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ON THE APPROXIMATE CONTROLLABILITY FOR FRACTIONAL NEUTRAL INCLUSION SYSTEMS WITH NONLOCAL CONDITIONS

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Abstract. The aim of this work is to study the approximate controllability for some fractional neutral inclusion system with nonlocal conditions. We establish a new variation of constant formula that helps us to formulate the problem of the approximate controllability. We assume that the linear system without the input functions is approximately controllable, then we prove with the lack of compactness, the approximate controllability for the whole nonlinear system. For illustrative purposes, we provide an application to the heat equation with memory.

1. Introduction

Numerous problems in science, engineering, and the economy are frequently modelled and described using fractional differential equations, see for instance [3, 9, 15, 19, 20, 23, 24, 25, 27] and the references therein. Particularly, fractional integro-differential equations have a significant impact on the modeling of a number of natural processes. The motivation comes from biological sciences, physics and other fields, including population dynamics, elasticity, forecasting human populations, radiation transport, torsion of a wire, oscillating magnetic field, control of the memory behavior of electrical sockets, Bernoulies problems, mortality of equipment problems and inverse problems of reaction diffusion equations, epidemiological systems, diminishing the spread of viruses, and many others examples, see for instance [1, 4, 34, 35, 39]. In addition, fractional differential equations can be more useful than classical differential equations for studying a variety of scientific phenomena since, in the real system, the derivative of the model sum may be exact in terms of fractional order. Numerous good monographs provide the crucial conceptual tools for the subjective evaluation of this research topic. For instance, we refer the readers to [2, 10, 11, 21, 26, 32, 37, 38, 41, 46, 47, 48, 49, 50] and some other references therein.

In general, the interest of modeling a phenomenon by a mathematical object is the possibility to understand it, but also to predict it thanks to simulation. Often, one seeks to study the possibility of acting on a given system, so that it functions in a desired goal,

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or at best, at the least cost, etc. It is the aim of the control theory which is a mathematical theory allowing as to determine laws of guidance and action on a given system. The concept of controllability for some partial differential equations has been studied in many works. The literature is rich and it is not the place here to recall the large number of issues devoted to this purpose. Exact controllability and approximate controllability are necessary to distinguish. In [40], Triggiani set forth that the concept of exact controllability is usually too strong in general infinite-dimensional spaces. Then, it is more practical to explore a comparatively weaker concept of controllability, known as approximate controllability. The approximate controllability ensures that it is possible to control a movement from any point to an arbitrary vicinity of any other point, but generally the trajectory never reaches the given end point.

In literature, there are many works that treat the approximate controllability for fractional differential equations under different conditions, see for instance [17, 18, 29, 30, 31, 33, 36, 43, 45]. In this work, we treat the approximate controllability of the following neutral delayed integrodifferential fractional inclusions:

$$
{}^{c}D_t^q N(t,x_t) \in AN(t,x_t) + \int_0^t G(t-s)N(s,x_s)ds + F(t,x(t),x_t) + Bu(t), \quad t \in [0,a] \quad (1)
$$

$$
x(t) = \phi(t) \qquad t \in [-r,0], \tag{2}
$$

$$
x(t) = g(x)(t) \qquad t \in [-r,0], \tag{3}
$$

where $g : \mathcal{C}([-r,a];X) \to \mathcal{C}$ is a continuous function; we denote by $\mathcal{C} = \mathcal{C}([-r,0];X)$, X is a reflexive Banach space, $A : D(A) \to X$ generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X, $B : U \to X$ is a bounded linear operator, U is a Hilbert space, $G(t) : D(A) \to X$ is a closed linear operator, $u(\cdot) \in \mathbb{L}^2([0,a];U)$ is the control function, $F : [0,a] \times X \times \mathcal{C} \to 2^X$ is a multi-valued function, $N : [0,a] \times \mathcal{C} \to X$ such that $N(t,\phi) = \phi(0) + h(t,\phi)$ for each $(t,\phi) \in [0,a] \times \mathcal{C}$, h is a continuous function that will be specified later, and ${}^{c}D_t^q$, $0 < q < 1$ is the Caputo fractional derivative of order q. In this paper, 2^X denotes the collection of all nonempty subsets of X. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators defined from X to a linear normed space Y. $\mathcal{L}(X,Y)$ is endowed with the norm $\|\cdot\|_{\mathcal{L}(X,Y)}$ defined by $\|T\|_{\mathcal{L}(X,Y)} = \sup\{\|Tx\|_Y : \|x\|_X = 1\}$, for $T \in \mathcal{L}(X,Y)$; we denote $\mathcal{L}(X)$ if $Y = X$. In return to the literature, R. Sakthivel et al. [33] studied the approximate controllability of equation (1) (with $G = 0$, $h = 0$, and $r = 0$) assuming that the semigroup generated by A is compact and under a principal assumption that is the corresponding linear part is approximately controllable. R.N Wang et al. [43] studied the existence and the approximate controllability of equation (1)–(2) (with $G = 0$, and $h = 0$) using the compactness of the semigroup generated by A. In particular, Qiao-Min Xiang and Peng-Xian Zhu in [45] studied the existence of the mild solution and the approximate controllability of equation (1)–(3) (with $G = 0$, and $h = 0$). They assumed that the semigroup generated by A is not compact, the state space and the control space are a Hilbert spaces, and F is a multi-valued function with closed convex values for which $F(t,\cdot,\cdot)$ is weakly upper semicontinuous for a.e $t \in [0,a]$, and $F(\cdot,x,v)$ has a $\mathbb{L}^p$-integral selection for each $(x,v) \in X \times \mathcal{C}$. It is assumed also that F satisfies the following condition:

$$\exists\ \mu \in \mathcal{C}([0,a];\mathbb{R}^+) \text{ such that } \chi(F(t,\Omega,\Lambda)) \leqslant \mu(t)(\chi(\Omega) + \sup_{s\in[-r,0]} \Lambda(s))$$

for a.e $t \in [0,a]$ and for each bounded sets Ω of X and Λ of $\mathcal{C}$, where χ is the measure of noncompactness. This statement might not be accurate, and even if it were, it would be challenging to prove for specific phenomena. In this paper, we treat a more general case when we study the approximate controllability of both equations (1)–(2) and (1)–(3) assuming that $G \neq 0$ which means that the semigroups theory does not remains sufficient to study this class of differential equations. Then, we use resolvent operators theory in the sens given by R. Grimmer [12, 13], see subsection 2.2. We prove our result under simple and basic assumptions on the system operators, particularly that the corresponding linear system is approximately controllable. The previous assumption on map F will be replaced by simple assumptions, see $(\mathbf{H_3})-(\mathbf{H_5})$ in section 3. An example in which these assumptions can be verified is given there. Also, the compactness of the nonlocal function is not required, unlike some other restrictions can be taken by consideration. The resolvent operator theory was used to construct a variation of constants formula in order to establish our main results. Briefly, this work is organized as follows: In section 2, we recall some pertinent properties relating to the theory of fractional calculus, resolvent operators theory and multi-valued analysis. In section 3, we establish our results concerning the existence of mild solution and the approximate controllability of both equations (1)–(2) and (1)–(3). In general, we present two approaches to prove that equation (1) is approximately controllable on $[0,a]$. In section 4, we apply our main results to the heat equation with memory. The last one is a conclusion.

2. Preliminaries

To achieve the main goal of this paper, we recall some key facts, concepts, and lemmas of fractional calculus, the theory of resolved operators, multivalued analysis, the noncompactness measure and the duality mapping. Let X be a Banach space equipped with the norm $\|\cdot\|$, and $\mathcal{C}([-r,a];X)$ denotes the space of continuous functions from $[-r,a]$ to X endowed with the norm $\|\cdot\|_\infty$.

2.1. Caputo fractional derivative

Under natural conditions on a function $v(\cdot)$, the Caputo fractional derivative of $v(\cdot)$ is defined as follows.

DEFINITION 2.1. [7] Let $q>0$, $b\in\mathbb{R}$ and $t>b$. The Caputo q-order fractional derivative of $v(\cdot)$ is defined by the following fractional operator:

$${}_b^cD_t^q v(t) = \begin{cases} \dfrac{1}{\Gamma(n-q)}\displaystyle\int_b^t \frac{v^{(n)}(s)}{(t-s)^{q+1-n}}ds, & \text{if } n-1<q<n, \quad n\in\mathbb{N}^* \\[2ex] \dfrac{d^n v(t)}{dt^n}, & \text{if } n=q. \end{cases}$$

2.2. Resolvent operators

We consider the following integrodifferential linear system:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t G(t-s)x(s)ds, & t \geqslant 0 \\ x(0) = x_0. \end{cases} \tag{4}$$

Since A is a closed map, the subspace $D(A)$ equipped with the graph norm $\|x\| + \|Ax\|$, $x \in D(A)$ is a Banach space.

DEFINITION 2.2. [12] A family of bounded linear operators $(R(t))_{t\geqslant 0}$ on X is called a resolvent operator of equation (4), if

a) $R(0) = I$, and for each $t \geqslant 0$, $\|R(t)\|_{\mathcal{L}(X)} \leqslant \tilde{M}e^{wt}$ for some constants $\tilde{M} \geqslant 1$ and $w \in \mathbb{R}$.

b) For each $x \in X$, $R(\cdot)x$ is continuous.

c) $R(t) \in \mathcal{L}(D(A))$ for $t \geqslant 0$. For each $x \in D(A)$, $R(\cdot)x \in \mathcal{C}(\mathbb{R}^+; D(A)) \cap \mathcal{C}^1(\mathbb{R}^+; X)$, and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t G(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)G(s)xds, \quad t \geqslant 0. \end{aligned}$$

We assume that:

($\mathbf{H_1}$) The operator A generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X.

($\mathbf{H_2}$) For each $t \geqslant 0$, $G(t) \in \mathcal{L}(D(A), X)$ and for each $x \in D(A)$, the function $G(\cdot)x$ is bounded, differentiable and the derivative $G'(\cdot)x$ is bounded and uniformly continuous on $\mathbb{R}^+$.

THEOREM 2.1. [8, Theorem 1] *Assume that* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. Then, equation* (4) *has a unique resolvent operator.*

THEOREM 2.2. [22] *Assume that* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. Let* $(R(t))_{t\geqslant 0}$ *be the resolvent operator of equation* (4). *Assume that* $(T(t))_{t\geqslant 0}$ *is norm-continuous for* $t > 0$, *then* $(R(t))_{t\geqslant 0}$ *is norm-continuous for* $t > 0$.

Let $(R(t))_{t\geqslant 0}$ be the resolvent operator of equation (4). We define $(Q_q(t))_{t\geqslant 0}$ and $(P_q(t))_{t\geqslant 0}$ on X by

$$Q_q(t)x = \int_0^{+\infty} \Psi_q(s)R(st^q)xds, \quad t \geqslant 0,\ x \in X,$$

and

$$P_q(t)x = \int_0^{+\infty} qs\Psi_q(s)R(st^q)xds, \quad t \geqslant 0,\ x \in X,$$

where,

$$\Psi_q(s) = \frac{1}{\pi q}\sum_{n=1}^{+\infty}(-s)^{n-1}\frac{\Gamma(1+qn)}{n!}\sin(n\pi q), \quad q \in]0,1[,$$

is the Wright type function defined on $(0,+\infty)$ in which $0 \leqslant \Psi_q(s)$ for $s \in (0,+\infty)$,

$$\int_0^{+\infty}\Psi_q(s)ds = 1, \quad \text{and} \quad \int_0^{+\infty}s^{\gamma}\Psi_q(s)ds = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma q)}, \quad \gamma \in [0,1].$$

Assume that there is $M \geqslant 1$ such that $\|R(t)\|_{\mathcal{L}(X)} \leqslant M$ for each $t \geqslant 0$.

LEMMA 2.1. *The families $(Q_q(t))_{t\geqslant 0}$ and $(P_q(t))_{t\geqslant 0}$ satisfy the following properties:*

i) For each $t \geqslant 0$, $Q_q(t)$ and $P_q(t)$ are linear and bounded on X. Moreover,

$$\|Q_q(t)x\| \leqslant M\|x\|, \text{ and } \|P_q(t)x\| \leqslant \frac{qM}{\Gamma(1+q)}\|x\|, \quad \text{for } x \in X.$$

ii) $Q_q(\cdot)$, and $P_q(\cdot)$ are strongly continuous on X.

iii) If $R(t)$ is norm-continuous for $t > 0$, then $Q_q(t)$ and $P_q(t)$ are norm-continuous for $t > 0$.

Proof.

i) Let $t \geqslant 0$, and $x \in X$. Then,

$$\begin{aligned}\|Q_q(t)x\| &\leqslant M\int_0^{+\infty}\Psi_q(s)ds\|x\| \\ &= M\|x\|.\end{aligned}$$

Using the fact that $\int_0^{+\infty}s\Psi_q(s)ds = \frac{1}{\Gamma(1+q)}$, we obtain that

$$\begin{aligned}\|P_q(t)x\| &\leqslant qM\int_0^{+\infty}s\Psi_q(s)ds\|x\| \\ &= \frac{qM}{\Gamma(1+q)}\|x\|.\end{aligned}$$

ii) Let $x \in X$, and $0 \leqslant t_0 < t$. Then,

$$\|Q_q(t)x - Q_q(t_0)x\| \leqslant \int_0^{+\infty}\Psi_q(s)\|R(st^q)x - R(st_0^q)x\|ds.$$

Since $(R(t))_{t\geqslant 0}$ is strongly continuous, by the Lebesgue dominated convergence Theorem, we deduce that

$$\lim_{t\to t_0^+}\|Q_q(t)x-Q_q(t_0)x\|=0.$$

If $0<t<t_0$, similarly we get that

$$\lim_{t\to t_0^-}\|Q_q(t)x-Q_q(t_0)x\|=0.$$

Thus, $(Q_q(t))_{t\geqslant 0}$ is strongly continuous on X. In the same manner, we prove that $(P_q(t))_{t\geqslant 0}$ is strongly continuous on X.

iii) Assume that $(R(t))_{t\geqslant 0}$ is norm-continuous for $t>0$. Let $t_0\in]0,+\infty[$, and $x\in X$. Then,

$$\|Q_q(t)x-Q_q(t_0)x\|\leqslant\int_0^{+\infty}\Psi_q(s)\|R(st^q)-R(st_0^q)\|_{\mathcal{L}(X)}ds\|x\|,$$

which implies that

$$\|Q_q(t)-Q_q(t_0)\|_{\mathcal{L}(X)}\leqslant\int_0^{+\infty}\Psi_q(s)\|R(st^q)-R(st_0^q)\|_{\mathcal{L}(X)}ds.$$

Since $\lim_{t\to t_0}\Psi_q(s)\|R(st^q)-R(st_0^q)\|_{\mathcal{L}(X)}=0$ for a.e $s\in[0,+\infty[$, and

$$\Psi_q(s)\|R(st^q)-R(st_0^q)\|_{\mathcal{L}(X)}\leqslant 2M\Psi_q(s),$$

using Lebesgue dominated convergence Theorem, we get that

$$\lim_{t\to t_0}\|Q_q(t)-Q_q(t_0)\|_{\mathcal{L}(X)}=0.$$

We argue as above, we can affirm that

$$\lim_{t\to t_0}\|P_q(t)-P_q(t_0)\|_{\mathcal{L}(X)}=0.$$

As a consequence, $Q_q(t)$ and $P_q(t)$ are norm-continuous for $t>0$. □

LEMMA 2.2. *Let $p>1$ be such that $pq>1$. Assume that $R(t)$ is norm-continuous for $t>0$. Define the operator*

$$\Phi:\mathbb{L}^p([0,a];X)\to\mathcal{C}([0,a];X)$$

by

$$\Phi(v)(t)=\int_0^t(t-s)^{q-1}P_q(t-s)v(s)ds,$$

for $v\in\mathbb{L}^p([0,a];X)$, and $t\in[0,a]$. Then, Φ takes each bounded set in $\mathbb{L}^p([0,a];X)$ to an equicontinuous one in $\mathcal{C}([0,a];X)$.

Proof. Let Ω be a bounded set in $\mathbb{L}^p([0,a];X)$, $v \in \Omega$, and $0 \leqslant t_1 < t_2 \leqslant a$. Then,

$$\begin{aligned}
&\|\Phi(v)(t_1) - \Phi(v)(t_2)\| \\
= &\left\| \int_0^{t_1} (t_1 - s)^{q-1} P_q(t_1 - s)v(s)ds - \int_0^{t_2} (t_2 - s)^{q-1} P_q(t_2 - s)v(s)ds \right\| \\
\leqslant &\int_0^{t_1} \| \left[(t_1 - s)^{q-1} P_q(t_1 - s) - (t_2 - s)^{q-1} P_q(t_2 - s) \right] v(s) \| ds \\
&+ \int_{t_1}^{t_2} \|(t_2 - s)^{q-1} P_q(t_2 - s)v(s)\| ds.
\end{aligned}$$

Let

$$I(t_1,t_2) = \int_0^{t_1} \|(t_1 - s)^{q-1} P_q(t_1 - s)v(s) - (t_2 - s)^{q-1} P_q(t_2 - s)v(s)\| ds.$$

Then,

$$\begin{aligned}
I(t_1,t_2) \leqslant &\int_0^{t_1} (t_1 - s)^{q-1} \|P_q(t_1 - s) - P_q(t_2 - s)\|_{\mathcal{L}(X)} \|v(s)\| ds \\
&+ \frac{Mq}{\Gamma(q+1)} \int_0^{t_1} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| \|v(s)\| ds \\
\leqslant &\left[\int_0^{t_1} [(t_1 - s)^{q-1} \|P_q(t_1 - s) - P_q(t_2 - s)\|_{\mathcal{L}(X)}]^{\frac{p}{p-1}} ds \right]^{\frac{p-1}{p}} \|v\|_{\mathbb{L}^p} \\
&+ \frac{Mq}{\Gamma(q+1)} \left[\int_0^{t_1} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}|^{\frac{p}{p-1}} ds \right]^{\frac{p-1}{p}} \|v\|_{\mathbb{L}^p} \\
= &\left[\int_0^{t_1} [s^{q-1} \|P_q(s) - P_q(t_2 - t_1 + s)\|_{\mathcal{L}(X)}]^{\frac{p}{p-1}} ds \right]^{\frac{p-1}{p}} \|v\|_{\mathbb{L}^p} \\
&+ \frac{Mq}{\Gamma(q+1)} \left[\int_0^{t_1} |s^{q-1} - (t_2 - t_1 + s)^{q-1}|^{\frac{p}{p-1}} ds \right]^{\frac{p-1}{p}} \|v\|_{\mathbb{L}^p}.
\end{aligned}$$

Since,

$$\lim_{t_2 \to t_1} s^{q-1} \|P_q(s) - P_q(t_2 - t_1 + s)\|_{\mathcal{L}(X)} = 0, \quad \text{a.e} \quad s \in [0,t_1],$$

$$\lim_{t_2 \to t_1} |s^{q-1} - (t_2 - t_1 + s)^{q-1}| = 0, \quad \text{a.e} \quad s \in [0,t_1],$$

$$s^{q-1} \|P_q(s) - P_q(t_2 - t_1 + s)\|_{\mathcal{L}(X)} \leqslant \frac{2qM}{\Gamma(1+q)} \frac{1}{s^{1-q}},$$

and

$$|s^{q-1} - (t_2 - t_1 + s)^{q-1}| \leqslant \frac{2}{s^{1-q}},$$

using the Lebesgue dominated convergence Theorem, we can affirm that $\lim_{t_2\to t_1} I(t_1,t_2) = 0$ uniformly for $g \in \Omega$. Now, let $J(t_1,t_2) = \int_{t_1}^{t_2} \|(t_2-s)^{q-1}P_q(t_2-s)v(s)\|ds$, then

$$J(t_1,t_2) \leqslant \frac{qM}{\Gamma(1+q)} \left(\int_{t_1}^{t_2} |t_2-s|^{\frac{(q-1)p}{p-1}}\right)^{\frac{p-1}{p}} \|v\|_{\mathbb{L}^p}$$
$$= \frac{qM}{\Gamma(1+q)} \sqrt[p]{\left(\frac{p-1}{pq-1}\right)^{p-1}} (t_2-t_1)^{(pq-1)/p}\|v\|_{\mathbb{L}^p},$$

which implies that $\lim_{t_2\to t_1} J(t_1,t_2) = 0$ uniformly for $v \in \Omega$. $\square$

Let consider the following linear equation:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \int_0^t G(t-s)x(s)ds + f(t), & t \geqslant 0 \\ x(0) = x_0, \end{cases} \tag{5}$$

where $f \in \mathbb{L}^p_{loc}(\mathbb{R}^+, X)$. By the definition of the Gamma function $\Gamma(\cdot)$, and the Caputo fractional derivative, we can rewrite the Cauchy problem (5) in the following equivalent form:

$$\begin{cases} x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[Ax(s) + \int_0^s G(s-r)x(r)dr + f(s)\right] ds, & t \geqslant 0 \\ x(0) = x_0, \end{cases} \tag{6}$$

provided that this integral exists.

THEOREM 2.3. *If x is a solution of equation (6), then*

$$x(t) = Q_q(t)x_0 + \int_0^t (t-s)^{q-1}P_q(t-s)f(s)ds, \quad t \geqslant 0. \tag{7}$$

Proof. Let $\xi > 0$. We apply the Laplace transform to equation (6), it follows that

$$\widehat{x}(\xi) = \frac{1}{\xi}x_0 + \frac{1}{\xi^q}A\widehat{x}(\xi) + \frac{1}{\xi^q}\widehat{G}(\xi)\widehat{x}(\xi) + \frac{1}{\xi^q}\widehat{f}(\xi),$$

where

$$\widehat{x}(\xi) = \int_0^{+\infty} e^{-\xi s}x(s)ds, \quad \widehat{f}(\xi) = \int_0^{+\infty} e^{-\xi s}f(s)ds,$$

and

$$\widehat{G}(\xi)x = \int_0^{+\infty} e^{-\xi s}G(s)xds, \quad x \in D(A).$$

Hence,

$$(\xi^q I - A - \widehat{G}(\xi))\widehat{x}(\xi) = \xi^{q-1}x_0 + \widehat{f}(\xi).$$

On one side, we have

$$\begin{aligned}
&\int_0^{+\infty} e^{-\xi^q s}R(s)(\xi^q I - A - \widehat{G}(\xi))\widehat{x}(\xi)ds \\
&= \int_0^{+\infty} \xi^q e^{-\xi^q s}R(s)\widehat{x}(\xi)ds - \int_0^{+\infty} e^{-\xi^q s}R(s)[A\widehat{x}(\xi) + \widehat{G}(\xi)\widehat{x}(\xi)]ds \\
&= \widehat{x}(\xi) + \int_0^{+\infty} e^{-\xi^q s}R'(s)\widehat{x}(\xi)ds - \int_0^{+\infty} e^{-\xi^q s}R(s)[A\widehat{x}(\xi) + \widehat{G}(\xi)\widehat{x}(\xi)]ds.
\end{aligned}$$

Since

$$R'(s)\widehat{x}(\xi) = R(s)A\widehat{x}(\xi) + \int_0^s R(s-r)G(r)\widehat{x}(\xi)dr, \quad s \geqslant 0,$$

it follows that

$$\begin{aligned}
&\int_0^{+\infty} e^{-\xi^q s}R(s)(\xi^q I - A - \widehat{G}(\xi))\widehat{x}(\xi)ds \\
&= \widehat{x}(\xi) + \int_0^{+\infty} e^{-\xi^q s}\left[-R(s)\widehat{G}(\xi)\widehat{x}(\xi)\int_0^s R(s-r)G(r)\widehat{x}(\xi)dr\right]ds \\
&= \widehat{x}(\xi) + \int_0^{+\infty} e^{-\xi s}e^{(\xi-\xi^q)s}\left[-R(s)\widehat{G}(\xi)\widehat{x}(\xi) + \int_0^s R(s-r)G(r)\widehat{x}(\xi)dr\right]ds \\
&= \widehat{x}(\xi) + \frac{I}{\xi^q} * \left[\widehat{R}(\xi)\widehat{G}(\xi) - \widehat{R}(\xi)\widehat{G}(\xi)\right]\widehat{x}(\xi) \\
&= \widehat{x}(\xi).
\end{aligned}$$

Since $\int_0^{+\infty} e^{-\xi\theta}\phi_q(\theta)d\theta = e^{-\xi^q}$, and $\Psi_q(\theta) = \frac{1}{q}\theta^{-1-1/q}\phi_q(\theta^{-1/q})$, $q \in (0,1)$ for

$$\phi_q(\theta) = \frac{1}{\pi}\Sigma_{n=1}^{+\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,+\infty),$$

it follows that

$$\begin{aligned}
&\int_0^{+\infty} \xi^{q-1}e^{-\xi^q s}R(s)x_0 ds \\
&= \int_0^{+\infty} q(\xi t)^{q-1}e^{-(\xi t)^q}R(t^q)x_0 dt \\
&= \int_0^{+\infty} -\frac{1}{\xi}\frac{d}{dt}[e^{-(\xi t)^q}]R(t^q)x_0 dt \\
&= \int_0^{+\infty} -\frac{1}{\xi}\frac{d}{dt}\left[\int_0^{+\infty} e^{-\xi t\theta}\phi_q(\theta)d\theta\right]R(t^q)x_0 dt \\
&= \int_0^{+\infty}\int_0^{+\infty} \theta\phi_q(\theta)e^{-\xi t\theta}d\theta R(t^q)x_0 dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} e^{-\xi s} \int_0^{+\infty} \phi_q(\theta) R(\frac{s^q}{\theta^q}) x_0 d\theta ds \\
&= \int_0^{+\infty} e^{-\xi s} \int_0^{+\infty} \frac{1}{q} \theta^{1-1/q} \phi_q(\theta^{-1/q}) R(s^q \theta) x_0 d\theta ds \\
&= \int_0^{+\infty} e^{-\xi s} \int_0^{+\infty} \Psi_q(\theta) R(s^q \theta) x_0 d\theta ds \\
&= \int_0^{+\infty} e^{-\xi s} Q_q(s) x_0 ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\xi^q s} R(s) \widehat{f}(\xi) ds \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} q\phi_q(\theta) e^{-\xi t\theta} R(t^q) e^{-\xi s} f(s) d\theta ds dt \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} q\phi_q(\theta) e^{-\xi(t+s)} R(\frac{t^q}{\theta^q}) \frac{t^{q-1}}{\theta^q} f(s) d\theta ds dt \\
&= \int_0^{+\infty} \left(\int_0^{+\infty} \int_0^{+\infty} q\phi_q(\theta) e^{-\xi(t+s)} R(\frac{t^q}{\theta^q}) \frac{t^{q-1}}{\theta^q} d\theta dt \right) f(s) ds \\
&= \int_0^{+\infty} \left(\int_s^{\infty} \int_0^{+\infty} q\phi_q(\theta) e^{-\xi t} R(\frac{(t-s)^q}{\theta^q}) \frac{(t-s)^{q-1}}{\theta^q} d\theta dt \right) f(s) ds \\
&= \int_0^{+\infty} e^{-\xi t} \left(q \int_0^t \int_0^{+\infty} \phi_q(\theta) R(\frac{(t-s)^q}{\theta^q}) \frac{(t-s)^{q-1}}{\theta^q} f(s) d\theta ds \right) dt \\
&= \int_0^{+\infty} e^{-\xi t} \left(\int_0^t (t-s)^{q-1} \left(\int_0^{+\infty} q\theta \Psi_q(\theta) R((t-s)^q \theta) d\theta \right) f(s) ds \right) dt \\
&= \int_0^{+\infty} e^{-\xi t} \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds dt.
\end{aligned}
$$

By the inverted Laplace transform, we get that

$$
x(t) = Q_q(t) x_0 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds, \quad \text{for } t \geqslant 0. \quad \square
$$

DEFINITION 2.3. A function $x \in \mathcal{C}([0,a];X)$ which satisfies (7) for $t \in [0,a]$ is called a mild solution of equation (5) on $[0,a]$.

LEMMA 2.3. *Assume that $R(t)$ is norm-continuous for $t > 0$. If D is a relatively compact subset of X and K is a bounded set of $\mathbb{L}^p([0,a];X)$. Then, the set of mild solutions of equation (5), $\{x(\cdot,x_0,f); x_0 \in D; f \in K\}$ is equicontinuous in $\mathcal{C}([0,a];X)$.*

Proof. Let define $\Omega(D \times K) = \{x(\cdot,x_0,f)\colon (x_0,f) \in D \times K\}$, where $x(\cdot,x_0,f)$ is the mild solution of equation (5) corresponding to $x_0 \in D$ and $f \in K$. Then,

$$
\Omega(D \times K) = \{Q_q(\cdot)x_0\colon x_0 \in D\} + \Phi(K),
$$

where Φ is the mapping defined in Lemma 2.2. The set $\{Q_q(\cdot)x_0\colon\ x_0 \in D\}$ is relatively compact. In fact let $y_n = Q_q(\cdot)x_0^n$ such that x_0^n is a bounded sequence in D. Since D is relatively compact, there exists a subsequence $x_0^{\psi(n)}$ of x_0^n such that $x_0^{\psi(n)}$ converges to some $x_0 \in D$. Let $y = Q_q(\cdot)x_0$, then

$$\|y_{\psi(n)} - y\|_\infty \leqslant M\|x_0^{\psi(n)} - x_0\|_X,$$

which implies that $\{Q_q(\cdot)x_0\colon\ x_0 \in D\}$ is relatively compact. Then, $\{Q_q(\cdot)x_0\colon\ x_0 \in D\}$ is equicontinous in $\mathcal{C}([0,a],X)$. Since K is bounded in $\mathbb{L}^p([0,a];X)$, by Lemma 2.2, we conclude that $\Omega(D \times K)$ is equicontinous in $\mathcal{C}([0,a];X)$. $\square$

2.3. Multivalued analysis

Let Y and Z be two metric spaces. Denote by

$$C(Z) = \{D \in 2^Z\colon\ D \text{ is closed }\},$$

$$C_v(Z) = \{D \in C(Z)\colon\ D \text{ is convex}\},$$

and

$$K(Z) = \{D \in C(Z)\colon\ D \text{ is compact }\}.$$

Let $\beta : Y \to 2^Z$ be a multi-valued map, and $Gra(\beta) = \{(w,y); y \in \beta(w)\}$ be the graph of β. Denote by $\beta^{-1}(D) = \{y \in Y; \beta(y) \cap D \neq \emptyset\}$ the complete preimage of D under β, where $D \in 2^Z$. Then,

i) β is called closed, if $Gra(\beta)$ is closed in $Y \times Z$.

ii) β is called quasi-compact, if $\beta(D)$ is relatively compact for each compact set D of Y.

iii) β is called upper semi-continuous (shortly u.s.c), if $\beta^{-}(D)$ is a closed subset of Y for each closed set D of Z, and lower semi-continuous (shortly l.s.c), if $\beta^{-1}(D)$ is an open subset of Y for each open set D of Z.

LEMMA 2.4. [16] *Let $\beta : Y \to K(Z)$ be a closed quasi-compact multi-valued map. Then, β is an u.s.c map.*

EXAMPLE 2.1. The multi-map $\beta : [0,1] \to 2^{[0,1]}$ defined as

$$\beta(x) = \begin{cases} [0,1/2], & \text{if } x \neq 1/2 \\ \\ [0,1], & \text{if } x = 1/2 \end{cases}$$

is u.s.c.

In what follows we consider Y and Z as Banach spaces. A multi-valued map $\beta : D \subset Y \to 2^Z$ is called weakly upper semi-continuous (shortly w.u.s.c), if $\beta^{-}(D^{'})$ is closed in D for each closed set $D^{'}$ of Z. It is clear that an u.s.c multi-valued map is a w.u.s.c one.

LEMMA 2.5. [44, Lemma 2.2(ii)] *Let* $\beta : D \subset Y \to 2^Z$ *be a multi-valued map with convex weakly compact values. Then,* β *is w.u.s.c if and only if for each sequence* $\{(y_n, z_n)\}_{n\geqslant 1} \subset D \times Z$ *such that* $y_n \to y$ *(strongly) in* Y *and* $z_n \in \beta(y_n)$, $n \geqslant 1$, *it follows that there exists a subsequence* $(z_{n_k})_{k\geqslant 1}$ *of* $(z_n)_{n\geqslant 1}$ *and* $z \in \beta(y)$ *such that* $z_{n_k} \rightharpoonup z$ *(weakly) in* Z*, as* $k \to +\infty$.

LEMMA 2.6. [6, Lemma 1] *Let* D *be a nonempty, compact and convex part of a Banach space, and* $\beta : D \to 2^D$ *is an u.s.c multi-valued map with a closed contractible values. Then,* β *has at least one fixed point.*

2.4. Measure of noncompactness

Next, we recall some results concerning the measure of noncompactness.

DEFINITION 2.4. Let Ω be a bounded set of X. The measure of noncompactness of Ω is given by the value

$$\nu(\Omega) = \inf\{\varepsilon > 0 : \ \Omega \text{ has a finite cover by closed balls of radius } < \varepsilon\}.$$

THEOREM 2.4. [16] *Let* Z *be a Banach space. Denote by* $\mathcal{P}(Z)$ *the collection of all subsets of* Z*. We have the following properties:*

i) *If* Ω_1, $\Omega_2 \in \mathcal{P}(Z)$ *such that* $\Omega_1 \subset \Omega_2$*, then* $\nu(\Omega_1) \leqslant \nu(\Omega_2)$.

ii) $\nu(\Omega) = 0$ *if and only if* Ω *is relatively compact.*

iii) $\nu(\Omega_1 + \Omega_2) \leqslant \nu(\Omega_1) + \nu(\Omega_2)$*, for* Ω_1, $\Omega_2 \in \mathcal{P}(Z)$.

iv) $\nu(\overline{conv}(\Omega)) = \nu(\Omega)$*, where* $\overline{conv}(\Omega)$ *is the closed convex hull of* $\Omega \in \mathcal{P}(Z)$.

v) *If* $L : Z \to Z$ *is a Lipschitzian map with* $Lip(L) > 0$*, then*

$$\nu(L(\Omega)) \leqslant Lip(L)\nu(\Omega), \quad \textit{for} \ \ \Omega \in \mathcal{P}(Z).$$

vi) $\nu(\kappa\Omega) = |\kappa|\nu(\Omega)$ *for* $\kappa \in \mathbb{R}$*, and* $\Omega \in \mathcal{P}(Z)$.

LEMMA 2.7. [6, Theorem 2] *For every bounded set* D *of* X *and* $\varepsilon > 0$*, there exists a sequence* $(w_n)_{n\geqslant 1} \subset D$ *such that*

$$\nu(D) \leqslant 2\nu(\{w_n : \ n \geqslant 1\}) + \varepsilon.$$

LEMMA 2.8. [16] *If* $(f_n)_{n\geqslant 1} \subset \mathbb{L}^1([0,a];X)$ *is a bounded sequence. Then,*

$$\nu\left(\left\{\int_0^t f_n(s)ds : \ n \geqslant 1\right\}\right) \leqslant 2\int_0^t \nu(\{f_n(s)ds : \ n \geqslant 1\})ds, \quad t \in [0,a].$$

2.5. Duality mapping

Let $(Z,\|\cdot\|_Z)$ be a real linear normed space and $(Z^*,\|\cdot\|_{Z^*})$ its dual. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between Z and Z^*.

DEFINITION 2.5. A normed linear space Z is called smooth if for every $x \in Z$, with $\|x\|_Z = 1$, there exists a unique $x^* \in Z^*$ such that $\|x^*\|_{Z^*} = 1$, and $\langle x^*, x\rangle = \|x\|_Z$.

DEFINITION 2.6. A normed linear space Z is called strictly convex if and only if $\|x+y\|_Z = \|x\|_Z + \|y\|_Z$ implies $x = c\cdot y$ for $c > 0$ whenever $x \neq 0$ and $y \neq 0$.

Without loss of generality and from Asplund Theorem (see [5, page: 36]), we can assume that X and X^* are simultaneously smooth and strictly convex. We denote by $\|\cdot\|_*$ the norm metric equipped to the de space X^*.

DEFINITION 2.7. [5] The operator $J : X \to \mathcal{P}(X^*)$ defined by

$$J(x) = \left\{x^* \in X^* : \ \langle x, x^*\rangle = \|x\|^2 = \|x^*\|_*^2\right\},$$

is called the duality mapping of X.

LEMMA 2.9. [5] *The duality mapping is single-valued. Moreover,*

1. *J is invertible from X to X^*, and $J^{-1} : X^* \to X$ is also a duality mapping.*
2. *J is demi-continuous i-e J is continuous from X with strong topology into X^* with weak topology.*
3. *J is strictly monotonic.*

REMARK 1. If X is a Hilbert space, then the duality mapping J is exactly the identity map.

LEMMA 2.10. *Let $(f_n)_{n\geqslant 1} \subset \mathbb{L}^p([0,a];X)$, $p > 1$ with $pq > 1$, and $(x_n)_{n\geqslant 1} \subset \mathcal{C}([0,a];X)$ be two sequences such that x_n is a mild solution of the following system:*

$$\begin{cases} {}^cD_t^q x_n(t) = Ax_n(t) + \displaystyle\int_0^t G(t-s)x_n(s)ds + f_n(t), \quad t \in [0,a] \\ x_n(0) = x_{n,0} \in X, \end{cases}$$

where $f_n \rightharpoonup f$ (weakly) in $\mathbb{L}^p([0,a];X)$, $x_n \rightharpoonup x$ (weakly) in $\mathcal{C}([0,a];X)$, and $x_{n,0} \rightharpoonup x_0$ in X. Then, x is a mild soltion of the following equation:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \displaystyle\int_0^t G(t-s)x(s)ds + f(t), \quad t \in [0,a] \\ x(0) = x_0. \end{cases} \tag{8}$$

Proof. Let define $\tilde{\Phi} : X \times \mathbb{L}^p([0,a];X) \to \mathcal{C}([0,a];X)$, by $\tilde{\Phi}(x_0, f) = x(\cdot, x_0, f)$ such that $x(\cdot, x_0, f)$ is the mild solution of equation (8). $\tilde{\Phi}$ is a linear bounded map. Moreover,

$$\|\tilde{\Phi}(x_0, f)\| \leqslant M\|x_0\| + \frac{qM}{\Gamma(q+1)} \sqrt[p]{\left(\frac{p-1}{pq-1}\right)^{p-1}} a^{(pq-1)/p} \|f\|_{\mathbb{L}^p},$$

for $x_0 \in X$, and $f \in \mathbb{L}^p([0,a];X)$. Hence,

$$x_n = \tilde{\Phi}(x_0, f_n) \rightharpoonup \tilde{\Phi}(x_0, f) \text{ weakly in } \mathcal{C}([0,a];X).$$

Due to the uniqueness of the weak limit, we can affirm that $x = \tilde{\Phi}(x_0, f)$. □

LEMMA 2.11. [14, Lemma 2] *Let $v(\cdot)$, $w(\cdot) : [0,a] \to [0,+\infty[$ be two continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\delta > 0$, $0 < \gamma < 1$ such that*

$$v(t) \leqslant w(t) + \delta \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds, \qquad t \in [0,a].$$

Then,

$$v(t) \leqslant \left(e^{\delta^n \Gamma(\gamma)^n t^{\gamma n}/\Gamma(n\gamma)}\right) \Sigma_{j=0}^{n-1} \left(\frac{\delta a^\gamma}{\gamma}\right)^j w(t),$$

for $t \in [0,a]$ and $n \in \mathbb{N}^*$ such that $n\gamma > 1$.

REMARK 2. Let $\bar{g} : [0,a] \to \mathbb{R}^+$ be a continuous function. Then, $t \to \sup_{\sigma \in [0,t]} \bar{g}(\sigma)$ is a continuous function from $[0,a]$ into $\mathbb{R}^+$.

LEMMA 2.12. [42, Corollary 1.3.1] *Let X be a reflexive Banach space, and $1 < p < +\infty$. A subset $\Omega \subset \mathbb{L}^p([0,a];X)$ is weakly relatively sequentially compact in $\mathbb{L}^p([0,a];X)$ if and only if Ω is bounded in $\mathbb{L}^p([0,a];X)$.*

DEFINITION 2.8. A mild solution of equation (1)–(2) is a function $x \in \mathcal{C}([-r,a];X)$ such that

$$x(t) = \phi(t), \quad \text{for} \quad t \in [-r,0],$$

and

$$x(t) = Q_q(t)[\phi(0) + h(0,\phi)] - h(t,x_t) + \int_0^t (t-s)^{q-1} P_q(t-s)[Bu(s) + f(s)]ds, \quad t \in [0,a],$$

for $f \in \mathbb{L}^p([0,a];X)$ such that $f(t) \in F(t,x(t),x_t)$ a.e $t \in [0,a]$, and $u \in \mathbb{L}^p([0,a];U)$, $p > 1$.

DEFINITION 2.9. A function $x : [-r,a] \to X$ is called a mild solution of equation (1)–(3), if

$$x(t) = g(x)(t), \quad \text{for} \quad t \in [-r,0],$$

and

$$x(t) = Q_q(t)[g(x)(0) + h(0,g(x))] - h(t,g(x)) + \int_0^t (t-s)^{q-1} P_q(t-s)[Bu(s) + f(s)]ds, \\ t \in [0,a],$$

for $f \in \mathbb{L}^p([0,a];X)$ such that $f(t) \in F(t,x(t),x_t)$ a.e $t \in [0,a]$, and $u \in \mathbb{L}^p([0,a];U)$, $p > 1$.

DEFINITION 2.10. Equation (1)–(2) is said to be approximately controllable on $[0,a]$ if for each $d \in X$, and $\varepsilon > 0$, there exists a control function $u_\varepsilon(\cdot) \in \mathbb{L}^p([0,a];X)$ such that the mild solution $x(\cdot,u_\varepsilon)$ of equation (1)–(2) corresponding to $u_\varepsilon(\cdot)$ satisfies $x(t,u_\varepsilon) = \phi(t)$ for $t \in [-r,0]$, and

$$\|x(a,u_\varepsilon) - d\| < \varepsilon.$$

DEFINITION 2.11. Equation (1)–(3) is said to be approximately controllable on $[0,a]$ if for each $d \in X$, and $\varepsilon > 0$, there exists a control function $u_\varepsilon(\cdot) \in \mathbb{L}^p([0,a];X)$ such that the mild solution $x(\cdot,u_\varepsilon)$ of equation (1)–(3) corresponding to $u_\varepsilon(\cdot)$ satisfies $x(t,u_\varepsilon) = g(x)(t)$ for $t \in [-r,0]$, and

$$\|x(a,u_\varepsilon) - d\| < \varepsilon.$$

3. Controllability results

This section focuses primarily on the approximate controllability of both equations (1)–(2) and (1)–(3). The following are the key hypotheses on map F for proving the main results.

($\mathbf{H_3}$) $F(t,\cdot,\cdot)$ is w.u.s.c for a.e $t \in [0,a]$ and $F(\cdot,x,\phi)$ has a $\mathbb{L}^p$-integral selection for $(x,\phi) \in X \times \mathcal{C}$.

($\mathbf{H_4}$) There is a bounded function $\eta : [0,a] \to \mathbb{R}^+$ such that

$$\begin{aligned} \|F(t,x,\phi)\| &= \sup\{\|f(t)\| : f(t) \in F(t,x,\phi)\} \\ &\leqslant \eta(t)(1 + \|x\| + \|\phi\|_{\mathcal{C}}), \end{aligned}$$

for a.e $t \in [0,a]$, and $(x,\phi) \in X \times \mathcal{C}$.

($\mathbf{H_5}$) There exists $\mu(\cdot) \in \mathcal{C}([0,a];\mathbb{R}^+)$ such that for each bounded subset Ω of X, and each bounded subset Σ of $\mathcal{C}$, we have

$$\nu(F(t,f,\phi)) \leqslant \mu(t)[\nu(\Omega) + \nu(\Sigma(\gamma_\phi))], \text{ for } t \in [0,a] \text{ and } (f,\phi) \in \Omega \times \Sigma,$$

where

$$\gamma_\phi = \min\{\theta \in [-r,0] : \|\phi\|_{\mathcal{C}} = \|\phi(\theta)\|_X\}, \text{ and } \Sigma(\gamma_\phi) = \{\psi(\gamma_\phi);\ \psi \in \Sigma\}, \quad \forall\, \phi \in \Sigma.$$

Let

$$\eta_0 = \sup\{\eta(t) :\ t \in [0,a]\} \text{ and } \mu_0 = \sup\{\mu(t) :\ t \in [0,a]\}.$$

An example satisfying $(\mathbf{H_3})$–$(\mathbf{H_5})$ is provided below.

EXAMPLE 3.1. Let $X = \mathbb{L}^2([0,1];\mathbb{R})$. Let Ω be a bounded subset of X, and Σ be a bounded subset of $\mathcal{C}$. Let $F : [0,a] \times X \times \mathcal{C} \to 2^X$ be such that

$$F(t,f,\phi) = \{y(t) \in X \text{ such that } \|y(t)\| \leqslant \max(\|f\|, \|\phi\|_{\mathcal{C}})\}, \text{ for } t \in [0,a] \text{ and } (f,\phi) \in \Omega \times \Sigma.$$

It is clear that $f, \phi(\gamma_\phi) \in F(t,f,\phi)$ then $F(t,f,\phi) \neq \emptyset$. We can also show that $(\mathbf{H_4})$ hold with $\eta(\cdot) = 1$. For $(\mathbf{H_3})$, it is easily seen that F is a multi-valued map with a convex weakly compact values. Let $(f_n,\phi_n)_{n\geqslant 1}$ be a sequence in $X \times \mathcal{C}$ such that $(f_n,\phi_n) \to (f,\phi)$ strongly in $X \times \mathcal{C}$. Let $y_n(t) \in F(t,f_n,\phi_n)$, then

$$\|y_n(t)\| \leqslant \max(\|f_n\|, \|\phi_n\|_{\mathcal{C}}),$$

which implies that $y_n(t)$ is bounded in X. Hence, then there exists a subsequence $(y_{n_k}(t))_{k\geqslant 0}$ of $(y_n(t))_{n\geqslant 0}$ such that $y_{n_k}(t)$ converges weakly to $y(t)$ in X as $k \to +\infty$. Moreover,

$$|\langle y_{n_k}(t), y(t)\rangle| \leqslant \max(\|f_{n_k}\|, \|\phi_{n_k}\|_{\mathcal{C}})\|y(t)\|.$$

Letting $k \to \infty$, we obtain that $\|y(t)\| \leqslant \max(\|f\|, \|\phi\|)$, then $y(t) \in F(t,f,\phi)$. Applying Lemma 2.5, we obtain that $F(t,\cdot,\cdot)$ is w.u.s.c. For $(\mathbf{H_5})$, let $(t,f,\phi) \in [0,a] \times \Omega \times \Sigma$ and $y(t) \in F(t,f,\phi)$.

1. If $\|y(t)\| \leqslant \|f\|$, then $y(t) \in \Omega$.
2. If $\|y(t)\| \leqslant \|\phi\|_{\mathcal{C}}$, let $g : [-r, t-\gamma_\phi] \to X$ such that

$$g(s) = \begin{cases} \dfrac{1+s}{1+t-\gamma_\phi} y(t), & \text{if} \quad s \in [0, t-\gamma_\phi] \\[2ex] \dfrac{1}{1+t-\gamma_\phi} y(t), & \text{if} \quad s \in [-r, 0[. \end{cases}$$

It is a simple matter to prove that g is a continuous function, $g(t-\gamma_\phi) = y(t)$, and

$$\|g\| = \sup_{s\in[-r,t-\gamma_\phi]} \|g(s)\| = \|y(t)\|.$$

Let $\psi : [-r,0] \to X$ such that

$$\psi(\theta) = \begin{cases} y(t), & \text{if} \quad \theta \in [-r, \gamma_\phi[\\ g(t-\theta), & \text{if} \quad \theta \in [\gamma_\phi, 0], \end{cases}$$

it is clear that ψ is a continuous function, and $\psi(\gamma_\phi) = g(t-\gamma_\phi) = y(t)$, moreover

$$\|\psi\|_{\mathcal{C}} \leqslant \|y(t)\| \leqslant \|\phi\|_{\mathcal{C}},$$

then $y(t) = \psi(\gamma_\phi) \in \Sigma(\gamma_\phi)$.

From 1) and 2), we deduce that $F(t,f,\phi) \subseteq \Omega \cup \Sigma(\gamma_\phi)$ for $t \in [0,a]$ and $(f,\phi) \in \Omega \times \Sigma$. Then $(\mathbf{H_5})$ is satisfied with $\mu(\cdot) = 1$.

We define the multi-valued map $Sel_F : \mathcal{C}([-r,a];X) \to 2^{\mathbb{L}^p([0,a];X)}$ by

$$Sel_F(x) = \{f \in \mathbb{L}^p([0,a];X) \text{ such that } f(t) \in F(t,x(t),x_t) \text{ a.e } t \in [0,a]\}.$$

LEMMA 3.1. [44, Lemma 3.3] *Let $p > 1$ be such that $pq > 1$, assume that $(\mathbf{H_3})$–$(\mathbf{H_4})$ hold. Then, Sel_F is w.u.s.c with a nonempty, convex, and weakly compact values.*

3.1. Existence of mild solutions for equation (1)–(2) with $h = 0$

In this subsection, we study the existence of the mild solution of equation (1)–(2) with $h = 0$. We recall that equation (1)–(2) with $h = 0$ takes the following form:

$$\begin{cases} {}^cD_t^q x(t) \in Ax(t) + \int_0^t G(t-s)x(s)ds + F(t,x(t),x_t) + Bu(t), & t \in [0,a] \\ x(t) = \phi(t), & t \in [-r,0]. \end{cases} \tag{9}$$

We consider the following result.

THEOREM 3.1. *Let $p > 1$ be such that $pq > 1$. Assume that $(\mathbf{H_3})$–$(\mathbf{H_5})$ hold, and $R(t)$ is norm-continuous for $t > 0$. Then, for each $\phi \in \mathcal{C}$, equation (9) has at least one mild solution on $[0,a]$.*

Proof. Let $\phi \in \mathcal{C}$. We define the following multi-valued map

$$\mathcal{H} : \mathcal{C}([-r,a];X) \to 2^{\mathcal{C}([-r,a];X)}$$

by

$$\mathcal{H}(x) = \mathcal{G}(Sel_F(x)),$$

where,

$$\mathcal{G}(f)(t) = \begin{cases} Q_q(t)\phi(0) + \int_0^t (t-s)^{q-1} P_q(t-s)[f(s) + Bu(s)]ds, & t \in [0,a], \\ \phi(t), & t \in [-r,0], \end{cases}$$

for $f \in Sel_F(x)$. Let $u(\cdot) \in \mathbb{L}^p([0,a];U)$. We split the proof into three steps.

Step 1: We show that there exists a nonempty, convex, and compact set $D \subset \mathcal{C}([-r,a];X])$ such that $\mathcal{H}(D) \subset 2^D$. Let $x \in \mathcal{C}([-r,a];X)$, and $f \in Sel_F(x)$. Let $z \in \mathcal{H}(x)$, then

$$\begin{aligned}\|z(t)\| &\leqslant \|Q_q(t)\phi(0)\| + \int_0^t (t-s)^{q-1}\|P_q(t-s)[f(s)+Bu(s)]\|ds \\ &\leqslant M\|\phi(0)\| + \frac{qM}{\Gamma(q+1)}\int_0^t (t-s)^{q-1}[\|f(s)\| + \|Bu(s)\|]ds.\end{aligned}$$

Therefore,

$$\|x_t\|_{\mathcal{C}} \leqslant \|\phi\|_{\mathcal{C}} + \sup\{\|x(s)\|,\ s\in[0,t]\}.$$

By assumption $(\mathbf{H_4})$, we have $\|f(s)\| \leqslant \eta(s)(1+\|x(s)\|+\|x_s\|_{\mathcal{C}})$. Then,

$$\begin{aligned}\|z(t)\| \leqslant{}& M\|\phi(0)\| + \frac{qM}{\Gamma(q+1)}\left[\int_0^t (t-s)^{\frac{pq-p}{p-1}}ds\right]^{\frac{p-1}{p}}\|B\|_{\mathcal{L}(U,X)}\|u\|_{\mathbb{L}^1} \\ &+\frac{qM}{\Gamma(q+1)}\int_0^t \eta(s)(t-s)^{q-1}(1+\|x(s)\|+\|x_s\|_{\mathcal{C}})ds \\ \leqslant{}& M\|\phi(0)\| + \frac{qM}{\Gamma(q+1)}\left[\sqrt[p]{\left(\frac{p-1}{pq-1}\right)^{p-1}}a^{(pq-1)/p}\right]\|B\|_{\mathcal{L}(U,X)}\|u\|_{\mathbb{L}^1} \\ &+\frac{qM}{\Gamma(q+1)}\left[\sqrt[p]{\left(\frac{p-1}{pq-1}\right)^{p-1}}a^{(pq-1)/p}\right][(1+\|\phi\|_{\mathcal{C}})\|\eta\|_{\mathbb{L}^p}] \\ &+\frac{qM}{\Gamma(q+1)}\int_0^t \eta(s)(t-s)^{q-1}(\|x(s)\| + \sup_{\sigma\in[0,s]}\|x(\sigma)\|)ds \\ \leqslant{}& a_1 + 2a_2\eta_0\int_0^t (t-s)^{q-1}\sup_{\sigma\in[0,s]}\|x(\sigma)\|ds,\end{aligned}$$

where,

$$\begin{aligned}a_1 ={}& M\|\phi(0)\| + a_2\left[\sqrt[p]{\left(\frac{p-1}{pq-1}\right)^{p-1}}a^{(pq-1)/p}\right] \\ &\times\left(\|B\|_{\mathcal{L}(U,X)}\|u\|_{\mathbb{L}^1} + (1+\|\phi\|_{\mathcal{C}})\|\eta\|_{\mathbb{L}^p}\right)\end{aligned}$$

and

$$a_2 = \frac{qM}{\Gamma(q+1)}.$$

Moreover, we have

$$\sup_{\sigma\in[0,t]} \|z(s)\| \leqslant a_1 + 2a_2\eta_0 \int_0^t (t-s)^{q-1} \sup_{\sigma\in[0,s]} \|x(\sigma)\| ds.$$

Let $S_0(\cdot) \in \mathcal{C}([0,a];\mathbb{R}^+)$ be the unique solution of the following integral equation:

$$S_0(t) = a_1 + 2a_2\eta_0 \int_0^t (t-s)^{q-1} S_0(s) ds.$$

Let

$$D_0 = \left\{ x \in \mathcal{C}([-r,a];X) \;\middle|\; \begin{array}{l} x(t) = \phi(t) \text{ for } t \in [-r,0] \\ \\ \sup_{\sigma\in[0,t]} \|x(\sigma)\| \leqslant S_0(t) \text{ for } t \in [0,a] \end{array} \right\}.$$

One can see that D_0 is nonempty, closed, bounded, convex, and $\mathcal{H}(D_0) \subset 2^{D_0}$. Let $D_{k+1} = \overline{conv}(\cup_{x\in D_k} \mathcal{H}(x))$ for each $k \geqslant 0$. Then,

1. $D_{k+1} \subset D_k$, for each $k \geqslant 0$.
2. D_k is nonempty, bounded, convex, and closed, for each $k \geqslant 0$.
3. D_k is equicontinuous, for each $k \geqslant 2$.

Indeed, let $x \in D_k$, by assumption $(\mathbf{H_4})$, we can affirm that $Sel_F(x)$ is bounded in $\mathbb{L}^p([0,a];X)$. Since $(R(t))_{t\geqslant 0}$ is equicontinuous for $t>0$, we use Lemma 2.3 it follows that

$$\mathcal{H}(x) = \{z(\cdot,\phi(0),f);\ f \in Sel_F(x)\} \cup \{\phi\}$$

is equicontinuous in $\mathcal{C}([-r,a];X)$. Therefore, we have

$$\cup_{x\in D_{k+1}} \mathcal{H}(x) \subset D_{k+2} \subset \cup_{x\in D_k} \mathcal{H}(x) \subset D_{k+1} \subset D_k \quad for \quad k \geqslant 0. \tag{10}$$

Then, for each $k \geqslant 2$, D_k is equicontinuous. Now, we set $D = \cap_{k\geqslant 0} D_k$, it is obvious that D is closed, convex, nonempty, bounded and equicontinuous. $D(t) = \{x(t);\ x \in D\}$ is relatively compact. In fact, if $t \in [-r,0]$, we obtain that $\nu(D(t)) = 0$. If $t \in]0,a]$, it follows from (10) that

$$\nu(\cup_{x\in D_{k+1}} \mathcal{H}(x)(t)) \leqslant \nu(D_{k+2}(t)) \leqslant \nu(\cup_{x\in D_k} \mathcal{H}(x)(t)) \quad for \quad k \geqslant 0. \tag{11}$$

Let $k \geqslant 2$. Then, $\mathcal{M} = \{\mathcal{G}(f)(t);\ f \in \cup_{x\in D_k} Sel_F(x)\}$ is bounded in X, thus for every $\varepsilon > 0$, using Lemma 2.7, we can find a sequence $(w_n)_{n\geqslant 1} \subset \mathcal{M}$ such that

$$\nu(\mathcal{M}) \leqslant 2\nu(\{w_n;\ n \geqslant 1\}) + \varepsilon.$$

Then, for each $\varepsilon > 0$, there exists $(f_n)_{n\geqslant 1} \subset \cup_{x\in D_k} Sel_F(x)$ such that

$$\nu(\mathcal{M}) \leqslant 2\nu(\{\mathcal{G}(f_n)(t);\ n \geqslant 1\}) + \varepsilon. \tag{12}$$

By Lemma 2.8, we infer that

$$\begin{aligned}\nu(\{\mathcal{G}(f_n)(t):\ n\geqslant 1\}) &= \nu\left(\left\{\int_0^t (t-s)^{q-1}P_q(t-s)[Bu(s)+f_n(s)]ds:\ n\geqslant 1\right\}\right)\\ &\leqslant 2\int_0^t (t-s)^{q-1}\nu\left(\left\{P_q(t-s)f_n(s)ds:\ n\geqslant 1\right\}\right)\\ &\leqslant 2a_2\int_0^t (t-s)^{q-1}\nu\left(\{f_n(s)ds:\ n\geqslant 1\}\right).\end{aligned}$$

From (12), we obtain that

$$\nu(\mathcal{M})\leqslant 4a_2\int_0^t (t-s)^{q-1}\nu\left(\{f_n(s)ds:\ n\geqslant 1\}\right)+\varepsilon,$$

which implies that

$$\begin{aligned}\nu(\cup_{x\in D_k}\mathcal{H}(x)(t)) &\leqslant 4a_2\int_0^t (t-s)^{q-1}\nu\left(\{f_n(s)ds:\ n\geqslant 1\}\right)+\varepsilon\\ &\leqslant 4a_2\int_0^t (t-s)^{q-1}\nu\left(F(s,x(s),x_s)\right)ds+\varepsilon\\ &\leqslant 8a_2\int_0^t (t-s)^{q-1}\mu(s)\sup_{\sigma\in[0,s]}\nu(D_k(\sigma))ds+\varepsilon.\end{aligned}$$

According to (11), we infer that

$$\nu(D_{k+2}(t))\leqslant 8a_2\int_0^t (t-s)^{q-1}\mu(s)\sup_{\sigma\in[0,s]}\nu(D_k(\sigma))ds+\varepsilon,$$

which implies that

$$\lim_{k\to+\infty}\sup_{\sigma\in[0,t]}\nu(D_{k+2}(\sigma))\leqslant 8a_2\mu_0\int_0^t (t-s)^{q-1}\lim_{k\to+\infty}\sup_{\sigma\in[0,s]}\nu(D_k(\sigma))ds.$$

By remark 2, we have $t\to\sup_{\sigma\in[0,t]}\nu(D_k(\sigma))$ is a continuous function from $[0,a]$ to $\mathbb{R}^+$. Since $D\subset D_{k+1}\subset D_k$ for each $k\geqslant 0$, it follows that

$$t\to\sup_{\sigma\in[0,t]}\nu(D(\sigma))\leqslant t\to\sup_{\sigma\in[0,t]}\nu(D_{k+1}(\sigma))\leqslant t\to\sup_{\sigma\in[0,t]}\nu(D_k(\sigma))$$

which implies that $t\to\sup_{\sigma\in[0,t]}\nu(D_k(\sigma))$ is a bounded decreasing function. Then,

$$t\to\lim_{k\to+\infty}\sup_{\sigma\in[0,t]}\nu(D_k(\sigma))$$

is a continuous function. Using Lemma 2.11, we obtain that

$$\lim_{k\to+\infty}\sup_{\sigma\in[0,t]}\nu(D_k(t))=0,$$

then $\nu(D(t)) = 0$, that is $D(t)$ is relatively compact for each $t \in [-r,a]$. Using Ascoli-Arzéla Theorem, we shall that D is compact and $\mathcal{H}(D) \subset 2^D$.

Step 2: We show that $\mathcal{H}$ has a closed contractible values. In fact, let $(v_n,w_n)_{n\geqslant 1}$ be a sequence in $Gra(\mathcal{H})$ such that $(v_n,w_n) \to (v,w)$ (strongly) when n goes to infinity. To prove that $\mathcal{H}$ has a closed values, we need to prove that $(v,w) \in Gra(\mathcal{H})$. We have $(w_n) \in \mathcal{H}(v_n)$, $n \geqslant 1$. Then, there exists $(f_n)_{n\geqslant 1} \subset Sel_F(v_n)$ such that $w_n = \mathcal{G}(f_n)$. By Lemma 3.1, we get that Sel_F is w.u.s.c with nonempty, convex and weakly compact values. Since $v_n \to v$ (strongly) in $\mathcal{C}([-r,a];X)$ and $f_n \in Sel_F(v_n)$, using Lemma 2.5, we can affirm that there exists a subsequence of f_n that we continue to denote by the same index $n \geqslant 1$ such that $f_n \rightharpoonup f$ (weakly) in $\mathbb{L}^p([0,a];X)$. Using Lemma 2.10, we get that $w = \mathcal{G}(f)$, and $f \in Sel_F(v)$. Thus, $w \in \mathcal{H}(v)$ which implies that $(v,w) \in Gra(\mathcal{H})$. Now, we show that $\mathcal{H}$ has a contractible values. Let $x \in D$, $\tilde{f} \in Sel_F(x)$, $\tilde{v} = \mathcal{G}(\tilde{f})$, and $\Theta : [0,1] \times \mathcal{H}(x) \to \mathcal{C}([0,a];X)$ be such that

$$\Theta(\xi,w)(t) = \begin{cases} w(t), & \text{if } t \in [0,\xi a], \\ \\ \bar{x}(t,\xi,w), & \text{if } t \in]\xi a, a], \end{cases}$$

where

$$\begin{aligned} \bar{x}(t,\xi,w) = {} & Q_q(t)\phi(0) + \int_0^{\xi t} (t-s)^{q-1} P_q(t-s)[\bar{f}(s) + Bu(s)]ds \\ & + \int_{\xi t}^{t} (t-s)^{q-1} P_q(t-s)[\tilde{f}(s) + Bu(s)]ds. \end{aligned}$$

Then, Θ is continuous, $\Theta(0,w) = \tilde{v}$, and $\Theta(1,w) = w$ for each $w \in \mathcal{H}(x)$. That is $\mathcal{H}(x)$ is contractible.

Step 3: Let $\beta : \mathcal{C}([-r,a];X) \to 2^{\mathcal{C}([-r,a];X)}$ be the multi-valued map defined by

$$\beta(x) = \begin{cases} \mathcal{H}(x) & \text{if } x \in D \\ \\ \{0\} & \text{if not}. \end{cases}$$

Then,

1. D is compact, and $\beta(D) \subset 2^D$.
2. β has a closed contractible values.
3. β is u.s.c (from Lemma 2.4).

Using Lemma 2.6, we can affirm that β has a fixed point in D. $\square$

3.2. Approximate controllability for equation (1)–(2) with $h = 0$

This subsection focuses on the approximate controllability of equations (9) with $h = 0$. We define the map Γ_0^a from X^* to X by

$$\Gamma_0^a x^* = \int_0^a (a-s)^{q-1} P_q(a-s) B B^* P_q^*(a-s) x^* ds,$$

where B^* and $P_q^*(t)$ are the adjoint operators of B and $P_q(t)$ respectively. Then, Γ_0^a is a bounded linear operator from X^* to X. Equation (9) is approximately controllable if for each $d \in X$ and $\lambda > 0$, there are $(x,u) \in \mathcal{C}([-r,a];X) \times \mathbb{L}^p([0,a];X)$ such that

$$\begin{cases} u(t) = u^\lambda(t) = B^* P^*(a-t) J(\lambda I + \Gamma_0^a J)^{-1} w(x), \quad for\; t \in [0,a] \\ x(t) = x^\lambda(t) = Q_q(t)\phi(0) + \displaystyle\int_0^t (t-s)^{q-1} P_q(t-s)[Bu(s) + f(s)]ds, \quad for\; t \in [0,a] \\ x(t) = \phi(t), \quad for\; t \in [-r,a], \end{cases}$$

where $w(x) = d - Q_q(a)\phi(0) - \displaystyle\int_0^a (a-s)^{q-1} P_q(a-s) f(s) ds$, for $f \in Sel_F(x)$.

THEOREM 3.2. *Let $p > 1$ be such that $pq > 1$. Assume that $(\mathbf{H_3})$–$(\mathbf{H_5})$ hold, and $R(t)$ is norm-continuous for $t > 0$. Then, for each $\phi \in \mathcal{C}$, the set of mild solutions of equation (9) is compact in $\mathcal{C}([-r,a];X)$.*

Proof. Let $(x_n)_{n\geqslant 1}$ be a sequence of mild solutions of equation (9), and $f_n \in Sel_F(x_n)$ such that $x_n = \mathcal{G}(f_n)$. From condition $(\mathbf{H_4})$, we have

$$\begin{aligned} \|f_n(t)\| &\leqslant \eta(t)\left(1 + \|\phi\|_{\mathcal{C}} + 2 \sup_{t\in[0,a]} \|x_n(t)\|\right) \\ &\leqslant \eta(t)\left(1 + \|\phi\|_{\mathcal{C}} + 2 \sup_{t\in[0,a]} S_0(t)\right). \end{aligned}$$

Then, $(f_n)_{n\geqslant 1}$ is bounded in $\mathbb{L}^p([0,a];X)$, using Lemma 2.3, we can affirm that $\{x_n : n \geqslant 1\}$ is equicontinuous in $\mathcal{C}([-r,a];X)$. Let $\mathcal{M} = \{x_n :\ n \geqslant 1\}$, since x_n, $n \geqslant 1$ is a mild solution of equation (9), it follows that

$$\sup_{\sigma\in[0,t]} \nu(\mathcal{M}(\sigma)) \leqslant 4a_2\mu_0 \int_0^t (t-s)^{q-1} \sup_{\sigma\in[0,s]} \nu(\mathcal{M}(\sigma)) ds.$$

By Lemma 2.11, we obtain that $\nu(\mathcal{M}(t)) = 0$, then $\{x_n :\ n \geqslant 1\}$ is relatively compact, which implies that there exists a subsequence $\{x_{n_k} :\ k \geqslant 1\}$ of $\{x_n :\ n \geqslant 1\}$ that converges strongly to a limit $x \in \mathcal{C}([-r,a];X)$. Applying Lemma 3.1, we get that Sel_F is w.u.s.c, with nonempty, convex, and weakly compact values. That is there exists a subsequence $(f_{n_{\varphi(k)}})_{k\geqslant 1}$ of $(f_n)_{n\geqslant 1}$ that converges weakly to a limit $f \in \mathbb{L}^p([0,a];X)$. Using Lemma 2.10, we obtain that x is a mild solution of equation (9). Consequently, the set of mild solutions of equation (9) is compact in $\mathcal{C}([-r,a];X)$. $\square$

THEOREM 3.3. *Assume that U is a separable Hilbert space, and the linear system associated to equation (9) is approximately controllable on $[0,a]$, then for each $\lambda > 0$, $(\lambda I + \Gamma_0^a J)^{-1}$ exists as an element of $\mathcal{L}(X)$. Moreover,*

$$\lim_{\lambda \to 0^+} \|\lambda(\lambda I + \Gamma_0^a J)^{-1})h\| = 0,$$

and

$$\|\lambda(\lambda I + \Gamma_0^a J)^{-1}h\| \leqslant \|h\|,$$

for each $h \in X$.

Proof. The proof is similar to that of Lemma 4.2 from [28]. □

REMARK 3. Under conditions of Theorem 3.2, we can show that system (9) is not exactly controllable on $[0,a]$. Indeed, let $C(\cdot,\phi,U)$ be the set of mild solutions of equation (9). We define the linear map $\mathcal{B}_a$ by

$$\begin{array}{rcl} \mathcal{B}_a : C(\cdot,\phi,U) & \longrightarrow & X \\ x & \mapsto & x(a). \end{array}$$

By absurd we assume that system (9) is exactly controllable on $[0,a]$. Then, $\mathcal{B}_a$ is surjective. Using Theorem 3.2, we can affirm that $\mathcal{B}_a$ is compact. We define the map $\tilde{\mathcal{B}_a}$ by

$$\begin{array}{rcl} \tilde{\mathcal{B}_a} : C(\cdot,\phi,U)/\mathcal{B}_a & \longrightarrow & X \\ \bar{x} & \mapsto & \tilde{\mathcal{B}_a}(\bar{x}) = \mathcal{B}_a(x). \end{array}$$

where $\bar{x} = \{y \in C(\cdot,\phi,U) : \; x(a) = y(a)\}$. Then, $\tilde{\mathcal{B}_a}$ is invertible and compact which implies that X is a finite dimensional space. That is a contradiction.

The following Theorem is the main result in this subsection.

THEOREM 3.4. *Assume that all assumptions of Theorem 3.1 hold, the linear system associated to equation (9) is approximately controllable on $[0,a]$, and U is a separable Hilbert space. Then, equation (9) is approximately controllable on $[0,a]$.*

Proof. Let $\lambda > 0$, we define the multi-valued map $\mathcal{H}^\lambda$, by

$$\begin{array}{rcl} \mathcal{H}^\lambda : \mathcal{C}([-r,a];X) & \longrightarrow & 2^{\mathcal{C}([-r,a];X)} \\ x & \mapsto & \mathcal{G}^\lambda(Sel_F(x)) \end{array}$$

where

$$\mathcal{G}^\lambda(f)(t) = \begin{cases} Q_q(t)\phi(0) + \displaystyle\int_0^t (t-s)^{q-1} P_q(t-s)[f(s) + Bu^\lambda(s)]ds, & t \in [0,a] \\ \phi(t), & t \in [-r,0] \end{cases}$$

for $f \in Sel_F(x)$. Let $(x^\lambda)_{\lambda>0}$ be a sequence of mild solutions of equation (9) corresponding to the following sequence of control functions:

$$u^\lambda(t) = B^* P_q(a-t)^* J(\lambda I + \Gamma_0^a J)^{-1} w(x^\lambda).$$

It is evident to see that

$$x^\lambda(a) - d = -\lambda(\lambda I + \Gamma_0^a J)^{-1} w(x^\lambda).$$

Moreover,

$$\nu\left(\left\{\int_0^t (t-s)^{q-1} P_q(t-s) f^\lambda(s)ds : \lambda > 0\right\}\right) \leqslant 4a_2\mu_0 \frac{a^q}{q}\left(\sup_{s\in[-r,a]} \nu(\{x^\lambda(s) : \lambda > 0\})\right).$$

By Theorem 3.2, we can see that $\{x^\lambda : \lambda > 0\}$ is relatively compact. Applying Ascoli-Arzéla Theorem we can prove that

$$\left\{\int_0^{\cdot} (\cdot - s)^{q-1} P_q(\cdot - s) f^\lambda(s)ds : f^\lambda \in Sel_F(x^\lambda) : \lambda > 0\right\}$$

is relatively compact in $\mathcal{C}([0,a];X)$, then there exists a sequence $\lambda_n \to 0$ as $n \to +\infty$, and $K_0 \in \mathcal{C}([0,a];X)$ such that

$$\lim_{n\to+\infty} \sup_{s\in[0,a]} \|\int_0^t (t-s)^{q-1} P_q(t-s) f^{\lambda_n}(s)ds - K_0(t)\| = 0.$$

Writing $w_a = d - Q_q(a)\phi(0) - K_0(a)$, we can assert that

$$\|x^{\lambda_n}(a) - d\| \leqslant \|\lambda_n(\lambda_n I + \Gamma_0^a J)^{-1} w_a\| + \|w(x^{\lambda_n}) - w_a\|.$$

It follows from Theorem 3.3, that the right term tends to 0, as $n \to +\infty$. $\square$

3.3. Approximate controllability for equation (1)–(3) with $h = 0$

We recall that equation (1)–(3) with $h = 0$ takes the following form:

$$\begin{cases} {}^cD_t^q x(t) \in Ax(t) + \int_0^t G(t-s)x(s)ds + F(t,x(t),x_t) + Bu(t), & t \in [0,a] \\ x(t) = g(x)(t), & t \in [-r,0]. \end{cases} \tag{13}$$

In this subsection, we extend the controllability results obtained in the previous subsection to equation (13). To achieve our results, some restrictions on map g must be imposed. We recall that the compactness of g is not necessary here.

($\mathbf{H_6}$) There is a function $\rho \in \mathcal{C}([-r,0];\mathbb{R}^+)$ such that $\|g(x)(\theta)\| \leqslant \rho(\theta)$ for $\theta \in [-r,0]$, and $x \in \mathcal{C}([-r,a];X)$.

$(\mathbf{H_7})$ There exists a function $l \in \mathcal{C}([-r,0];\mathbb{R}^+)$ such that

$$\nu(g(D)(\theta)) \leqslant l(\theta) \sup_{s\in[-r,a]} \nu(D(s)),$$

for $\theta \in [-r,0]$, and for each bounded set D of $\mathcal{C}([-r,a];X)$.

$(\mathbf{H_8})$ For any bounded set D of $\mathcal{C}([-r,a];X)$, $g(D)$ is equicontinuous in $\mathcal{C}$.

An example in which $(\mathbf{H_6})$, $(\mathbf{H_7})$, and $(\mathbf{H_8})$ are satisfied is provided below.

EXAMPLE 3.2. Let $X = \mathbb{L}^2(0,1)$, $\{c_i(\cdot)\}_{1\leqslant i\leqslant m} \subset \mathcal{C}([-r,0];\mathbb{R})$, $\xi \in (0,1)$, and $\theta \in [-r,0]$ such that

$$g(x)(\theta)(\xi) = \sum_{i=1}^{m} c_i(\theta) sin(x(\theta_i + r)(\xi)), \quad x \in \mathcal{C}([-r,a];X),$$

where $-2r < \theta_1 < \cdots < \theta_m < a - r$. Then,

$$\|g(x)(\theta)\| \leqslant \sum_{i=1}^{m} |c_i(\theta)|, \quad \theta \in [-r,0],$$

which means that $(\mathbf{H_6})$ holds with $\rho(\theta) = \sum\limits_{i=1}^{m} |c_i(\theta)|$. Moreover,

$$\|g(x)(\theta) - g(y)(\theta)\| \leqslant 2^{m/2} \left(\sum_{i=1}^{m} \|c_i\|_{\mathcal{C}}^2 \right)^{1/2} \max_{1\leqslant i\leqslant m} \|x(\theta_i + r) - y(\theta_i + r)\|,$$

for $x,y \in \mathcal{C}([-r,a];X)$. Let D be a bounded set of $\mathcal{C}([-r,a];X)$. Let $\varepsilon > 0$, and $r_\varepsilon = \max\limits_{1\leqslant i\leqslant m} \nu(D(\theta_i + r)) + \varepsilon$. Then, there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$\cup_{i=1}^{m} D(\theta_i + r) \subset \cup_{j=1}^{n} \bar{B}(x_j, r_\varepsilon).$$

We define the functions sequence $(y_j)_{1\leqslant j\leqslant n}$ from $[-r,a]$ to X by $y_j(t) = x_j$ for $j \in \{1,2,\cdots,n\}$ and $t \in [-r,a]$. Let $x \in D$, and $j \in \{1,2,\cdots,n\}$. Then,

$$\begin{aligned}
\|g(x)(\theta) - g(y_j)(\theta)\| &\leqslant 2^{m/2} \left(\sum_{i=1}^{m} \|c_i\|_{\mathcal{C}}^2 \right)^{1/2} \max_{1\leqslant i\leqslant m} \|x(\theta_i + r) - x_j\| \\
&\leqslant 2^{m/2} \left(\sum_{i=1}^{m} \|c_i\|_{\mathcal{C}}^2 \right)^{1/2} r_\varepsilon.
\end{aligned}$$

Which implies that

$$g(D)(\theta) \subset 2^{m/2} \left(\sum_{i=1}^{m} \|c_i\|_{\mathcal{C}}^2 \right)^{1/2} \cup_{j=1}^{n} \bar{B}(g(y_j)(\theta), r_\varepsilon).$$

Then,

$$\begin{aligned}\nu(g(D)(\theta)) &\leqslant 2^{m/2}\left(\sum_{i=1}^{m}\|c_i\|_{\mathcal{C}}^2\right)^{1/2}\max_{1\leqslant i\leqslant m}\nu(D(\theta_i+r))\\ &\leqslant 2^{m/2}\left(\sum_{i=1}^{m}\|c_i\|_{\mathcal{C}}^2\right)^{1/2}\sup_{s\in[-r,a]}\nu(D(s)).\end{aligned}$$

Hence, $(\mathbf{H_7})$ holds with

$$l(\theta)=2^{m/2}\left(\sum_{i=1}^{m}\|c_i\|_{\mathcal{C}}^2\right)^{1/2}=cte.$$

In addition, for each $-r\leqslant t_1<t_2\leqslant 0$, we have

$$\|g(x)(t_1)-g(x)(t_2)\|\leqslant\sum_{i=1}^{m}|c_i(t_1)-c_i(t_2)|,$$

for $x\in\mathcal{C}([-r,a];X)$, which implies that $(\mathbf{H_8})$ holds.

Where the assumption $(\mathbf{H_7})$ is satisfied, we denote by $l_0=\sup\{l(\theta):\ \theta\in[-r,0]\}$. The following Theorem is needed to show that equation (13) is approximately controllable on $[0,a]$.

THEOREM 3.5. *Assume that $(\mathbf{H_3})$–$(\mathbf{H_5})$, and $(\mathbf{H_7})$ hold. Assume that there exists K an equicontinous set of $\mathcal{C}([-r,a];X)$ such that*

$$\{x\in\mathcal{C}([-r,a];X):\ x \textit{ is a mild solution of equation } (13)\}\subseteq K.$$

If

$$\max\left[l_0,\left(Ml(0)+4a_2\mu_0\frac{a^q}{q}\right)\right]<1.$$

Then, the set of mild solutions of equation (13) is relatively compact.

Proof. Let $(x_n)_{n\geqslant 0}$ be a sequence of mild solutions of equation (13). Then,

$$x_n(t)=g(x_n)(t),\quad \text{for}\quad t\in[-r,0],$$

and

$$x_n(t)=Q_q(t)g(x_n)(0)+\int_0^t(t-s)^{q-1}[f_n(s)+Bu(s)]ds,$$

for $t\in[0,a]$ and $f_n\in Sel_F(x_n)$. Note that

$$\nu(\{x_n(t):\ n\geqslant 0\})\leqslant l_0\sup_{s\in[-r,a]}\nu(\{x_n(s):\ n\geqslant 0\}),\quad \text{for } t\in[-r,0],$$

and

$$\nu(\{x_n(t):\ n \geqslant 0\}) \leqslant \left(Ml(0) + 4a_2\mu_0 \frac{a^q}{q}\right) \sup_{s\in[-r,a]} \nu(\{x_n(s):\ n \geqslant 0\}), \quad \text{for } t \in [0,a].$$

Then,

$$\sup_{s\in[-r,a]} \nu(\{x_n(s):\ n \geqslant 0\}) \leqslant \max\left(l_0, Ml(0) + 4a_2\mu_0 \frac{a^q}{q}\right) \sup_{s\in[-r,a]} \nu(\{x_n(s):\ n \geqslant 0\}).$$

If $\max\left(l_0, Ml(0) + 4a_2\mu_0 \frac{a^q}{q}\right) < 1$, it follows that $(x_n(t))_{n\geqslant 0}$ is relatively compact. Since $(x_n(t))_{n\geqslant 0}$ is equicontinuous, it follows that $(x_n)_{n\geqslant 0}$ is relatively compact, which means that there exists a subsequence of $(x_n)_{n\geqslant 0}$ that converges in $\mathcal{C}([-r,a];X)$. □

Bellow, we find the main result in this subsection.

THEOREM 3.6. *Let $p > 1$ such that $pq > 1$. Assume that assumptions $(\mathbf{H_3})$–$(\mathbf{H_8})$ hold, $R(t)$ is norm-continuous for $t > 0$, the linear system associated to equation (13) is approximately controllable on $[0,a]$, and U is a separable Hilbert space. If*

$$\max\left[l_0, \left(Ml(0) + 4a_2\mu_0 \frac{a^q}{q}\right)\right] < 1,$$

then equation (13) is approximately controllable on $[0,a]$.

Proof. We define the multi-valued map $\mathcal{K} : \mathcal{C}([-r,a];X) \to 2^{\mathcal{C}([-r,a];X)}$ by

$$\mathcal{K}(x) = \mathcal{G}(Sel_F(x))$$

where,

$$\mathcal{G}(f)(t) = \begin{cases} Q_q(t)g(x)(0) + \int_0^t (t-s)^{q-1} P_q(t-s)[Bu(s) + f(s)], & t \in [0,a] \\ x(t) = g(x)(t), \quad t \in [-r,0], \end{cases}$$

for $f \in Sel_F(x)$. Firstly, we prove that system (13) has a mild solution. Let $x \in \mathcal{C}([-r,a];X)$, and $f \in Sel_F(x)$. Let $\tilde{z} \in \mathcal{K}(x)$, using $(\mathbf{H_4})$ and $(\mathbf{H_6})$, we can affirm that

$$\begin{aligned} \|\tilde{z}(t)\| &\leqslant \|Q_q(t)g(x)(0)\| + \int_0^t (t-s)^{q-1} P_q(t-s)[f(s) + Bu(s)]ds \\ &\leqslant M\rho(0)a_1 - M\|\phi(0)\| + 2a_2 \int_0^t (t-s)^{q-1} \eta(s) \sup_{\sigma\in[0,s]} \|x(\sigma)\| ds \\ &= a_1' + 2a_2\eta_0 \int_0^t (t-s)^{q-1} \sup_{\sigma\in[0,s]} \|x(\sigma)\| ds, \end{aligned}$$

where $a_1' = M\rho(0)a_1 - M\|\phi(0)\|$. Therefore, we have

$$\sup_{\sigma\in[0,t]}\|\tilde{z}(\sigma)\| \leqslant a_1^{'} + 2a_2\eta_0\int_0^t (t-s)^{q-1}\sup_{\sigma\in[0,s]}\|x(\sigma)\|ds.$$

Let $S_1 \in \mathcal{C}([0,a];\mathbb{R}^+)$ be the unique solution of the following integral equation:

$$S_1(t) = a_1^{'} + 2a_2\eta_0\int_0^t (t-s)^{q-1}S_1(s)ds.$$

Let

$$N_0 = \left\{ x\in\mathcal{C}([-r,a];X) \;\middle|\; \begin{array}{l} x(t) = g(x)(t) \text{ if } t\in[-r,0] \\ \\ \sup\limits_{\sigma\in[0,t]}\|x(\sigma)\| \leqslant S_1(t) \text{ if } t\in[0,a] \end{array} \right\}.$$

It is easily seen that N_0 is nonempty, closed, bounded, convex , and $\mathcal{K}(N_0)\subset N_0$. Let

$$N_{k+1} = \overline{conv}\left(\cup_{x\in N_k}\mathcal{K}(x)\right);\quad k\geqslant 0.$$

Then,

1. $N_{k+1}\subset N_k$, for each $k\geqslant 0$.
2. N_k is nonempty, closed, bounded, and convex, for each $k\geqslant 0$.
3. N_k is equicontinuous, for each $k\geqslant 2$.

Indeed, an analysis similar to that in the proof of Theorem 3.1 shows that

$$\mathcal{K}(x) = \{z(\cdot,g(x)(0),f);\; f\in Sel_F(x)\}\cup\{g(x)\}$$

is equicontinuous in $\mathcal{C}([-r,a];X)$. Therefore, we have

$$\cup_{x\in N_{k+1}}\mathcal{K}(x)\subset N_{k+2}\subset\cup_{x\in N_k}\mathcal{K}(x)\subset N_{k+1}, \tag{14}$$

which implies that N_k is equicontinuous for each $k\geqslant 2$. Let $N=\cap_{k\geqslant 0}N_k$, then N is nonempty, convex, closed, bounded, and equicontinuous. To prove that N is compact, it is sufficient to show that $N(t)=\{x(t);\; x\in N\}$ is relatively compact for $t\in[-r,a]$, that is $\nu(N(t))=0$, for $t\in[-r,a]$. For $t\in]0,a]$, using (14) , we obtain that

$$\nu(\cup_{x\in N_{k+1}}\mathcal{K}(x)(t))\subset\nu(N_{k+2}(t))\subset\nu(\cup_{x\in N_k}\mathcal{K}(x)(t)). \tag{15}$$

Let $x\in N_k$, then $\Pi=\{\mathcal{G}(f)(t):\; f\in\cup_{x\in N_k}Sel_F(x)\}$ is bounded in X. By Lemma 2.7, we can affirm that for every $\varepsilon>0$, there is $(w_n)_{n\geqslant 1}\subset\Pi$ such that

$$\nu(\Pi)\leqslant 2\nu(\{w_n:\; n\geqslant 1\})+\varepsilon.$$

As in the proof of Theorem 3.1, we can show that $\nu(N(t))=0$. Now, for $t\in[-r,0]$, we have $N(t)=\{g(x)(t);\; x\in N\}=g(N)(t)$. Then,

$$\nu(N(t))\leqslant l(t)\sup_{s\in[-r,a]}\nu(N(s))\leqslant l_0\sup_{s\in[-r,a]}\nu(N(s)) = l_0\sup_{s\in[-r,0]}\nu(N(s)),$$

which implies that

$$\sup_{t\in[-r,0]} \nu(N(t)) \leqslant l_0 \sup_{t\in[-r,0]} \nu(N(t)).$$

If $l_0 < 1$, we obtain that

$$\sup_{t\in[-r,0]} \nu(N(t)) = 0.$$

Hence, $N(t)$ is relatively compact, moreover $\mathcal{K}(N) \subset 2^N$. Consequently, N is compact. By step 2 in the proof of Theorem 3.1, and Lemma 2.6, we can prove that $\mathcal{K}$ has a fixed point. Proceed analogously to the proof of Theorem 3.4, we show that system (13) has a sequence of mild solutions $(x^\lambda)_{\lambda>0}$ corresponding to the following sequence of control functions

$$u^\lambda(t) = B^* P_q(a-t)^* J(\lambda I + \Gamma_0^a J)^{-1} w(x^\lambda),$$

where,

$$w(x^\lambda) = d - Q_q(a)g(x^\lambda)(0) - \int_0^a (a-s)^{q-1} P_q(a-s) f^\lambda(s)ds, \text{ for } f^\lambda \in Sel_F(x^\lambda).$$

By Theorem 3.5, we can see that $\{x^\lambda : \lambda > 0\}$ is relatively compact. Then, there exists $\lambda_n \to 0$ as $n \to +\infty$ such that $x^{\lambda_n} \to x^* \in \mathcal{C}([-r,a];X)$ as $n \to +\infty$. We can also show that

$$\left\{ \int_0^{\cdot} (\cdot - s)^{q-1} P_q(\cdot - s) f^{\lambda_n}(s)ds : \ f^{\lambda_n} \in Sel_F(x^{\lambda_n}) : \ n \geqslant 1 \right\}$$

is relatively compact in $\mathcal{C}([0,a];X)$, then there exists a sequence $\lambda_{\varphi(n)} \to 0$ as $n \to +\infty$, and $K_1 \in \mathcal{C}([0,a];X)$ such that

$$\lim_{n\to+\infty} \sup_{s\in[0,a]} \| \int_0^t (t-s)^{q-1} P_q(\cdot - s) f^{\lambda_{\varphi(n)}}(s)ds - K_1(t)\| = 0.$$

Taking $w_a = d - Q_q(a)g(x^*)(0) - K_1(a)$, the rest of the proof runs as to that of Theorem 3.4. □

3.4. Approximate controllability for equation (1)–(2) with $h \neq 0$

Similarly, in this subsection, we study the approximate controllability of the following neutral equation:

$$\begin{cases} {}^c D_t^q N(t,x_t) \in AN(t,x_t) + \displaystyle\int_0^t G(t-s)N(s,x_s)ds + F(t,x(t),x_t) + Bu(t), & t \in [0,a] \\ x(t) = \phi(t), \quad t \in [-r,0]. \end{cases} \tag{16}$$

We assume the following.

$(\mathbf{H_9})$ The linear system:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \int_0^t G(t-s)x(s)ds + Bu(t), & t \in [0,a] \\ x(0) = x_0 \end{cases}$$

is approximately controllable on $[0,a]$ for every $a > 0$.

On the map h, we shall make two standing assumptions under consideration.

$(\mathbf{H_{10}})$ There exists a constant $\delta_0 \in]0,1[$ such that

$$\|h(t,\phi_1) - h(t,\phi_2)\| \leqslant \delta_0 \|\phi_1 - \phi_2\|_{\mathcal{C}},$$

for $\phi_1, \phi_2 \in \mathcal{C}$, and $t \in [0,a]$.

$(\mathbf{H_{11}})$ For each bounded subset Ω of $\mathcal{C}([-r,a];X)$, $\{t \to h(t,x_t) : x \in \Omega\}$ is equicontinuous in $\mathcal{C}([0,a];X)$.

The following Theorem is the main result in this subsection.

THEOREM 3.7. *Suppose that all assumptions of Theorem 3.1 are satisfied, and* $(\mathbf{H_9})$–$(\mathbf{H_{11}})$ *hold. Then, equation* (16) *is approximately controllable on* $[0,a]$.

Proof. Let $d \in X$. We define the multi-valued map $\mathcal{F}$ by

$$\begin{array}{rcl} \mathcal{F} : \mathcal{C}([-r,a];X) & \longrightarrow & 2^{\mathcal{C}([-r,a];X)} \\ x & \mapsto & \mathcal{G}(Sel_F(x)) \end{array}$$

where $\mathcal{G}(f)(t) = \phi(t)$ for $t \in [-r,0]$, and

$$\mathcal{G}(f)(t) = Q_q(t)[\phi(0) + h(0,\phi)] - h(t,x_t) + \int_0^t (t-s)^{q-1} P_q(t-s)[f(s) + Bu(s)]ds$$

for $t \in [0,a]$ and $f \in Sel_F(x)$. Let $u(\cdot) \in \mathbb{L}^p([0,a],U)$. The proof is divided in two step, firstly we prove that system (16) has a mild solution, and secondly we prove that system (16) is approximately controllable on $[0,a]$. Let $x \in \mathcal{C}([-r,a];X)$, $z \in \mathcal{F}(x)$, and $t \in [0,a]$. Then,

$$\sup_{\sigma\in[0,t]} \|z(\sigma)\| \leqslant b_1 + \delta_0 \sup_{\sigma\in[0,t]} \|x(\sigma)\| + 2a_2\eta_0 \int_0^t (t-s)^{q-1} \sup_{\sigma\in[0,s]} \|x(\sigma)\| ds,$$

where $h_a = \sup_{t\in[0,a]} \|h(t,0)\|$, and

$$\begin{aligned} b_1 &= M[\|\phi(0)\| + \|h(0,\phi)\|] + \delta_0\|\phi\|_{\mathcal{C}} + h_a \\ &\quad + a_2 \left[\sqrt[p]{\left(\frac{p-1}{pq-1}\right)^{p-1}} a^{(pq-1)/p} \right] [(1 + \|\phi\|_{\mathcal{C}})\|\eta\|_{\mathbb{L}^p} + \|Bu\|]. \end{aligned}$$

Let $S_2 \in \mathcal{C}([0,a];\mathbb{R}^+)$ be the unique solution of the following equation:

$$S_2(t) = \frac{b_1}{1-\delta_0} + \frac{2a_2}{1-\delta_0}\eta_0 \int_0^t (t-s)^{q-1} S_2(s)ds.$$

Let

$$L_0 = \left\{ x \in \mathcal{C}([-r,a];X) \;\middle|\; \begin{array}{l} x(t) = \phi(t) \text{ if } t \in [-r,0] \\ \\ \sup\limits_{\sigma\in[0,t]} \|x(\sigma)\| \leqslant S_2(t) \text{ if } t \in [0,a] \end{array} \right\},$$

and

$$L_{k+1} = \overline{conv}(\cup_{x\in L_k} \mathcal{F}(x)).$$

Arguing as in the proof of Theorem 3.1, we can see that $L = \cap_{k\geqslant 0} L_k$ is compact in $\mathcal{C}([-r,a];X)$, and system (16) has a mild solution for each $u \in \mathbb{L}^p([0,a];X)$. Let $x(\cdot) = x(\cdot,\phi,0)$ be the mild solution of equation (16) corresponding to $u = 0$, then we have

$$x(t) = Q_q(t)[\phi(0)+h(0,\phi)] - h(t,x_t) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s)ds,$$

for $f \in Sel_F(x)$. Take $0 < a_n < a$ such that $a_n \to a$ as $n \to +\infty$. We denote $x^n = x(a_n)$. Let consider the following system:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \int_0^t G(t-s)x(s)ds + Bu(t), \quad t \in [0,a] \\ x(0) = x^n. \end{cases}$$

It follows from assumption $(\mathbf{H_9})$, that this system is approximately controllable on $[0,a-a_n]$. Then, there is a control function $w_n(\cdot) \in \mathbb{L}^p([0,a-a_n];U)$ such that

$$\lim_{n\to+\infty} \|Q_q(a-a_n)x^n + \int_0^{a-a_n} (a-a_n-s)^{q-1} P_q(a-a_n-s)Bw_n(s)ds - d\| = 0.$$

We define

$$u_n(s) = \begin{cases} w_n(s-a_n), \quad \text{if } a_n < s \leqslant a \\ \\ 0, \quad \text{if } 0 \leqslant s \leqslant a_n. \end{cases}$$

We consider the following equation:

$$\begin{cases} z(t) = Q_q(t)[\phi(0)+h(0,\phi)] - h(t,z_t) + \int_0^t (t-s)^{q-1} P_q(t-s)[f(s)+Bu_n(s)]ds, \\ \hfill t \in [0,a] \\ z_0 = \phi. \end{cases}$$

Using $(\mathbf{H_{10}})$, we can show that, this system has a unique solution, that will be denoted by y^n. It is evident to see that $y^n(t) = x(t)$ for each $t \in [0,a_n]$.
Let $\beta_n = a - a_n$, and

$$\Delta_n = Q_q(a)[\phi(0)+h(0,\phi)] - h(a,y_a^n) - Q_q(\beta_n)x^n + \int_0^{a_n} (a-s)^{q-1} P_q(a-s) f(s)ds.$$

Then,

$$\|y^n(a)-d\| \leqslant \|\int_0^{\beta_n}(\beta_n-s)^{q-1}P_q(\beta_n-s)Bw_n(s)ds+Q_q(\beta_n)x^n-d\| \\ +\|\int_{a_n}^{a}(a-s)^{q-1}P_q(a-s)f(s)ds\|+\|\Delta_n\|.$$

We have

$$\lim_{n\to+\infty}\|\int_{a_n}^{a}(a-s)^{q-1}P_q(a-s)f(s)ds\|=0, \tag{17}$$

and

$$\lim_{n\to+\infty}\|\int_0^{\beta_n}(\beta_n-s)^{q-1}P_q(\beta_n-s)Bw_n(s)ds+Q_q(\beta_n)x^n-d\|=0. \tag{18}$$

In addition,

$$\lim_{n\to+\infty}\|[Q_q(a)-Q_q(\beta_n)Q_q(a_n)](\phi(0)+h(0,\phi))\|=0, \tag{19}$$

and

$$\lim_{n\to+\infty}\|\int_0^{a_n}[(a-s)^{q-1}P_q(a-s)-Q_q(\beta_n)(a_n-s)^{q-1}P_q(a_n-s)]f(s)ds\|=0. \tag{20}$$

Let $N_0\in\mathbb{N}$ such that if $n\geqslant N_0$, we have

$$0<a_n+\theta<a+\theta\leqslant a_n<a, \text{ for each } \theta\in[-r,0[.$$

then,

$$y_a^n(\theta)=y^n(a+\theta)=x(a+\theta)=x_a(\theta), \text{ for } \theta\in[-r,0[, \text{ and } n\geqslant N_0,$$

which implies that

$$y_a^n(\theta)-x_a(\theta)=0, \text{ for } \theta\in[-r,0[, \text{ and } n\geqslant N_0.$$

For $\theta=0$, we have

$$\|y^n(a)-x(a)\|\leqslant\|y_a^n-y_{a_n}^n\|+\|x(a_n)-x(a)\|.$$

Then, for $n\geqslant N_0$, we obtain

$$\|y_a^n-x_a\|_{\mathcal{C}}=\sup_{\theta\in[-r,0]}\|y_a^n(\theta)-x_a(\theta)\| = \|y^n(a)-x(a)\| \\ \leqslant \|y_a^n-y_{a_n}^n\|+\|x(a_n)-x(a)\|.$$

Taking in account that

$$\|y_a^n-x_{a_n}\|_{\mathcal{C}}\leqslant\|y_a^n-x_a\|_{\mathcal{C}}+\|x_a-x_{a_n}\|_{\mathcal{C}},$$

since h is a continuous function, we can deduce that

$$\lim_{n\to+\infty}\|h(a,y_a^n)-Q_q(\beta_n)h(a_n,x_{a_n})\|=0. \tag{21}$$

We replace x^n by its value in Δ_n, using (17), (18), (19), (20), and (21), we can prove that

$$\lim_{n\to+\infty}\|y^n(a)-d\|=0. \quad \square$$

REMARK 4. The above approach is the controllability result in general Banach space. If $h=0$, we can use this approach to show that systems (9) and (13) are approximately controllable on $[0,a]$.

3.5. Approximate controllability for equation (1)–(3) with $h\neq 0$

In this subsection, we study the approximate controllability of the following neutral equation:

$$\begin{cases} {}^cD_t^q N(t,x_t)\in AN(t,x_t)+\displaystyle\int_0^t G(t-s)N(s,x_s)ds+F(t,x(t),x_t)+Bu(t), \quad t\in[0,a] \\ x(t)=g(x)(t), \quad t\in[-r,0]. \end{cases} \tag{22}$$

We assume the following.

($\mathbf{H_{12}}$) For each bounded set Λ of $\mathcal{C}$, $\{t\to h(t,\phi):\ \phi\in\Lambda\}$ is equicontinuous in $\mathcal{C}([0,a];X)$.

($\mathbf{H_{13}}$) There exists $L_g\in]0,1[$ such that

$$\|g(x)-g(y)\|_{\mathcal{C}}\leqslant L_g\|x-y\|_\infty$$

for $x,y\in\mathcal{C}([-r,a];X)$.

Let

$$C(a)=\max\left(4a_2\mu_0\frac{a^q}{q},2a_2\sup_{t\in[0,a]}\int_0^t(t-s)^{q-1}\eta(s)ds\right).$$

THEOREM 3.8. *Let $p>1$ such that $pq>1$. Assume that* ($\mathbf{H_3}$)–($\mathbf{H_5}$) *hold, $R(t)$ is norm-continuous for $t>0$,* ($\mathbf{H_8}$)–($\mathbf{H_{10}}$), ($\mathbf{H_{12}}$)–($\mathbf{H_{13}}$) *are verified, and U is a separable Hilbert space. If*

$$[M(1+\delta_0)+\delta_0]L_g+C(a)<1,$$

then, system (22) *is approximately controllable on* $[0,a]$.

Proof. Let $d\in X$. We can use two approach. But firstly, we show that equation (22) has a mild solution. We define the following multi-valued map:

$$\begin{array}{rcl}\Lambda : \mathcal{C}([-r,a];X) & \longrightarrow & 2^{\mathcal{C}([-r,a];X)} \\ x & \mapsto & \mathcal{G}(Sel_F(x))\end{array}$$

where $\mathcal{G}(f)(t) = g(x)(t)$, for $t \in [-r,0]$ and

$$\mathcal{G}(f)(t) = Q_q(t)[g(x)(0) + h(0,g(x))] - h(t,g(x)) + \int_0^t (t-s)^{q-1} P_q(t-s)[f(s) + Bu(s)]ds$$

for $t \in [0,a]$, and $f \in Sel_F(x)$. Let $u(\cdot) \in \mathbb{L}^p([0,a];U)$, $x \in \mathcal{C}([-r,a];X)$, and $z \in \Lambda(x)$. Then,

$$\|z(t)\| \leqslant L_g \|x\|_\infty + \|g(0)\|, \quad \text{for } t \in [-r,0],$$

and

$$\|z(t)\| \leqslant \left([M(1+\delta_0) + \delta_0]L_g + 2a_2 \sup_{t\in[0,a]} \int_0^t (t-s)^{q-1}\eta(s)ds \right) \|x\|_\infty + C_1, \quad t \in [0,a],$$

for some constant $C_1 \geqslant 0$. Then, there exists $R > 0$ such that $\|z\|_\infty \leqslant R$, for each $x \in \mathcal{C}([-r,a];X)$ in which $\|x\|_\infty \leqslant R$. Let,

$$\Omega_0 = \left\{ x \in \mathcal{C}([-r,a];X) \;\middle|\; \begin{array}{l} x(t) = g(x)(t) \text{ if } t \in [-r,0] \\ \\ \|x\|_\infty \leqslant R. \end{array} \right\},$$

and

$$\Omega_{k+1} = \overline{conv}(\cup_{x\in\Omega_k} \Lambda(x)).$$

Arguing as in the proof of Theorem 3.1, we can see that $\Omega = \cap_{k\geqslant 0}\Omega_k$ is a compact subset of $\mathcal{C}([-r,a];X)$, and system (22) has a mild solution for each $u \in \mathbb{L}^p([0,a];X)$.

Approach 1: Let $x(\cdot) = x(\cdot,\phi,0)$ be the mild solution of equation (22) corresponding to $u = 0$. Then,

$$x(t) = Q_q(t)[g(x)(0) + h(0,g(x))] - h(t,g(x)) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s)ds, \quad t \in [0,a],$$

for some $f \in Sel_F(x)$.

Take $0 < a_n < a$ such that $a_n \to a$, as $n \to +\infty$. We denote by $x^n = x(a_n)$. We consider the following system:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \int_0^t G(t-s)x(s)ds + Bu(t) \\ \\ x(0) = x^n. \end{cases}$$

By the assumption $(\mathbf{H_9})$, we can see that this system is approximately controllable on $[0, a-a_n]$, then there exists a control function $w_n(\cdot) \in \mathbb{L}^2([0,a-a_n];U)$ such that

$$\lim_{n\to+\infty}\|Q_q(a-a_n)x^n+\int_0^{a-a_n}(a-a_n-s)^{q-1}P_q(a-a_n-s)Bw_n(s)ds-d\|=0.$$

We define

$$u_n(s)=\begin{cases} w_n(s-a_n) & \text{if } a_n<s\leqslant a\\ 0 & \text{if } 0\leqslant s\leqslant a_n.\end{cases}$$

Let consider the following equation:

$$\begin{cases} z(t)=Q_q(t)[g(z)(0)+h(0,g(z))]-h(t,g(z))+\displaystyle\int_0^t(t-s)^{q-1}P_q(t-s)[f(s)+Bu_n(s)]ds, \\ \qquad\qquad t\in[0,a]\\ z(t)=g(z)(t)\quad t\in[-r,0],\end{cases}$$

using $(\mathbf{H_{10}})$, and $(\mathbf{H_{13}})$ it follows that this system has a unique mild solution denoted by y^n. It is immediate to see that $y^n(t)=x(t)$ for $t\in[0,a_n]$. The rest of the proof is similar to that of Theorem 3.7.

Approach 2: Let $(x^n)_{n\geqslant 0}$ be a sequence of mild solutions of equation (22). In the same reasoning applied in the previous proofs we can show that $(x^n)_{n\geqslant 0}$ is equicontinuous in $\mathcal{C}([-r,a];X)$, then

$$\sup_{s\in[-r,a]}\nu(\{x^n(s):\ n\geqslant 0\})=\inf\{\varepsilon>0 \text{ such that } (x^n)_{n\geqslant 0} \text{ has a } \varepsilon\text{-net }\}.$$

Let $\varepsilon>0$, and $r=\sup_{s\in[-r,a]}\nu(\{x^n(s):\ n\geqslant 0\})+\varepsilon$. Then, there are $x_1,x_2,\ldots,x_p$ in $\mathcal{C}([-r,a];X)$ such that

$$\{x^n(s);n\geqslant 0\}\subset\cup_{i=1}^p\bar{B}(x_i(s),r).$$

In addition,

$$\|g(x^n)-g(x_i)\|_{\mathcal{C}}\leqslant L_g\|x^n-x_i\|_\infty,$$

and

$$\|h(t,g(x^n))-h(t,g(x_i))\|\leqslant L_g\delta_0\|x^n-x_i\|_\infty.$$

Moreover,

$$\nu\left(\left\{\int_0^t(t-s)^{q-1}P_q(t-s)f^n(s)ds:\ n\geqslant 0\right\}\right)\leqslant 4a_2\mu_0\frac{a^q}{q}\sup_{s\in[-r,a]}\nu(\{x^n(s):\ n\geqslant 0\}),$$

which implies that

$$\sup_{s\in[-r,a]}\nu(\{x^n(s):\ n\geqslant 0\})\leqslant\left([M(1+\delta_0)+\delta_0]L_g+4a_2\mu_0\frac{a^q}{q}\right)\sup_{s\in[-r,a]}\nu(\{x^n(s):\ n\geqslant 0\}).$$

Then, $\{x^n(t):\ n\geqslant 0\}$ is relatively compact for $t\in[-r,a]$. Thus $(x^n)_{n\geqslant 0}$ is relatively compact. Arguing as in the proof of Theorem 3.2, we can prove that the set of mild solutions of equation (22) is relatively compact. Let $(x^\lambda)_{\lambda>0}$ be a sequence of mild solutions of equation (22), with the following sequence of control functions

$$u^{\lambda}(t) = B^*P^*(a-t)J(\lambda I + \Gamma_0^a J)^{-1} w(x^{\lambda}), \quad \text{for } t \in [0,a]$$

where

$$w(x^{\lambda}) = d - Q_q(a)[g(x^{\lambda})(0) + h(0, g(x^{\lambda}))] + h(a, g(x^{\lambda}) - \int_0^a (a-s)^{q-1} P_q(a-s) f^{\lambda}(s)ds$$

and $f^{\lambda} \in Sel_F(x^{\lambda})$. Then $(x^{\lambda})_{\lambda>0}$ is relatively compact, which implies that there exists $(\lambda_n)_{n\geqslant 1}$ converges to 0 as $n \to +\infty$ such that $x^{\lambda_n} \to x^*$ (strongly) in $\mathcal{C}([-r,a];X)$ as $n \to +\infty$. The following set

$$\left\{ \int_0^{\cdot} (\cdot - s)^{q-1} P_q(\cdot - s) f^{\lambda_n}(s)ds : \ f^{\lambda_n} \in Sel_F(x^{\lambda_n}) : \ n \geqslant 1 \right\}$$

is relatively compact in $\mathcal{C}([0,a];X)$, then there exists a sequence $\lambda_{\varphi(n)} \to 0$ as $n \to +\infty$, and $K_3 \in \mathcal{C}([0,a];X)$ such that

$$\lim_{n\to+\infty} \sup_{t\in[0,a]} \| \int_0^t (t-s)^{q-1} P_q(t-s) f^{\varphi(n)}(s)ds - K_3(t)\| = 0.$$

Let

$$w_a = d - Q_q(a)[g(x^*)(0) + h(0, g(x^*))] + h(a, g(x^*)) - K_3(a),$$

the rest of the proof is similar to that of Theorem 3.4. □

4. Application

To apply our basic results, we consider the following neutral fractional inclusion:

$$\begin{cases} {}^cD_t^q N(t,x_t) \in AN(t,x_t) + \displaystyle\int_0^t \delta(t-s) AN(s,x_s)ds + F(t,x(t),x_t) + Bu(t), \quad t \in [0,a] \\ x(t) = \sum_{k=1}^m c_i(t) x(t_i - r), \quad t \in [-r,0], \end{cases} \tag{23}$$

where $0 < t_1 < \ldots < t_m < a$, $\frac{1}{2} < q < 1$, $\delta(t) = e^{-t}$, for $t \geqslant 0$, $c_i \in \mathcal{C}([-r,0];\mathbb{R}^+)$ with quite small real values, $X = \mathbb{L}^2(0,\pi)$, and $U = \mathbb{L}^2(0,\frac{\pi}{2})$. U is a separable Hilbert space and X is a reflexive Banach space. We denote by $\widehat{l}(\lambda)$ the Laplace transforms of a given function $l(t)$. The following Theorem will be needed to give an explicit form to $(R(t))_{t\geqslant 0}$.

THEOREM 4.1. [13, Theorem 3.1] *Assume that the following conditions are satisfied:*

a) A generates an analytic semigroup $(T(t))_{t\geqslant 0}$ and satisfies the following estimate:

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{M'}{|\lambda|}, \ Re(\lambda) > 0, \ \textit{and} \ M' \geqslant 1.$$

b) $\|G(t)\|_{\mathcal{L}(Y,X)} \leqslant b(t)$ *for some* $b(\cdot) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$ *and* $G(t)x$ *is strongly measurable for each* $x \in Y$.

c) For $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > 0$, $\widehat{G}(\lambda)$ *exists as an element of* $\mathcal{L}(Y,X)$ *and*

$$\|\widehat{G}(\lambda)\|_{\mathcal{L}(Y,X)} \leqslant \frac{N}{|\lambda|^\beta}, \text{ for } \beta > 0, \text{ and } N \geqslant 1.$$

Then, equation (4) *has a unique resolvent operator defined on* X *by*

$$R(t)x = \begin{cases} T(t)x + R_1(t)x & \text{if } \ t > 0 \\ x \ \text{ if } \ t = 0, \end{cases}$$

where

$$R_1(t)x = \frac{1}{2\pi i}\int_{\zeta - i\infty}^{\zeta + i\infty} e^{\lambda t} \sum_{j=1}^{+\infty} \left[(\lambda I - A)^{-1}\widehat{G}(\lambda)\right]^j (\lambda I - A)^{-1} x d\lambda,$$

in which $\zeta^\beta > 2(2M' + 1)N$.

Let define A by

$$Af = f'' \qquad \text{for} \qquad f \in H^1_0(0,\pi) \cap H^2(0,\pi).$$

Here, $G(t) = \delta(t)A$ for $t \geqslant 0$. The operators A and $G(t)$, $t \geqslant 0$ satisfy the assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, respectively. Moreover, they satisfy conditions a), b) and c) in Theorem 4.1. Indeed, it is easily seen that condition a) holds with $M' = 1$. For condition b), let $x \in D(A)$, then $G(t)x = e^{-t}Ax$ is strongly measurable. Moreover,

$$\|G(t)x\| \leqslant e^{-t}(\|x\| + \|Ax\|),$$

which implies that condition b) holds with $b(t) = e^{-t}$. For condition c), let $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, then $\widehat{G}(\lambda) = \dfrac{1}{\lambda + 1}A$ as an element of $\mathcal{L}(D(A),X)$. Moreover, $\|\widehat{G}(\lambda)x\| \leqslant \dfrac{1}{|\lambda|}(\|x\| + \|Ax\|)$, which implies that condition c) holds with $N = \beta = 1$.

The analytic semigoup generated by A is given by

$$T(t)w = \sum_{n=1}^{+\infty} e^{-n^2 t}\langle w, e_n\rangle e_n, \quad \text{for} \quad w = \sum_{n=1}^{+\infty} \langle w, e_n\rangle e_n \in X,$$

where $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ for each $n \geqslant 1$. Let $w = \sum_{n=1}^{+\infty} w_n e_n \in X$, then

$$(\lambda I - A)^{-1} w = \sum_{n=1}^{+\infty} \frac{1}{\lambda + n^2} w_n e_n,$$

and

$$(\lambda I - A)^{-1}Aw = \sum_{n=1}^{+\infty} \frac{-n^2}{\lambda + n^2} w_n e_n .$$

Hence,

$$R(t)w = \sum_{n=1}^{+\infty} b_n(t) w_n e_n ,$$

where

$$\begin{aligned} b_n(t) &= e^{-n^2 t} + \frac{1}{2\pi i} \int_{\zeta - i\infty}^{\zeta + i\infty} e^{\lambda t} \sum_{j=1}^{+\infty} \left[\frac{-n^2}{(\lambda + n^2)(\lambda + 1)} \right]^j \frac{1}{(\lambda + n^2)} d\lambda \\ &= e^{-n^2 t} + \frac{1}{2\pi i} \int_{\zeta - i\infty}^{\zeta + i\infty} \frac{-n^2 e^{\lambda t}}{(\lambda + n^2)[n^2 + (\lambda + n^2)(\lambda + 1)]} d\lambda \\ &= e^{-n^2 t} + \frac{1}{2\pi i} \int_{\zeta - i\infty}^{\zeta + i\infty} F_n(\lambda) e^{\lambda t} d\lambda , \end{aligned}$$

for

$$F_n(\lambda) = \frac{-n^2}{(\lambda + n^2)[n^2 + (\lambda + n^2)(\lambda + 1)]} , \quad n \geqslant 1 .$$

With a standard calculus, we get that

$$F_n(\lambda) = \begin{cases} \dfrac{-1}{\lambda + 1} + \dfrac{1}{2(\lambda - (-i - 1))} + \dfrac{1}{2(\lambda - (i - 1))}, & \text{for } n = 1 \\[2ex] \dfrac{-1}{(\lambda + 4)} - \dfrac{\frac{-3 - \sqrt{7}i}{2\sqrt{7}i}}{\lambda - (\frac{-5 - \sqrt{7}i}{2})} - \dfrac{\frac{3 - \sqrt{7}i}{2\sqrt{7}i}}{\lambda - (\frac{-5 + \sqrt{7}i}{2})}, & \text{for } n = 2 \\[2ex] \dfrac{-1}{7(\lambda + n^2)} - \dfrac{b_n}{\lambda - \gamma_{1,n}} - \dfrac{c_n}{\lambda - \gamma_{2,n}}, & \text{for } n \geqslant 3, \end{cases}$$

where

$$\left\{ \begin{aligned} b_n &= \frac{1 - n^2 - \sqrt{n^4 - 6n^2 + 1}}{14\sqrt{n^4 - 6n^2 + 1}} - \frac{1}{7}, \\ c_n &= \frac{-1 + n^2 + \sqrt{n^4 - 6n^2 + 1}}{14\sqrt{n^4 - 6n^2 + 1}}, \\ \gamma_{1,n} &= \frac{-(n^2 + 1) - \sqrt{n^4 - 6n^2 + 1}}{2}, \\ \gamma_{2,n} &= \frac{-(n^2 + 1) + \sqrt{n^4 - 6n^2 + 1}}{2}, \end{aligned} \right. \qquad \text{for } n \geqslant 3.$$

By the inverted Laplace transform, we infer that

$$b_n(t) = \begin{cases} \cos(t)e^{-t}, & \text{for } n = 1 \\ -\dfrac{1}{\sqrt{7}}\left(3\sin(\frac{\sqrt{7}}{2}t) - \sqrt{7}\cos(\frac{\sqrt{7}}{2}t)\right)e^{-\frac{5}{2}t}, & \text{for } n = 2 \\ \dfrac{6}{7}e^{-n^2 t} - b_n e^{\gamma_{1,n}t} - c_n e^{\gamma_{2,n}t}, & \text{for } n \geqslant 3. \end{cases}$$

Thus,

$$\begin{aligned} \|R(t)w\| &\leqslant \left[|b_1(t)|^2|\langle w, e_1\rangle|^2 + |b_2(t)|^2|\langle w, e_2\rangle|^2 + \sum_{n=3}^{+\infty}|b_n(t)|^2|\langle w, e_n\rangle|^2\right]^{\frac{1}{2}} \\ &\leqslant \max\left(|b_1(t)|^2, |b_2(t)|^2, \max_{n\geqslant 3}|b_n(t)|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{+\infty}|\langle w, e_n\rangle|^2\right)^{\frac{1}{2}} \\ &= \max\left(1, 1 + \frac{3}{\sqrt{7}}, \max_{n\geqslant 3} u_n\right)e^{-t}\left(\sum_{n=1}^{+\infty}|\langle w, e_n\rangle|^2\right)^{\frac{1}{2}}, \end{aligned}$$

where

$$u_n = \frac{8}{7} + \frac{n^2 - 1}{7\sqrt{n^4 - 6n^2 + 1}}.$$

It is not hard to show that $(u_n)_{n\geqslant 3}$ is a positive decreasing sequence, then

$$0 \leqslant u_n \leqslant u_3, \quad \text{for } n \geqslant 3.$$

Since $u_3 = 1.358836\cdots$, it follows that

$$\|R(t)w\| \leqslant \left(1 + \frac{3}{\sqrt{7}}\right)e^{-t}\|w\|.$$

Let F be the multi-valued function given in example 3.1 by

$$F(t, f, \phi) = \{y(t) \in X \text{ such that } \|y(t)\| \leqslant \max(\|f\|, \|\phi\|_{\mathcal{C}})\}.$$

Recall that F satisfies the assumptions $(\mathbf{H_3})$–$(\mathbf{H_5})$. Let define the linear bounded operator B from U to X, by

$$Bu = \chi_{[0,\frac{\pi}{2}]}u, \quad \text{for } u \in U.$$

Let h be a function defined from $[0, a] \times \mathcal{C}$ to X by

$$h(t, \phi) = \frac{e^{-t}}{2r}\int_{-r}^{0}\phi(s)ds, \quad \text{for each } t \in [0, a], \text{ and } \phi \in \mathcal{C}.$$

It is clear that h is continuous from $[0,a]\times\mathcal{C}$ into X. Let ϕ_1, $\phi_2\in\mathcal{C}$, and $t\in[0,a]$, then

$$\begin{aligned}\|h(t,\phi_1)-h(t,\phi_2)\| &\leqslant \frac{e^{-t}}{2}\|\phi_1-\phi_2\|_{\mathcal{C}}\\ &\leqslant \frac{1}{2}\|\phi_1-\phi_2\|_{\mathcal{C}},\end{aligned}$$

which implies that condition $(\mathbf{H_{10}})$ is verified with $\delta_0=\dfrac{1}{2}$. For $(\mathbf{H_{12}})$, let $\phi\in\mathcal{C}$, and $0\leqslant t_2<t_1\leqslant a$. Then,

$$\begin{aligned}\|h(t_1,\phi)-h(t_2,\phi)\| &= \|\frac{e^{-t_1}}{2r}\int_{-r}^{0}\phi(s)ds-\frac{e^{-t_2}}{2r}\int_{-r}^{0}\phi(s)ds\|\\ &\leqslant \frac{(e^{-t_2}-e^{-t_1})}{2}\|\phi\|_{\mathcal{C}},\end{aligned}$$

which implies that $(\mathbf{H_{12}})$ is satisfied. Also, one we can see that function g given by

$$g(x)(t)=\textstyle\sum_{k=1}^{m}c_i(t)x(t_i-r),\quad \text{for } t\in[-r,0],\text{ and } x\in\mathcal{C}([-r,a];X)$$

satisfies assumptions $(\mathbf{H_8})$ and $(\mathbf{H_{13}})$, with $L_g=\sum_{k=1}^{m}\|c_i\|_{\mathcal{C}}$. The following linear equation:

$$\begin{cases}{}^{c}D_t^{q}x(t)=Ax(t)+\displaystyle\int_0^t b(t-s)Ax(s)ds+Bu(t), & t\in[0,a]\\ x(0)=x_0\end{cases}$$

is approximately controllable on $[0,a]$. Applying Theorem 3.8, we can deduce that system (23) is approximately controllable on $[0,a]$ for every

$$a\in\left]0,\sqrt[q]{\frac{q[2-3(M+1)L_g]}{8a_2}}\right[,\quad \text{where}\quad M=\left(1+\frac{3}{\sqrt{7}}\right).$$

REMARK 5. The values $c_i(t)$ are chosen quite small enough, and

$$a\in\left]0,\sqrt[q]{\frac{q[2-3(M+1)L_g]}{8a_2}}\right[$$

such that the following condition

$$\left(\frac{3M+1}{2}L_g+4a_2\frac{a^q}{q}\right)<1$$

is satisfied.

5. Conclusion

In this work, we establish several results and the approximate controllability for some integrodifferential fractional neutral inclusion with delay and nonlocal conditions. Firstly, we establish a new variation of constant formula for the mild solutions. With the lack of compactness, we prove under sufficient conditions the approximate controllability for integrodifferential fractional neutral inclusion with delay and nonlocal conditions. For that goal, we use several results from the resolvent operators theory and fixed point theory combined with the measure of noncompactness.

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