1-\zeta}}dt\right]\\
&+\left(\frac{b-x}{b-a}\right)\left[\frac{(b-x)^{1-\zeta}}{b-x}\int_x^b\frac{\{t-\nu\}\varphi'(t)}{(b-t)^{1-\zeta}}dt\right],
\end{aligned}$$

$\forall x \in [\mu, \nu]$ and $\varepsilon \in [0,1]$. Next by using $\eta : I \subset [0,\infty) \to \mathbb{R}$, the $h-s$-convex(concave) function, we get

$$\begin{aligned}
&\eta \left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^{\zeta}\varphi(b) \right. \\
&\left. + J_a^{\zeta-1}(P_2(x,b)\varphi(b)) + \frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}} J_a^0 \varphi(a) \right] \\
\leqslant (\geqslant) & \left[\frac{b-a}{x-a} h\left(\frac{x-a}{b-a}\right)\right]^{-s} \eta \left[\frac{(b-x)^{1-\zeta}}{x-a} \int_a^x \frac{\{t-\mu\}\varphi'(t)}{(b-t)^{1-\zeta}} dt \right] \\
& + \left[\frac{b-a}{b-x} h\left(\frac{b-x}{b-a}\right)\right]^{-s} \eta \left[\frac{(b-x)^{1-\zeta}}{b-x} \int_x^b \frac{\{t-\nu\}\varphi'(t)}{(b-t)^{1-\zeta}} dt \right],
\end{aligned}$$

$\forall x \in [\mu, \nu]$ and $\varepsilon \in [0,1]$. Applying Jensen's integral inequality [8], we get the inequality (19). $\square$

REMARK 3. In Theorem 7, if we put $\varepsilon = 0$, in (19). we get

$$\begin{aligned}
&\eta \left[\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^{\zeta}\varphi(b) + J_a^{\zeta-1}(P_1(x,b)\varphi(b)) \right] \\
\leqslant (\geqslant) & \frac{(b-x)^{1-\zeta}}{(x-a)^{\zeta}} \left[\frac{b-a}{x-a} h\left(\frac{x-a}{b-a}\right)\right]^{-s} \int_a^x \eta \left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}} \right] dt \\
& + \frac{1}{(b-x)^{\zeta}} \left[\frac{b-a}{b-x} h\left(\frac{b-x}{b-a}\right)\right]^{-s} \int_x^b \eta \left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}} \right] dt.
\end{aligned}$$

COROLLARY 4. *In Theorem 7, one can see the following.*

1. *If one takes $h(t) = t^{\frac{s}{s+1}}$ where $s \in [0,1]$ in (19), then one has the fractional Ostrowski type inequality for h-convex(concave) functions:*

$$\begin{aligned}
&\eta \left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^{\zeta}\varphi(b) \right. \\
&\left. + J_a^{\zeta-1}(P_2(x,b)\varphi(b)) + \frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}} J_a^0 \varphi(a) \right] \\
\leqslant (\geqslant) & h\left(\frac{x-a}{b-a}\right) \left[\frac{(b-x)^{1-\zeta}}{x-a} \int_a^x \eta \left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}} \right] dt \right. \\
& \left. + h\left(\frac{b-x}{b-a}\right) \left[\frac{1}{(b-x)^{\zeta}} \int_x^b \eta \left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}} \right] dt \right]. \right.
\end{aligned} \tag{20}$$

REMARK 4. If we choose $\varepsilon = 0$ in (20), we get

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^\zeta \varphi(b) + J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant (\geqslant)\ h\left(\frac{x-a}{b-a}\right)\left[\frac{(b-x)^{1-\zeta}}{x-a}\int_a^x \eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right]$$
$$+h\left(\frac{b-x}{b-a}\right)\left[\frac{1}{(b-x)^\zeta}\int_x^b \eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].$$

2. *If one takes $h(t) = t^2$ and $s \in [0,1)$ in (19), then one has the Ostrowski inequality for Godunova-Levin s-convex(concave) functions:*

$$\eta\left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^\zeta \varphi(b)\right.$$
$$\left. + J_a^{\zeta-1}(P_2(x,b)\varphi(b)) + \frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}} J_a^0 \varphi(a)\right]$$
$$\leqslant (\geqslant)\ \frac{(b-a)^s(b-x)^{1-\zeta}}{(x-a)^{1+s}} \int_a^x \eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt$$
$$+\frac{(b-a)^s}{(b-x)^{\zeta+s}} \int_x^b \eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt. \tag{21}$$

REMARK 5. If we choose $\varepsilon = 0$ in (21), we get

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^\zeta \varphi(b) + J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant (\geqslant)\ \frac{(b-a)^s(b-x)^{1-\zeta}}{(x-a)^{1+s}} \int_a^x \eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt$$
$$+\frac{(b-a)^s}{(b-x)^{\zeta+s}} \int_x^b \eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt.$$

3. *If one takes $h(t) = t^2$ and $s = 1$ in (19), then one has the fractional Ostrowski type inequality for Godunova-Levin convex(concave) function:*

$$\eta\left[(1-\varepsilon)\varphi(x) - \frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta} J_a^\zeta \varphi(b)\right.$$
$$\left. + J_a^{\zeta-1}(P_2(x,b)\varphi(b)) + \frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}} J_a^0 \varphi(a)\right]$$
$$\leqslant (\geqslant)\ \frac{(b-a)(b-x)^{1-\zeta}}{(x-a)^2} \int_a^x \eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt$$
$$+\frac{(b-a)}{(b-x)^{\zeta+1}} \int_x^b \eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt. \tag{22}$$

REMARK 6. If we choose $\varepsilon = 0$ in (22), we get

$$\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^{\zeta}\varphi(b)+J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant(\geqslant)\ \frac{(b-a)(b-x)^{1-\zeta}}{(x-a)^2}\int_a^x\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt$$
$$+\frac{(b-a)}{(b-x)^{\zeta+1}}\int_x^b\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt.$$

4. *If one takes $h(t)=1$ and $s\in(0,1]$ in (19), then one has the fractional Ostrowski type inequality for s-convex(concave) functions in 2^{nd} kind:*

$$\eta\left[(1-\varepsilon)\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^{\zeta}\varphi(b)\right.$$
$$\left.+J_a^{\zeta-1}(P_2(x,b)\varphi(b))+\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}}J_a^0\varphi(a)\right]$$
$$\leqslant(\geqslant)\ \frac{(x-a)^{s-1}(b-x)^{1-\zeta}}{(b-a)^s}\int_a^x\eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt$$
$$+\frac{(b-x)^{s-\zeta}}{(b-a)^s}\int_x^b\eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt. \tag{23}$$

REMARK 7. If we choose $\varepsilon = 0$ in (23), we get

$$\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^{\zeta}\varphi(b)+J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant(\geqslant)\ \frac{(x-a)^{s-1}(b-x)^{1-\zeta}}{(b-a)^s}\int_a^x\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt$$
$$+\frac{(b-x)^{s-\zeta}}{(b-a)^s}\int_x^b\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt.$$

5. *If one takes $s=0$ in (19), then one has the fractional Ostrowski type inequality for P-convex (concave) functions:*

$$\eta\left[(1-\varepsilon)\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^{\zeta}\varphi(b)\right.$$
$$\left.+J_a^{\zeta-1}(P_2(x,b)\varphi(b))+\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}}J_a^0\varphi(a)\right]$$
$$\leqslant(\geqslant)\ \frac{(b-x)^{1-\zeta}}{(x-a)}\int_a^x\eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\frac{1}{(b-x)^{\zeta}}\int_x^b\eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt. \tag{24}$$

REMARK 8. If we choose $\varepsilon=0$ in (24), we get

$$\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^\zeta\varphi(b)+J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant(\geqslant)\ \frac{(b-x)^{1-\zeta}}{(x-a)}\int_a^x\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\frac{1}{(b-x)^\zeta}\int_x^b\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt.$$

6. *If one takes* $h(t)=s=1$ *in* (19), *then one has the fractional Ostrowski type inequality for convex(concave) functions:*

$$\begin{aligned}&\eta\left[(1-\varepsilon)\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^\zeta\varphi(b)\right.\\&\left.+J_a^{\zeta-1}(P_2(x,b)\varphi(b))+\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}}J_a^0\varphi(a)\right]\\&\leqslant(\geqslant)\ \frac{(b-x)^{1-\zeta}}{b-a}\left[\int_a^x\eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\int_x^b\eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].\end{aligned}\tag{25}$$

REMARK 9. If we choose $\varepsilon=0$ in (25), we get

$$\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^\zeta\varphi(b)+J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant(\geqslant)\ \frac{(b-x)^{1-\zeta}}{b-a}\left[\int_a^x\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\int_x^b\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].$$

7. *If one takes* $s=1$ *and* $h(t)=2\sqrt{t(1-t)}$ *in* (19), *then one has the fractional Ostrowski type inequality for MT -convex(concave) functions:*

$$\begin{aligned}&\eta\left[(1-\varepsilon)\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^\zeta\varphi(b)\right.\\&\left.+J_a^{\zeta-1}(P_2(x,b)\varphi(b))+\frac{\varepsilon(b-x)^{1-\zeta}}{2(b-a)^{1-\zeta}}J_a^0\varphi(a)\right]\\&\leqslant(\geqslant)\ \frac{(b-x)^{\frac{1}{2}-\zeta}}{2\sqrt{(x-a)}}\left[\int_a^x\eta\left[\frac{(t-\mu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\int_x^b\eta\left[\frac{(t-\nu)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].\end{aligned}\tag{26}$$

REMARK 10. If we choose $\varepsilon=0$ in (26), we get

$$\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{b-a}(b-x)^{1-\zeta}J_a^\zeta\varphi(b)+J_a^{\zeta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leqslant(\geqslant)\ \frac{(b-x)^{\frac{1}{2}-\zeta}}{2\sqrt{(x-a)}}\left[\int_a^x\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\int_x^b\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].$$

3. Applications of midpoint inequalities

If we replace φ by $-\varphi$ and $x=\frac{a+b}{2}$ in Theorem 7, we get

THEOREM 8. *Let $\varphi:[a,b]\to\mathbb{R}$ be differentiable on (a,b), $\varphi':[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $\eta: I\subset\mathbb{R}\to\mathbb{R}$, be a $h-s$-convex(concave) function, then*

$$\begin{aligned}&\eta\left[\frac{\Gamma(\zeta)\left(\frac{b-a}{2}\right)^{1-\zeta}}{b-a}J_a^{\zeta}\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right.\\&\left.-J_a^{\zeta-1}\left(P_2\left(\frac{a+b}{2},b\right)\varphi(b)\right)-\frac{\varepsilon}{2^{2-\zeta}}J_a^0\varphi(a)\right]\\\leqslant(\geqslant)\ &\frac{2^{\zeta}\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{(b-a)^{\zeta}}\left[\int_{\frac{a+b}{2}}^{a}\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right.\\&\left.+\int_b^{\frac{a+b}{2}}\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].\end{aligned}\tag{27}$$

$\forall\varepsilon\in[0,1]$.

REMARK 11. In Theorem 8, one can see the following.

1. If we put $\varepsilon=0$, in (27). we get

$$\begin{aligned}&\eta\left[\frac{\Gamma(\zeta)\left(\frac{b-a}{2}\right)^{1-\zeta}}{b-a}J_a^{\zeta}\varphi(b)-\varphi\left(\frac{a+b}{2}\right)-J_a^{\zeta-1}\left(P_1\left(\frac{a+b}{2},b\right)\varphi(b)\right)\right]\\\leqslant(\geqslant)\ &\frac{2^{\zeta}\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{(b-a)^{\zeta}}\left[\int_{\frac{a+b}{2}}^{a}\eta\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt+\int_b^{\frac{a+b}{2}}\eta\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\zeta}}\right]dt\right].\end{aligned}$$

2. If we put $\zeta=1$ in (27). we get

$$\begin{aligned}&\eta\left[(\varepsilon-1)\varphi\left(\frac{a+b}{2}\right)-\varepsilon\frac{\varphi(a)+\varphi(b)}{2}+\frac{1}{b-a}\int_a^b\varphi(t)dt\right]\\\leqslant(\geqslant)\ &\frac{2\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\int_{\frac{a+b}{2}}^{b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right].\end{aligned}$$

3. If we put $\varepsilon=0,\ \zeta=1$ in (27). we get

$$\begin{aligned}&\eta\left[\frac{1}{b-a}\int_a^b\varphi(t)dt-\varphi\left(\frac{a+b}{2}\right)\right]\\\leqslant(\geqslant)\ &\frac{2\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[(\mu-t)\varphi'(t)\right]dt+\int_{\frac{a+b}{2}}^{b}\eta\left[(\nu-t)\varphi'(t)\right]dt\right].\end{aligned}$$

REMARK 12. Assume that $\eta : I \subset [0,\infty) \to \mathbb{R}$ be an $h-s$-convex(concave) function:

1. If we take $\zeta = 1$, $\varphi(t) = \frac{1}{t}$ in inequality (28) where $t \in [a,b] \subset (0,\infty)$, then we have

$$\eta\left[\frac{A(a,b)+(\varepsilon-1)L(a,b)}{A(a,b)L(a,b)} - \varepsilon\frac{A(a,b)}{G^2(a,b)}\right]$$
$$\leqslant (\geqslant)\ \frac{2\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[\frac{t-a}{t^2}\right]dt + \int_{\frac{a+b}{2}}^b \eta\left[\frac{t-b}{t^2}\right]dt\right].$$

2. If we take $\zeta = 1$, $\varphi(t) = -\ln t$ in inequality (28), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\eta\left[\ln\left(\frac{\exp[\varepsilon A(\ln a,\ln b)]A^{(1-\varepsilon)}(a,b)}{I(a,b)}\right)\right]$$
$$\leqslant (\geqslant)\ \frac{2\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[\frac{t-a}{t}\right]dt + \int_{\frac{a+b}{2}}^b \eta\left[\frac{t-b}{t}\right]dt\right].$$

3. If we take $\zeta = 1$, $\varphi(t) = t^p$, $p \in \mathbb{R}\setminus\{0,-1\}$ in inequality (28), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\eta\left[L_p^p(a,b)+(\varepsilon-1)A^p(a,b)-\varepsilon A(a^p,b^p)\right]$$
$$\leqslant (\geqslant)\ \frac{2\left[2\,h\left(\frac{1}{2}\right)\right]^{-s}}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[\frac{p\,(a-t)}{t^{1-p}}\right]dt + \int_{\frac{a+b}{2}}^b \eta\left[\frac{p\,(b-t)}{t^{1-p}}\right]dt\right].$$

REMARK 13. In Theorem 5, one can see the following.

1. Let $g(t) = t$, $x = \frac{a+b}{2}$, $\zeta = 1$, $0 < a < b$, $q \geqslant 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(t) = t^n$ in (12). Then

$$\left|A^n(a,b) - L_n^n(a,b)\right|$$
$$\leqslant \frac{M\,(b-a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1 t\left(\left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s}\right)dt\right)^{\frac{1}{q}}.$$

2. Let $g(t) = t$, $x = \frac{a+b}{2}$, $\zeta = 1$, $0 < a < b$, $q \geqslant 1$ and $\varphi : (0,1] \to \mathbb{R}$, $\varphi(t) = -\ln t$ in (12). Then

$$\left|\ln\left(\frac{A\,(a,b)}{I\,(a,b)}\right)\right| \leqslant \frac{M\,(b-a)}{(2)^{2-\frac{1}{q}}}\left(\int_0^1 t\left(\left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s}\right)dt\right)^{\frac{1}{q}}.$$

REMARK 14. In Theorem 6, one can see the following.

1. Let $g(t)=t,\ x=\frac{a+b}{2},\ \zeta=1,\ 0<a<b,\ p^{-1}+q^{-1}=1$ and $\varphi:\mathbb{R}\to\mathbb{R}^{+}$, $\varphi(t)=t^{n}$ in (15). Then

$$\begin{aligned}&\left|A^{n}(a,b)-L_{n}^{n}(a,b)\right|\\&\leqslant \frac{M(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_{0}^{1}\left(\left(\frac{h(t)}{t}\right)^{-s}+\left(\frac{h(1-t)}{1-t}\right)^{-s}\right)dt\right)^{\frac{1}{q}}.\end{aligned}$$

2. Let $g(t)=t,\ x=\frac{a+b}{2},\ \zeta=1,\ 0<a<b,\ p^{-1}+q^{-1}=1$ and $\varphi:(0,1]\to\mathbb{R}$, $\varphi(t)=-\ln t$ in (15). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right|\leqslant \frac{M(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_{0}^{1}\left(\left(\frac{h(t)}{t}\right)^{-s}+\left(\frac{h(1-t)}{1-t}\right)^{-s}\right)dt\right)^{\frac{1}{q}}.$$

4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, the generalization of Fractional Ostrowski inequality via generalized Montgomery identity [16] with $h-s$-convex functions, which we used the very first time here. This class of functions contains many important classes including class of h-convex [33], Godunova-Levin s-convex [9], s-convex in the 2^{nd} kind [4] (and hence contains class of convex and MT-convex functions [3]). It also contains class of P-convex functions [17] and class of Godunova-Levin functions [20]. We have stated our main result in section 2, which is the generalization of Ostrowski inequality via generalized Montgomery identity by fractional integrals for $h-s$-convex functions. Further, we used different techniques including Hölder's inequality [35] and power mean inequality [34] for generalization of Ostrowski inequality. In second last section we have given some applications in terms of special means including arithmetic, geometric, harmonic, logarithmic, identric and p-logarithmic means by using the midpoint inequalities.

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