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SOME RESULTS ON MIXED FRACTIONAL INTEGRODIFFERENTIAL EQUATION IN MATRIX MB–SPACE

V. V. KHARAT AND ANAND R. RESHIMKAR*

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Abstract. In this article, we study the best approximation of nonlinear mixed fractional integrodifferential equation with Caputo fractional derivative by using a class of stochastic matrix control functions. Next, using the fixed point method, we study the Ulam-Hyers and Ulam-Hyers-Rassias stability of the non-linear fractional integrodifferential equation of the mixed type in MB-space.

1. Introduction

Fractional calculus is an extension of natural number order calculus, which involves study of integrals and derivatives of any real or complex order. It has proved its importance in describing many physical phenomena in much better sense that leads to grab the attention from researchers of other fields like physics, chemistry, biology etc. Researchers have developed different initial and boundary value problems involving fractional derivatives and study their existence, uniqueness and stability of solutions to the problems via both analytical and numerical approach.

In [17], authors have studied the existence, uniqueness and boundedness of solutions of non-linear mixed fractional integrodifferential equation with fractional non-separated boundary conditions in Banach spaces. The Ulam stability for a class of nonlinear mixed fractional integrodifferential equations was studied in [19]. The existence of solutions for the mixed iterative fractional integrodifferential equations has been studied in [16]. The existence and uniqueness of solutions of fractional quasilinear mixed integrodifferential equations with nonlocal conditions in Banach spaces has been studied by the authors in [3]. The authors in [15] have studied nonlinear mixed fractional integrodifferential equations with nonlocal condition in Banach spaces. The existence and uniqueness and Wright stability results of the Riemann-Liouville fractional integrodifferential equations by Random controllers in MB-spaces have been studied by the authors in [22].

In [20], the authors used a stochastic controller to stabilize pseudo stochastic Lie bracket (derivation, derivation) in complex MB-algebras and obtained approximation by a stochastic Lie bracket and calculated maximum error of the estimate.

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* Corresponding author.

In this article, we study the distribution functions with the ranges in a class of matrix algebras with the generalized triangular norms, to define MB-space and introduce a new class of matrix control functions.

Motivated by the above cited work, aim of this work is to study the Ulam-Hyers and Ulam-Hyers-Rassias stability of fractional non-linear integrodifferential equation of the type,

$$\begin{cases} {}^{C}D^{\alpha}_{0+}y(t) = F(t,y(t)) + \int_0^t H(t,s,y(s))ds + \int_0^T K(t,s,y(s))ds \\ \quad y(0) = m \end{cases} \tag{1.1}$$

with $t \in [0,T]$ and a continuous function $F(t,y)$, also $H(t,s,y)$, $K(t,s,y)$ are continuous functions with respect to t,s and y on $[0,T]\times\mathbb{R}\times\mathbb{R}$, m is a fixed number, ${}^{C}D^{\alpha}_{0+}y(\cdot)$ denotes Caputo fractional derivative of order α, where $0 \leqslant \alpha < 1$.

2. Preliminaries

Let $\Theta_1 = [0,T]$, $T > 0, \Theta_2 = (0,\infty)$, $\Theta_3 = (0,1]$, $\Theta_4 = [0,\infty]$, $\Theta = [0,1]$, $i(\Theta_5) = (0,1) =$ interior of Θ_5.

Let

$$diagM_n(\Theta_5) = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} = diag[a_1,\cdots,a_n], a_1,\cdots,a_n \in \Theta_5 \right\},$$

where $diagM_n(\Theta_5)$ is equipped with the partial order relation:

$$\mathbf{A} := diag[a_1,\cdots,a_n], \mathbf{B} := [b_1,\cdots,b_n] \in diagM_n(\Theta_5),$$

$\mathbf{A} \preceq \mathbf{B} \Leftrightarrow a_j \leqslant b_j$ for every $j = 1,\cdots,n$.

Also, denote $\mathbf{A} \prec \mathbf{B} \Leftrightarrow \mathbf{A} \neq \mathbf{B}$; $\mathbf{A} \ll \mathbf{B}, a_j < b_j$ for every $j = 1,\cdots,n$.

Define $\mathbf{K} = diag[k,\cdots,k]$ in $diagM_n(\Theta_5)$ where $k \in \Theta_5$. Hence, we can write $1 = diag[1,\cdots,1]$, $0 = diag[0,\cdots,0]$. In this paper, we use extension of the concept of triangular norms on $diagM_n(\Theta_5)$ as in ([11], [23]).

DEFINITION 2.1. A generalized triangular norm (in short GTN) on $diagM_n(\Theta_5)$ is an operation $\circledast : diagM_n(\Theta_5) \times diagM_n(\Theta_5) \to diagM_n(\Theta_5)$ satisfying the following conditions:

1. $A \circledast 1 = A$, $\forall A \in diagM_n(\Theta_5)$. (Boundary condition)
2. $A \circledast B = B \circledast A$, $\forall (A,B) \in diagM_n(\Theta_5)^2$. (Commutativity)
3. $[A \circledast (B \circledast C)] = [(A \circledast B) \circledast C]$, $\forall\ (A,B,C) \in diagM_n(\Theta_5)^3$. (Associativity)
4. $A \preceq A'$and $B \preceq B' \Rightarrow A \circledast B \preceq A' \circledast B'$, $\forall\ (A,A',B,B') \in diagM_n(\Theta_5)^4$. (Monotonocity)

Also, for every $A,B \in diagM_n(\Theta_5)$ and all sequences $\{A_n\},\{B_n\}$ converging to A and B, respectively, suppose that

$$\lim_k (A_k \circledast B_k) = A \circledast B,$$

then $\circledast$ on $diagM_n(\Theta_5)$ is continuous GTN (in short CGTN). We will see some examples of CGTN.

1. $\circledast_M : diag(\Theta_5) \times diag(\Theta_5) \to diag(\Theta_5)$ defined by

$$\begin{aligned} A \circledast_M B &= diag[a_1,\cdots,a_n] \circledast_M diag[b_1,\cdots,b_n] \\ &= diag[\min\{a_1,b_1\},\cdots,\min\{a_n,b_n\}], \end{aligned}$$

   then $\circledast_M$ is a CGTN, known is minimum CGTN.

2. $\circledast_P : diag(\Theta_5) \times diag(\Theta_5) \to diag(\Theta_5)$ defined by

$$A \circledast_P B = diag[a_1,\cdots,a_n] \circledast_P diag[b_1,\cdots,b_n] = diag[a_1 \cdot b_1,\cdots,a_n \cdot b_n],$$

   then $\circledast_P$ is a CGTN, called as product CGTN.

3. $\circledast_L : diag(\Theta_5) \times diag(\Theta_5) \to diag(\Theta_5)$ defined by

$$\begin{aligned} A \circledast_L B &= diag[a_1,\cdots,a_n] \circledast_P diag[b_1,\cdots,b_n] \\ &= diag[\max\{a_1+b_1-1,0\},\cdots,\max\{a_n+b_n-1,0\}], \end{aligned}$$

   then $\circledast_L$ is a CGTN, known as Lukasiewicz CGTN.

Now, we have some numerical examples.

$$\begin{aligned} diag\left[\frac{1}{3},1,\frac{2}{5},\frac{4}{5}\right] \circledast_M diag\left[\frac{2}{3},0,\frac{1}{6},\frac{1}{8}\right] &= \begin{bmatrix} \frac{1}{3} & & & \\ & 1 & & \\ & & \frac{2}{5} & \\ & & & \frac{4}{5} \end{bmatrix} \circledast_M \begin{bmatrix} \frac{2}{3} & & & \\ & 0 & & \\ & & \frac{1}{6} & \\ & & & \frac{1}{8} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & & & \\ & 0 & & \\ & & \frac{1}{6} & \\ & & & \frac{1}{8} \end{bmatrix} \\ &= diag\left[\frac{1}{3},0,\frac{1}{6},\frac{1}{8}\right]. \end{aligned}$$

$$diag\left[\frac{1}{3},1,\frac{2}{5},\frac{4}{5}\right] \circledast_P diag\left[\frac{2}{3},0,\frac{1}{6},\frac{1}{8}\right] = \begin{bmatrix} \frac{1}{3} & & & \\ & 1 & & \\ & & \frac{2}{5} & \\ & & & \frac{4}{5} \end{bmatrix} \circledast_P \begin{bmatrix} \frac{2}{3} & & & \\ & 0 & & \\ & & \frac{1}{6} & \\ & & & \frac{1}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{9} & & & \\ & 0 & & \\ & & \frac{1}{15} & \\ & & & \frac{1}{10} \end{bmatrix}$$

$$= diag\left[\frac{2}{9},0,\frac{1}{15},\frac{1}{10}\right].$$

$$diag\left[\frac{1}{3},1,\frac{2}{5},\frac{4}{5}\right] \circledast_L diag\left[\frac{2}{3},0,\frac{1}{6},\frac{1}{8}\right]. = \begin{bmatrix} \frac{1}{3} & & & \\ & 1 & & \\ & & \frac{2}{5} & \\ & & & \frac{4}{5} \end{bmatrix} \circledast_L \begin{bmatrix} \frac{2}{3} & & & \\ & 0 & & \\ & & \frac{1}{6} & \\ & & & \frac{1}{8} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$= diag[0,0,0,0].$$

Let Σ^+ denote a set of all matrix-distribution-function-valued (MDF-valued), left continuous and increasing functions $\psi : \mathbb{R} \cup \{-\infty, \infty\} \to diagM_n(\Theta_5)$ such that $\psi_0 = \mathbf{0}$ and $\psi_{+\infty} = \mathbf{1}$. Also, let $O^+ \subseteq \Sigma^+$ are all proper mappings $\psi \in \Sigma^+$ such that $l^-\psi_\infty = \mathbf{1}$ where $l^-\psi_\infty$ denotes the left hand limit of ψ_τ as $t \to \tau$.

In Σ^+, define "$\succeq$" as given below:

$$\psi \succeq \phi \Leftrightarrow \psi_\tau \geqslant \phi_\tau, \ \forall \tau \in \mathbb{R}.$$

Further, for each $t \in \mathbb{R}$,

$$\vartheta_\tau^t = \begin{cases} 0 & if \ \tau \leqslant t, \\ 1 & if \ \tau > t. \end{cases}$$

DEFINITION 2.2. Consider CGTN $\circledast$, a linear space V and MDF-valued $\Omega : V \to O^+$. Define a matrix Menger normed space (MMN-space) $(W, \Omega, \circledast)$ as follows:

- $(MMN1)$ $\Omega_\tau^v = \vartheta_\tau^0$, $\forall \ \tau > 0$ iff $w = 0$.

- $(MMN2)$ $\Omega_{\tau}^{\alpha v} = \Omega_{\frac{\tau}{\alpha}}^{v}$, $\forall\, v \in V, \alpha \in \mathbb{C}$ with $\alpha \neq 0$.

- $(MMN3)$ $\Omega_{\tau_1+\tau_2}^{v_1+v_2} \succeq \Omega_{\tau_1}^{v_1} \circledast \Omega_{\tau_2}^{v_2}$, $\forall\, v_1,\ v_2 \in V,\ \tau_1,\ \tau_2 \geqslant 0$.

In [9, 21], approximation of equations have been studied in MN-spaces, fuzzy metric spaces and random multi-normed spaces. The stability results and other properties for stochastic fractional differential and integral equations have been studied in ([1, 2, 5, 6, 10], [12]–[15], [18, 24, 25, 27]).

THEOREM 2.1. ([4, 8]) *Let $(\mathcal{C}, d)$ be a complete Θ_4-valued metric space and let $P : \mathcal{C} \to \mathcal{C}$ be a strictly contractive function with Lipschitz constant $l < 1$. Then, for a given element $\eta \in \mathcal{C}$, either $d(P^n\eta,\ P^{n+1}\eta) = \infty$, for each $n \in \mathbb{N}$ or there is $n_0 \in \mathbb{N}$ such that*

1. *$d(P^n\eta,\ P^{n+1}\eta) < \infty$ for every $n \geqslant n_0$;*

2. *the fixed point η^* of P is the limit point of the sequence $\{P^n\eta\}$;*

3. *in the set $U = \{\eta \in \mathcal{C} : d(P^{n_0}\eta,\ \eta) < \infty\}$, $\eta^\star$ is the unique fixed point of P;*

4. *$(1-l)d(\eta,\ \eta^\star) \leqslant d(\eta,\ P\eta)$ for every $\eta \in \mathcal{C}$.*

DEFINITION 2.3. the Caputo derivative of fractional of order α is defined by

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}}\, ds, \quad n-1 < \alpha < n,\ \ n = [\alpha]+1, \tag{2.1} $$

where $[\alpha]$ denotes the integer part of the real number α.

DEFINITION 2.4. If for every continuously differentiable function $y(t)$ and MDF-valued ψ satisfying

$$ \Omega_{\tau}^{\left({}^{C}D_{0+}^{\alpha} y(t) - F(t,y(t)) - \int_0^t H(t,s,y(s))ds - \int_0^T K(t,s,y(s))ds\right)} \succeq \phi_{\tau}^{t}, $$

for each $t \in \Theta_1$ and $\tau \in \Theta_2$, there exists a solution $x(t)$ of the equation (1.1) and a fixed number $\lambda > 0$ with

$$ \Omega_{\tau}^{y(t)-x(t)} \preceq \phi^{t}{}_{\frac{\tau}{\lambda}}, $$

for each $t \in \Theta_1$ and $\tau \in \Theta_2$, where λ is independent of $y(t)$ and $x(t)$, then (1.1) has Ulam-Hayers-Rassias stability.

3. Main results

Consider the following hypothesis:

$(H1)$ Assume that $M,\ L_F,\ L_H,\ L_K$ are positive real numbers with $3M(\max\{L_F, L_H, L_K\}) \in \Theta_5$ and let $F:\Theta_1\times\mathbb{R}\to\mathbb{R}$ and $H,\ K:\Theta_1\times\Theta_1\times\mathbb{R}\to\mathbb{R}$, be continuous functions satisfying

$$\Omega_\tau^{(F(t,\ y)-F(t,\ \bar{y}))}\succeq\Omega^{y-\bar{y}}_{\frac{\tau}{L_F}},\tag{3.1}$$

for all $t\in\Theta_1,\ y,\ \bar{y}\in\mathbb{R}$ and $\tau\in\Theta_2$ and

$$\Omega_\tau^{(H(t,\ s,\ y)-H(t,\ s,\ \bar{y}))}\succeq\Omega^{y-\bar{y}}_{\frac{\tau}{L_H}}\tag{3.2}$$

and

$$\Omega_\tau^{(K(t,\ s,\ y)-K(t,\ s,\ \bar{y}))}\succeq\Omega^{y-\bar{y}}_{\frac{\tau}{L_K}}.\tag{3.3}$$

THEOREM 3.1. *Suppose that the hypothesis (H_1) holds consider a CDF $y:\Theta_1\to\mathbb{R}$ satisfying*

$$\Omega_\tau^{{}^C D^\alpha_{0+}y(t)-F(t,\ y(t))-\int_0^t H(t,\ s,\ y(s))ds-\int_0^T K(t,\ s,\ y(s))ds}\succeq\phi^t_\tau,\tag{3.4}$$

$\forall\, t,\ s\in\Theta_1,\ y\in\mathbb{R}$ *and* $\tau\in\Theta_2$ *and* ϕ *is MDF-valued, satisfying*

$$\Omega_\tau^{y(t)}\succeq\phi^t_\tau\Rightarrow\Omega_\tau^{{}^{RL}I^\alpha_{0+}y(t)}\succeq\phi^t_{\frac{\tau}{M}},\quad\inf_{\nu\in\Theta_1}\phi^\nu_{\frac{\tau}{T}}\succeq\phi^t_\tau,\tag{3.5}$$

for each $t\in\Theta_1$ *and* $\tau\in\Theta_2$. *Then, there exists a unique CF* $x:\Theta_1\to\mathbb{R}$ *such that*

$$x(t)=m+{}^{RL}I^\alpha_{0+}F(t,\ x(t))+{}^{RL}I^\alpha_{0+}\int_0^\nu H(t,\ s,\ x(s))ds+{}^{RL}I^\alpha_{0+}\int_0^T K(t,\ s,\ x(s))ds.\tag{3.6}$$

with

$$\Omega_\tau^{y(t)-x(t)}\succeq\phi^t_{\frac{M\tau}{1-3M(\max\{L_F,\ L_H,\ L_K\})}}.\tag{3.7}$$

Proof. Let $y,\ \bar{y}\in\mathcal{C}$, we define

$$d(y,\bar{y})=inf\Big\{\lambda\in\Theta_4\mid\Omega_\tau^{y(t)-\bar{y}(t)}\succeq\phi^t_{\frac{\tau}{\lambda}}\Big\},\tag{3.8}$$

for each $t\in\Theta_1$ and $\tau\in\Theta_2$, where $\mathcal{C}=\{y:\Theta_1\to\mathbb{R}$ is a CF$\}$. Let $P:\mathcal{C}\to\mathcal{C}$ is given by

$$P(y(t))=m+{}^{RL}I^\alpha_{0+}F(t,\ y(t))+{}^{RL}I^\alpha_{0+}\int_0^\nu H(t,\ s,\ y(s))ds+{}^{RL}I^\alpha_{0+}\int_0^T K(t,\ s,\ y(s))ds,\tag{3.9}$$

$\forall\, y,\ t\in\Theta_1$.

Claim 1: P is strictly contractive on $\mathcal{C}$.

Let $\lambda_{y\bar{y}} \in \Theta_4$ be a fixed number with $d(y,\ \bar{y}) \leqslant \lambda_{y\bar{y}},\ \forall\ y,\ \bar{y} \in \mathcal{C}$. Then from equation (3.8), we get

$$\Omega_{\tau}^{y(t)-\bar{y}(t)} \succeq \phi^{t}_{\frac{\tau}{\lambda_{y\bar{y}}}}. \tag{3.10}$$

Let $0 = s_1 < s_2 < \cdots < s_n = T$, $\Delta\nu_i = s_i - s_{i-1} = \dfrac{|T-0|}{n}, i = 1,2,\cdots,n$ and $||\Delta\nu|| = \max_{1\leqslant i\leqslant n}(\Delta s_i)$, for each $t,\ \nu \in \Theta_1$ and $\tau \in \Theta_2$. Using equations (3.2), (3.5) and (3.10), we have

$$\begin{aligned}
&\Omega_{\tau}^{\left[\int_0^{\nu}\left(H(t,\ s,\ y(s))-H(t,\ s,\ \bar{y}(s))ds\right)\right]}\\
&= \Omega_{\tau}^{\lim_{||\Delta\nu||\to 0}\Sigma_{i=1}^{n}\left[\left(H(t,\ s_i,\ y(s_i))-H(t,\ s_i,\ \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&= \lim_{||\Delta\nu||\to 0} \Omega_{\tau}^{\Sigma_{i=1}^{n}\left[\left(H(t,\ s_i,\ y(s_i))-H(t,\ s_i,\ \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \lim_{||\Delta\nu||\to 0} \circledast_M \Omega_{\frac{\tau}{n}}^{\left[\left(H(t,\ s_i,\ y(s_i))-H(t,\ s_i,\ \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \Omega_{\frac{\tau}{n\Delta\nu_i}}^{\left[\left(H(t,\ \nu,\ y(\nu))-H(t,\ \nu,\ \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{t\in\Theta_1} \Omega_{\frac{\tau}{T}}^{\left[\left(H(t,\ \nu,\ y(\nu))-H(t,\ \nu,\ \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \phi^{\nu}_{\frac{\tau}{T\lambda_{y\bar{y}}L_H}}\\
&\succeq \phi^{t}_{\frac{\tau}{T\lambda_{y\bar{y}}L_H}}.
\end{aligned} \tag{3.11}$$

Also, from equations (3.3), (3.5) and (3.10), we have

$$\begin{aligned}
&\Omega_{\tau}^{\left[\int_0^{\nu}\left(K(t,\ s,\ y(s))-K(t,\ s,\ \bar{y}(s))ds\right)\right]}\\
&= \Omega_{\tau}^{\lim_{||\Delta\nu||\to 0}\Sigma_{i=1}^{n}\left[\left(K(t,\ s_i,\ y(s_i))-K(t,\ s_i,\ \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&= \lim_{||\Delta\nu||\to 0} \Omega_{\tau}^{\Sigma_{i=1}^{n}\left[\left(K(t,\ s_i,\ y(s_i))-K(t,\ s_i,\ \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \lim_{||\Delta\nu||\to 0} \circledast_M \Omega_{\frac{\tau}{n}}^{\left[\left(K(t,\ s_i,\ y(s_i))-K(t,\ s_i,\ \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \Omega_{\frac{\tau}{n\Delta\nu_i}}^{\left[\left(K(t,\ \nu,\ y(\nu))-K(t,\ \nu,\ \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{t\in\Theta_1} \Omega_{\frac{\tau}{T}}^{\left[\left(K(t,\ \nu,\ y(\nu))-K(t,\ \nu,\ \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \phi^{\nu}_{\frac{\tau}{T\lambda_{y\bar{y}}L_K}}\\
&\succeq \phi^{t}_{\frac{\tau}{T\lambda_{y\bar{y}}L_K}}.
\end{aligned} \tag{3.12}$$

Now, by using equations (3.1), (3.5), (3.6), (3.10), (3.11), (3.12) we will have

$$
\begin{aligned}
&\Omega_{\tau}^{(Py)(t)-(P\bar{y})(t)}\\
&=\Omega_{\tau}^{{}^{RL}I^{\alpha}_{0+}[F(t,y(t))-F(t,\bar{y}(t))]+{}^{RL}I^{\alpha}_{0+}\left[\int_0^t\left(H(t,s,y(s))-H(t,s,\bar{y}(s))\right)ds\right]+{}^{RL}I^{\alpha}_{0+}\left[\int_0^T\left(K(t,s,y(s))-K(t,s,\bar{y}(s))\right)ds\right]}\\
&\succeq \Omega_{\frac{\tau}{3}}^{{}^{RL}I^{\alpha}_{0+}[F(t,\,y(t))-F(t,\,\bar{y}(t))]} \circledast \Omega_{\frac{\tau}{3}}^{{}^{RL}I^{\alpha}_{0+}\left[\int_0^t\left(H(t,\,s,\,y(s))-H(t,\,s,\,\bar{y}(s))\right)ds\right]}\\
&\quad\circledast \Omega_{\frac{\tau}{3}}^{{}^{RL}I^{\alpha}_{0+}\left[\int_0^T\left(K(t,\,s,\,y(s))-K(t,\,s,\,\bar{y}(s))\right)ds\right]}\\
&\succeq \phi^t_{\frac{\tau}{3M\lambda_{y\bar{y}}L_F}} \circledast \phi^t_{\frac{\tau}{3M\lambda_{y\bar{y}}L_H}} \circledast \phi^t_{\frac{\tau}{3M\lambda_{y\bar{y}}L_K}}\\
&\succeq \phi^t_{\frac{\tau}{3M\lambda_{y\bar{y}}\left[\max\{L_F,L_H,L_K\}\right]}}.
\end{aligned}
\tag{3.13}
$$

So, we get, $d(Py,\, P\bar{y}) \leqslant 3M\lambda_{y\bar{y}}\left[\max\{L_F,\, L_H,\, L_K\}\right]$, $\forall\, t\in\Theta_1, \tau\in\Theta_2$.

Therefore, we deduce that $d(Py,\, P\bar{y}) \leqslant [3M\left(\max\{L_F,L_H,L_K\}\right)]d(y,\bar{y})$ for any $y,\ \bar{y}\in\mathcal{C}$ where $3M\left[\max\{L_F,\, L_H,\, L_K\}\right]\in i(\Theta_5)$. From equation (3.6), we can find a fixed number $\lambda\in\Theta_2$ such that

$$
\begin{aligned}
\Omega_{\tau}^{y(t)-y_0(t)} &= \Omega_{\tau}^{\left[m+{}^{RL}I^{\alpha}_{0+}F(t,\,y(t))+{}^{RL}I^{\alpha}_{0+}\int_0^t H(t,\,s,\,y(s))ds+{}^{RL}I^{\alpha}_{0+}\int_0^T K(t,\,s,\,y(s))ds-y_0(t)\right]}\\
&\succeq \phi^t_{\frac{\tau}{\lambda}},
\end{aligned}
$$

for arbitrary $\bar{y}_0\in\mathcal{C}$, $\forall\, t\in\Theta_1, \tau\in\Theta_2$.

Since $F(t,\ \bar{y}_0(t))$, $H(t,\ s,\ \bar{y}_0(s))$, $K(t,\ s,\ \bar{y}_0(s))$, $\bar{y}_0(t)$, $\min_{t\in\Theta_1}\phi^t_\tau>0$, are bounded, using equation (3.8) will imply that $d(Py,\ y_0)<\infty$. Then from Theorem 2.1, there exists a CF $x:\Theta_1\to\mathbb{R}$ such that $P^n x\to x$ in $(\mathcal{C},\ d)$ and $Px=x$.

As $\bar{y},\ x$ both are bounded on Θ_1, for each $\bar{y}\in\mathcal{C}$ and $\min_{t\in\Theta_1}\phi^t_\tau>0$, we get a fixed number $\lambda_{\bar{y}}\in\Theta_4$ such that

$$\Omega_{\tau}^{(\bar{y}_0(t)-\bar{y}(t))}\succeq\phi^t_{\frac{\tau}{\lambda_{\bar{y}}}},$$

for any $t\ \in\Theta_1,\ \tau\in\Theta_2$. Hence, $d(\bar{y}_0,\ \bar{y})<\infty$ for any $\bar{y}\in\mathcal{C}$. Therefore, we get $\mathcal{C}=\{\bar{y}\in\mathcal{C}\mid d(\bar{y}_0,\ \bar{y})<\infty\}$. Further, from Theorem 2.1 and equation (3.6) will imply the uniqueness of y_0.

Now, using the equations (3.3), (3.5) and (3.9), we get

$$\Omega_{\tau}^{y(t)-m-{}^{RL}I^{\alpha}_{0+}F(t,\,y(t))-{}^{RL}I^{\alpha}_{0+}\int_0^t H(t,\,s,\,y(s))ds-{}^{RL}I^{\alpha}_{0+}\int_0^T K(t,\,s,\,y(s))ds}\succeq\phi^t_{\frac{\tau}{M}}.$$

Thus, we get

$$\Omega_{\tau}^{y(t)-Py(t)}\succeq\phi^t_{\frac{\tau}{M}},$$

for any $t\in\Theta_1$ and $\tau\in\Theta_2$ which gives

$$d(y,\ Py)\leqslant M. \tag{3.14}$$

Also, from Theorem 2.1 and equation (3.14), we deduce that

$$d(y,\ x) \leqslant \frac{1}{1-3M(\max\{L_F,\ L_H,\ L_K\})} d(Py,\ y) \leqslant \frac{M}{1-3M(\max\{L_F,\ L_H,\ L_K\})},$$

which implies equation (3.7). □

THEOREM 3.2. *Let $L_F,\ L_H,\ L_K \in \Theta_2$ be fixed numbers such that $3M(\max\{L_F,\ TL_H,\ TL_K\}) \in i(\Theta_5)$. Consider CFs $F:\Theta_1\times\mathbb{R}\to\mathbb{R}$, $H:\Theta_1\times\Theta_1\times\mathbb{R}\to\mathbb{R}$ satisfying equations (1.1), (3.2) and (3.3) respectively. If for $\varepsilon \geqslant 0$, $\tau\in\Theta_2$, $\varepsilon_\tau := diag\left[e^{-\frac{\varepsilon}{\tau}},\ \cdots,\ e^{-\frac{\varepsilon}{\tau}}\right]$ and a CDF $y:\Theta_1\to\mathbb{R}$ satisfies*

$$\Omega_\tau^{{}^C D_{0+}^{\alpha} y(t) - F(t,\ y(t)) - \int_0^t H(t,\ s,\ y(s))ds - \int_0^T K(t,\ s,\ y(s))ds} \succeq \varepsilon_\tau,$$

for all $t,\ s\in\Theta_1$, $y\in\mathbb{R}$ and $\tau\in\Theta_2$, then we there exists a unique CF $x:\Theta_1\to\mathbb{R}$ satisfying equation (3.6) and

$$\Omega_\tau^{(y(t)-x(t))} \succeq \varepsilon_{\frac{M\tau}{1-3M(\max\{L_F,\ TL_H,\ TL_K\})}},\ \forall t\in\Theta_1,\ \forall y\in\mathbb{R}. \tag{3.15}$$

Proof. Let $y,\ \bar{y}\in\mathcal{C}$, we define

$$d(y,\ \bar{y}) = inf\left\{\lambda\in\Theta_4 \mid \Omega_\tau^{y(t)-\bar{y}(t)} \succeq \varepsilon_{\frac{\tau}{\lambda}}\right\}, \tag{3.16}$$

for each $t\in\Theta_1$ and $\tau\in\Theta_2$, where $\mathcal{C} = \{y:\Theta_1\to\mathbb{R}$ is a CF$\}$. Let $P:\mathcal{C}\to\mathcal{C}$ is given by

$$P(y(t)) = m + {}^{RL}I_{0+}^{\alpha} F(t,\ y(t)) + {}^{RL}I_{0+}^{\alpha}\int_0^v H(t,\ s,\ y(s))ds + {}^{RL}I_{0+}^{\alpha}\int_0^T K(t,\ s,\ y(s))ds, \tag{3.17}$$

$\forall y,\ t\in\Theta_1$. First, we prove that the operator P is strictly contractive on $\mathcal{C}$. Let $\lambda_{y\bar{y}}\in\Theta_4$ be a fixed number with $d(y,\ \bar{y}) \leqslant \lambda_{y\bar{y}}$, $\forall y,\ \bar{y}\in\mathcal{C}$. Then from equation (3.16), we get

$$\Omega_\tau^{y(t)-\bar{y}(t)} \succeq \varepsilon_{\frac{\tau}{\lambda_{y\bar{y}}}}, \tag{3.18}$$

Let $0=s_1<s_2<\cdots<s_n=T$, $\Delta v_i = s_i - s_{i-1} = \dfrac{|T-0|}{n}$, $i=1,\ 2,\ \cdots,\ n$ and $||\Delta v|| = \max_{1\leqslant i\leqslant n}(\Delta s_i)$, for each $t,\ v\in\Theta_1$ and $\tau\in\Theta_2$. Using equations (3.2) and (3.17), we have

$$\begin{aligned}
&\Omega_\tau^{\left[\int_0^v \left(H(t,\ s,\ y(s)) - H(t,\ s,\ \bar{y}(s))ds\right)\right]}\\
&= \Omega_\tau^{\lim_{||\Delta v||\to 0}\Sigma_{i=1}^n\left[\left(H(t,\ s_i,\ y(s_i)) - H(t,\ s_i,\ \bar{y}(s_i))\right)\Delta v_i\right]}\\
&= \lim_{||\Delta v||\to 0}\Omega_\tau^{\Sigma_{i=1}^n\left[\left(H(t,\ s_i,\ y(s_i)) - H(t,\ s_i,\ \bar{y}(s_i))\right)\Delta v_i\right]}
\end{aligned}$$

$$
\begin{aligned}
&\succeq \lim_{||\Delta\nu||\to 0} \circledast_M \Omega_{\frac{\tau}{n}}^{\left[\left(H(t,\, s_i,\, y(s_i))-H(t,\, s_i,\, \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \Omega_{\frac{\tau}{n\Delta\nu_i}}^{\left[\left(H(t,\, \nu,\, y(\nu))-H(t,\, \nu,\, \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{t\in\Theta_1} \Omega_{\frac{\tau}{T}}^{\left[\left(H(t,\, \nu,\, y(\nu))-H(t,\, \nu,\, \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \varepsilon^{\nu}_{\frac{\tau}{T\lambda_{y\bar{y}}L_H}}\\
&\succeq \varepsilon^{t}_{\frac{\tau}{T\lambda_{y\bar{y}}L_H}}.
\end{aligned}
\tag{3.19}
$$

Also, from equations (3.3) and (3.18), we have

$$
\begin{aligned}
&\Omega_{\tau}^{\left[\int_0^{\nu}\left(K(t,\, s,\, y(s))-K(t,\, s,\, \bar{y}(s))ds\right)\right]}\\
&= \Omega_{\tau}^{\lim_{||\Delta\nu||\to 0}\Sigma_{i=1}^{n}\left[\left(K(t,\, s_i,\, y(s_i))-K(t,\, s_i,\, \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&= \lim_{||\Delta\nu||\to 0} \Omega_{\tau}^{\Sigma_{i=1}^{n}\left[\left(K(t,\, s_i,\, y(s_i))-K(t,\, s_i,\, \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \lim_{||\Delta\nu||\to 0} \circledast_M \Omega_{\frac{\tau}{n}}^{\left[\left(K(t,\, s_i,\, y(s_i))-K(t,\, s_i,\, \bar{y}(s_i))\right)\Delta\nu_i\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \Omega_{\frac{\tau}{n\Delta\nu_i}}^{\left[\left(K(t,\, \nu,\, y(\nu))-K(t,\, \nu,\, \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{t\in\Theta_1} \Omega_{\frac{\tau}{T}}^{\left[\left(K(t,\, \nu,\, y(\nu))-K(t,\, \nu,\, \bar{y}(\nu))\right)\right]}\\
&\succeq \inf_{\nu\in\Theta_1} \varepsilon^{\nu}_{\frac{\tau}{T\lambda_{y\bar{y}}L_H}}\\
&\succeq \varepsilon^{t}_{\frac{\tau}{T\lambda_{y\bar{y}}L_K}}.
\end{aligned}
\tag{3.20}
$$

Now, by using equations (3.1), (3.17) and (3.18), we will have

$$
\begin{aligned}
&\Omega_{\tau}^{(Py)(t)-(P\bar{y})(t)}\\
&= \Omega_{\tau}^{{}^{RL}I_{0+}^{\alpha}[F(t,y(t))-F(t,\bar{y}(t))]+{}^{RL}I_{0+}^{\alpha}\left[\int_0^t\left(H(t,s,y(s))-H(t,s,\bar{y}(s))\right)ds\right]+{}^{RL}I_{0+}^{\alpha}\left[\int_0^T\left(K(t,s,y(s))-K(t,s,\bar{y}(s))\right)ds\right]}\\
&\succeq \Omega_{\frac{\tau}{3}}^{{}^{RL}I_{0+}^{\alpha}[F(t,\, y(t))-F(t,\bar{y}(t))]} \circledast \Omega_{\frac{\tau}{3}}^{{}^{RL}I_{0+}^{\alpha}\left[\int_0^t\left(H(t,\, s,\, y(s))-H(t,\, s,\, \bar{y}(s))\right)ds\right]}\\
&\quad \circledast \Omega_{\frac{\tau}{3}}^{{}^{RL}I_{0+}^{\alpha}\left[\int_0^T\left(K(t,\, s,\, y(s))-K(t,\, s,\, \bar{y}(s))\right)ds\right]}\\
&\succeq \varepsilon_{\frac{\tau}{3M\lambda_{y\bar{y}}L_F}} \circledast \varepsilon_{\frac{\tau}{3M\lambda_{y\bar{y}}TL_H}} \circledast \varepsilon_{\frac{\tau}{3M\lambda_{y\bar{y}}TL_K}}\\
&\succeq \varepsilon_{\frac{\tau}{3M\lambda_{y\bar{y}}\left[\max\{L_F,\, TL_H,\, TL_K\}\right]}}.
\end{aligned}
\tag{3.21}
$$

for each $t \in \Theta_1$ and $\tau \in \Theta_2$. So, we get, $d(Py, P\bar{y}) \leqslant 3M\lambda_{y\bar{y}}\left[\max\{L_F, TL_H, TL_K\}d(y,\bar{y})\right]$, $\forall\, t \in \Theta_1,\ \tau \in \Theta_2$, for any $y,\ \bar{y} \in \mathcal{C}$, where $3M(\max\{L_F,\ TL_H,\ TL_K\}) \in i(\Theta_5)$. From equation (3.6), we can find a fixed number $\lambda \in \Theta_2$ such that

$$\begin{aligned}\Omega_\tau^{Py(t)-\bar{y}_0(t)} &= \Omega_\tau^{\left[m+{}^{RL}I_{0+}^{\alpha}F(t,\ y(t))+{}^{RL}I_{0+}^{\alpha}\int_0^t H(t,\ s,\ y(s))ds+{}^{RL}I_{0+}^{\alpha}\int_0^T K(t,\ s,\ y(s))ds-\bar{y}_0(t)\right]}\\ &\succeq \varepsilon_{\frac{\tau}{\lambda}},\end{aligned}$$

for arbitrary $y_0 \in \mathcal{C}$, $\forall\, t \in \Theta_1,\ \tau \in \Theta_2$.

As $F(t,\ \bar{y}_0(t))$, $H(t,\ s,\ \bar{y}_0(s))$, $K(t,\ s,\ \bar{y}_0(s))$, $\bar{y}_0(t)$ are bounded and using equation (3.16) will imply that $d(Py,\ \bar{y}_0) < \infty$. Then from Theorem 2.1, there exists a CF $x : \Theta_1 \to \mathbb{R}$ such that $P^n x \to x$ in $(\mathcal{C},\ d)$ and $Px = x$. By using the method that is followed in the proof of Theorem 3.1, we obtain $\mathcal{C} = \{y \in \mathcal{C} \mid d(\bar{y}_0,\ y) < \infty\}$. Further, the Theorem 2.1 and equation (3.9), we get the uniqueness of y_0.

Now, using the equation (3.3) and Theorem 5 in [7]

$$\Omega_\tau^{y(t)-m-{}^{RL}I_{0+}^{\alpha}F(t,\ x(t))-{}^{RL}I_{0+}^{\alpha}\int_0^v H(t,\ s,\ x(s))ds-{}^{RL}I_{0+}^{\alpha}\int_0^T K(t,\ s,\ x(s))ds} \succeq \varepsilon_{\frac{\tau}{M}}.$$

for any $t \in \Theta_1$ and $\tau \in \Theta_2$ which gives

$$d(y,\ Py) \leqslant M. \tag{3.22}$$

Also, from Theorem 2.1 and equation (3.14), we deduce that

$$\Omega_\tau^{(y(t)-x(t))} \succeq \varepsilon_{\frac{M\tau}{1-3M(\max\{L_F,\ TL_H,\ TL_K\})}}$$

which implies equation (3.15) for all $t \in \Theta_1$. □

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V. V. Kharat
Department of Mathematics
N. B. Navale Sinhgad College of Engg.
Kegaon, Solapur-413255, India (M.S.)
e-mail: vvkvinod9@gmail.com

Anand R. Reshimkar
Department of Mathematics
D. B. F. Dayanand College of Arts and Science
Solapur-413002, India (M.S.)
e-mail: anand.reshimkar@gmail.com