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SOLVABILITY FOR ITERATIVE SYSTEMS OF HADAMARD FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider an iterative system of Hadamard fractional boundary value problems. The sufficient conditions are derived for the existence of at least one positive solution by utilizing Guo–Krasnosel'skii fixed point theorem. The obtained results in the paper are illustrated with an example for their feasibility.

1. Introduction

Many real-world problems benefit significantly from using fractional calculus (FC) in mathematical modeling. A variety of materials and processes with characteristics of heredity and memory can be accurately described by the nonlocal nature of the FC [12, 10]. There are numerous applications in a variety of scientific disciplines, including biomathematics [6], random processes [13], viscoelasticity [14], non-Newtonian fluid mechanics [2], and characterization of anomalous diffusion [15].

It is evident from the literature that the majority of the research in the field of fractional differential equations (FDEs) focuses on Riemann-Liouville or Caputo type derivatives. The literature on FDEs of the Hadamard type is not enriched yet. The Hadamard fractional derivative, first proposed in 1892, is distinct from the aforementioned type of derivatives which includes a logarithmic function with any exponent as part of the kernel of the integral [7]. For a detailed description of Hadamard fractional derivative and integral, see [1, 3, 4, 8, 9]. In different industries, such as telecommunication equipment, synthetic chemicals, automobiles, and pharmaceuticals, BVPs are frequently used. In these processes, positive solutions seem to be beneficial; see [16, 17].

Recently, Thiramanus et al. [18], Pei et al. [19], Tariboon et al. [20], Wang et al. [21], Zhang et al. [24, 25] studied the existence, uniqueness, and multiplicity results on positive solutions to various types of Hadamard FBVPs. In [22], Yang investigated the

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existence of at least one positive solution for the coupled system of Hadamard FDEs for $z \in (1,e)$:

$$\begin{cases} {}^H\mathfrak{D}^{\alpha} u_1(z) + \lambda f(z, u_1(z), u_2(z)) = 0, \\ {}^H\mathfrak{D}^{\beta} u_2(z) + \lambda g(z, u_1(z), u_2(z)) = 0, \\ u_1^{(j)}(1) = u_2^{(j)}(1) = 0, \quad 0 \leqslant j \leqslant n-2, \\ u_1(e) = a u_2(\xi), \quad u_2(e) = b u_1(\eta), \end{cases}$$

where $\alpha, \beta \in (n-1, n]$, $n \geqslant 3$, $\eta, \xi \in (1,e)$, and ${}^H\mathfrak{D}^{\bullet}$ is the Hadamard fractional derivative.

In [23], Zhai et al. established the existence and uniqueness of solutions for the FBVP for $z \in (1,e)$:

$$\begin{cases} {}^H\mathfrak{D}^{\alpha} u_1(z) + f\big(z, u_2(z)\big) = \ell_f, \\ {}^H\mathfrak{D}^{\alpha} u_2(z) + g\big(z, u_1(z)\big) = \ell_g, \\ u_1^{(j)}(1) = u_2^{(j)}(1) = 0, \quad 0 \leqslant j \leqslant n-2, \\ u_1(e) = a u_2(\xi), \quad u_2(e) = b u_1(\eta), \end{cases}$$

where $\alpha, \beta \in (n-1, n]$, $n \geqslant 3$, $\eta, \xi \in (1,e)$, ℓ_f, ℓ_g are constants and ${}^H\mathfrak{D}^{\bullet}$ is the Hadamard fractional derivative.

However, to the best of our knowledge, the existence of positive solutions to iterative systems of Hadamard FBVP has yet to be explored. Motivated by aforementioned works, in this paper we consider the iterative system of FBVPs

$$\left.\begin{aligned} {}^H\mathfrak{D}^q_{1+} u_1(z) + \lambda_1 p_1(z) g_1\big(u_2(z)\big) &= 0, \\ {}^H\mathfrak{D}^q_{1+} u_2(z) + \lambda_2 p_2(z) g_2\big(u_3(z)\big) &= 0, \\ {}^H\mathfrak{D}^q_{1+} u_3(z) + \lambda_2 p_3(z) g_3\big(u_4(z)\big) &= 0, \\ &\cdots \\ {}^H\mathfrak{D}^q_{1+} u_{n-1}(z) + \lambda_n p_{n-1}(z) g_{n-1}\big(u_n(z)\big) &= 0, \\ {}^H\mathfrak{D}^q_{1+} u_n(z) + \lambda_n p_n(z) g_n\big(u_{n+1}(z)\big) &= 0, \\ u_{n+1}(z) = u_1(z), \quad z &\in (1,e), \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} u_1(1) = 0,\ u_1'(1) = 0,\ b_1 u_1(e) + b_2\, {}^H\mathfrak{D}^r_{1+} u_1(e) &= 0, \\ u_2(1) = 0,\ u_2'(1) = 0,\ b_1 u_2(e) + b_2\, {}^H\mathfrak{D}^r_{1+} u_2(e) &= 0, \\ u_3(1) = 0,\ u_3'(1) = 0,\ b_1 u_3(e) + b_2\, {}^H\mathfrak{D}^r_{1+} u_3(e) &= 0, \\ &\cdots \\ u_{n-1}(1) = 0,\ u_{n-1}'(1) = 0,\ b_1 u_{n-1}(e) + b_2\, {}^H\mathfrak{D}^r_{1+} u_{n-1}(e) &= 0, \\ u_n(1) = 0,\ u_n'(1) = 0,\ b_1 u_n(e) + b_2\, {}^H\mathfrak{D}^r_{1+} u_n(e) &= 0, \end{aligned}\right\} \tag{2}$$

where $q \in (2,3], r \in (1,2]$, b_1, b_2 are positive constants and ${}^H\mathfrak{D}^q_{1+}, {}^H\mathfrak{D}^r_{1+}$ are the Hadamard fractional derivatives.

We assume the following conditions are true in the entire paper:

(H$_1$) $p_k : [1,e] \to \mathbb{R}^+$ is continuous and p_k does not vanish identically on any closed subinterval of $[1,e]$, for $k = \overline{1,n}$,

(H$_2$) $\mathrm{g}_k : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $k = \overline{1,n}$,

(H$_3$) $b_2(r-1) > \dfrac{b_1\Gamma(q-r)}{\Gamma(q)}$ and $\Delta = b_2\Gamma(q) + b_1\Gamma(q-r)$,

(H$_4$) each of $\mathrm{g}_{k0} = \lim\limits_{x\to 0^+} \dfrac{\mathrm{g}_k(x)}{x}$ and $\mathrm{g}_{k\infty} = \lim\limits_{x\to\infty} \dfrac{\mathrm{g}_k(x)}{x}$, for $1 \leqslant k \leqslant n$, exists as positive real numbers.

The rest of the paper is organized as follows. In Sect. 2, we construct the Kernel and its bounds. In Sect. 3, we establish criteria for determining an optimal eigenvalue interval for which the FBVP (1)–(2) has at least one positive solution. In Sect. 4, an example is coined in support of validity of the results obtained in the previous sections.

2. Preliminaries, kernel and bounds

DEFINITION 1. [10] The Hadamard fractional integral of order $q \in \mathbb{R}^+$ of the function $\mathfrak{h}(\mathrm{z})$ is defined as

$$ {}^{H}\mathrm{I}_1^q\mathfrak{h}(\mathrm{z}) = \frac{1}{\Gamma(q)} \int_1^{\mathrm{z}} \left(\ln\frac{\mathrm{z}}{\mathrm{y}}\right)^{q-1} \mathfrak{h}(\mathrm{y})\frac{d\mathrm{y}}{\mathrm{y}}, \quad \mathrm{z} \in [1,e]. $$

DEFINITION 2. [10] The Hadamard fractional derivative of order $q \in (n-1,n]$, $n \in \mathbb{Z}^+$ of the function $\mathfrak{h}(\mathrm{z})$ is defined as

$$ {}^{H}\mathfrak{D}_1^q\mathfrak{h}(\mathrm{z}) = \frac{1}{\Gamma(n-q)} \left(\mathrm{z}\frac{d}{d\mathrm{z}}\right)^n \int_1^{\mathrm{z}} \left(\ln\frac{\mathrm{z}}{\mathrm{y}}\right)^{n-q+1} \mathfrak{h}(\mathrm{y})\frac{d\mathrm{y}}{\mathrm{y}}, \quad \mathrm{z} \in [1,e]. $$

LEMMA 1. [10] *If* $\mathrm{a}, q, \varpi > 0$, *then*

$$ \left({}^{H}\mathfrak{D}_{\mathrm{a}}^q\left(\ln\frac{\mathrm{z}}{\mathrm{a}}\right)^{\varpi-1}\right)(\mathrm{y}) = \frac{\Gamma(\varpi)}{\Gamma(\varpi-q)}\left(\ln\frac{\mathrm{y}}{\mathrm{a}}\right)^{\varpi-q-1}. $$

LEMMA 2. *Let* $\mathfrak{h}(\mathrm{z}) \in C\big([1,e],\mathbb{R}\big)$ *and* $\Delta \neq 0$. *Then* $u_1(\mathrm{z}) \in C\big([1,e],\mathbb{R}\big)$ *is a solution of the FBVP*

$$ {}^{H}\mathfrak{D}_{1^+}^q u_1(\mathrm{z}) + \mathfrak{h}(\mathrm{z}) = 0, \quad \mathrm{z} \in (1,e), \tag{3} $$

$$ \left.\begin{aligned} &u_1(1) = 0, \quad u_1'(1) = 0, \\ &b_1 u_1(e) + b_2\, {}^{H}\mathfrak{D}_{1^+}^r u_1(e) = 0 \end{aligned}\right\} \tag{4} $$

if and only if

$$ u_1(\mathrm{z}) = \int_1^e \mathrm{G}(\mathrm{z},\mathrm{y})\mathfrak{h}(\mathrm{y})\frac{d\mathrm{y}}{\mathrm{y}}, $$

where

$$\mathtt{G}(\mathrm{z},\mathrm{y}) = \begin{cases} \mathtt{G}_1(\mathrm{z},\mathrm{y}), & 1 \leqslant \mathrm{z} \leqslant \mathrm{y} \leqslant e, \\ \mathtt{G}_2(\mathrm{z},\mathrm{y}), & 1 \leqslant \mathrm{y} \leqslant \mathrm{z} \leqslant e, \end{cases} \tag{5}$$

$$\mathtt{G}_1(\mathrm{z},\mathrm{y}) = \frac{1}{\Delta}\left[b_2(1-\ln \mathrm{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln \mathrm{z})^{q-1}(1-\ln \mathrm{y})^{q-1},$$

$$\mathtt{G}_2(\mathrm{z},\mathrm{y}) = \mathtt{G}_1(\mathrm{z},\mathrm{y}) - \frac{1}{\Gamma(q)}\left(\ln\frac{\mathrm{z}}{\mathrm{y}}\right)^{q-1}.$$

Proof. Let $u_1(\mathrm{z}) \in C^{[q]+1}[1,e]$ be a solution of FBVP (3)–(4) and is uniquely expressed as

$$u_1(\mathrm{z}) = \sum_{k=1}^{3} \mathtt{c}_k(\ln \mathrm{z})^{q-k} - \int_1^{\mathrm{z}} \left(\ln\frac{\mathrm{z}}{\mathrm{y}}\right)^{q-1} \frac{\mathfrak{h}(\mathrm{y})}{\Gamma(q)}\frac{d\mathrm{y}}{\mathrm{y}}. \tag{6}$$

By the condition (4), we get $\mathtt{c}_3 = 0,\ \mathtt{c}_2 = 0$ and

$$\mathtt{c}_1 = \frac{1}{\Delta}\int_1^e \left[b_2(1-\ln \mathrm{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](1-\ln \mathrm{y})^{q-1}\mathfrak{h}(\mathrm{y})\frac{d\mathrm{y}}{\mathrm{y}}.$$

Hence the unique solution of FBVP (3)–(4) is

$$\begin{aligned} u_1(\mathrm{z}) &= \begin{cases} \dfrac{1}{\Delta}\displaystyle\int_1^e \left[b_2(1-\ln \mathrm{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln \mathrm{z})^{q-1} \\ \times(1-\ln \mathrm{y})^{q-1}\mathfrak{h}(\mathrm{y})\dfrac{d\mathrm{y}}{\mathrm{y}} - \displaystyle\int_1^{\mathrm{z}} \left(\ln\frac{\mathrm{z}}{\mathrm{y}}\right)^{q-1}\frac{\mathfrak{h}(\mathrm{y})}{\Gamma(q)}\frac{d\mathrm{y}}{\mathrm{y}} \end{cases} \\ &= \int_1^e \mathtt{G}(\mathrm{z},\mathrm{y})\mathfrak{h}(\mathrm{y})\frac{d\mathrm{y}}{\mathrm{y}}, \end{aligned}$$

where $\mathtt{G}(\mathrm{z},\mathrm{y})$ is given in (5). The converse follows by direct computation. The proof is completed. □

LEMMA 3. *Assume that* (H_3) *holds. The Kernel* $\mathtt{G}(\mathrm{z},\mathrm{y})$ *have the following properties:*

(a) $\mathtt{G}(\mathrm{z},\mathrm{y}) \geqslant 0,\ \forall\, \mathrm{z},\mathrm{y} \in [1,e]$,

(b) $\mathtt{G}(\mathrm{z},\mathrm{y}) \leqslant \mathtt{G}(e,\mathrm{y}),\ \forall\, \mathrm{z},\mathrm{y} \in [1,e]$,

(c) $\mathtt{G}(\mathrm{z},\mathrm{y}) \geqslant \left(\frac{1}{4}\right)^{q-1}\mathtt{G}(e,\mathrm{y}),\ \forall \mathrm{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right], \mathrm{y} \in (1,e)$.

Proof. The Kernel $\mathtt{G}(\mathrm{z},\mathrm{y})$ is given in (5). Let $1 \leqslant \mathrm{z} \leqslant \mathrm{y} \leqslant e$. Then

$$\mathtt{G}(\mathrm{z},\mathrm{y}) = \frac{1}{\Delta}\left[b_2(1-\ln \mathrm{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln \mathrm{z})^{q-1}(1-\ln \mathrm{y})^{q-1} \geqslant 0.$$

Let $1 \leqslant y \leqslant z \leqslant e$. Then

$$G(z,y) = \begin{cases} \frac{1}{\Delta}\left[b_2(1-\ln y)^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln z)^{q-1}(1-\ln y)^{q-1} \\ -\frac{1}{\Gamma(q)}\left(\ln\frac{z}{y}\right)^{q-1} \end{cases}$$

$$\geqslant \begin{cases} \frac{1}{\Delta}\left[b_2(1-\ln y)^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln z)^{q-1}(1-\ln y)^{q-1} \\ -\frac{1}{\Delta}\left[b_2 + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln z)^{q-1}(1-\ln y)^{q-1} \end{cases}$$

$$\geqslant \frac{b_2}{\Delta}\left[(1-\ln y)^{-r} - 1\right](\ln z)^{q-1}(1-\ln y)^{q-1} \geqslant 0.$$

Hence (a) holds. We prove (b). Let $1 \leqslant z \leqslant y \leqslant e$. Then

$$\frac{\partial G}{\partial z} = \frac{1}{\Delta}\left[b_2(1-\ln y)^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\alpha_1 - 1)\frac{(\ln z)^{q-2}}{z}(1-\ln y)^{q-1} \geqslant 0.$$

Therefore $G(z,y)$ is increasing w.r.t. z yielding $G(z,y) \leqslant G(e,y)$. Let $1 \leqslant y \leqslant z \leqslant e$. Then

$$\frac{\partial G}{\partial z} = \begin{cases} \frac{1}{\Delta}\left[b_2(1-\ln y)^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](q-1\frac{(\ln z)^{q-2}}{z}(1-\ln y)^{q-2} \\ -\frac{1}{\Gamma(q)}(q-1)\left(\ln\frac{z}{y}\right)^{q-2}\frac{1}{z} \end{cases}$$

$$\geqslant \begin{cases} \frac{(q-1)(\ln z)^{q-2}}{z\Delta}\left[b_2(1-\ln y)^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](1-\ln y)^{q-1} \\ -\frac{1}{z\Delta}\left[b_2 + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](1-\ln y)^{q-2} \end{cases}$$

$$= \begin{cases} \frac{(q-1)(\ln z)^{q-2}}{z\Delta}\left[\left(b_2(-r-1) - b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right)(\ln y) + (\ln y)^2\right] \\ \times(1-\ln y)^{q-2} \end{cases}$$

$$\geqslant 0.$$

Therefore $G(z,y)$ is increasing w.r.t. z yielding $G(z,y) \leqslant G(e,y)$. Hence the inequality

(b). We establish the inequality (c). Let $1 \leqslant \mathtt{z} \leqslant \mathtt{y} \leqslant e$ and $\mathtt{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]$. Then

$$\begin{aligned}
\mathtt{G}(\mathtt{z},\mathtt{y}) &= \frac{1}{\Delta}\left[b_2(1-\ln \mathtt{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln \mathtt{z})^{q-1}(1-\ln \mathtt{y})^{q-1} \\
&= \frac{(\ln \mathtt{z})^{q-1}}{\Delta}\left[b_2(1-\ln \mathtt{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](1-\ln \mathtt{y})^{q-1} \\
&\geqslant \left(\frac{1}{4}\right)^{q}\mathtt{G}(e,\mathtt{y}).
\end{aligned}$$

Let $1 \leqslant \mathtt{y} \leqslant \mathtt{z} \leqslant e$ and $\mathtt{z} \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]$. Then

$$\mathtt{G}(\mathtt{z},\mathtt{y}) = \begin{cases} \frac{1}{\Delta}\left[b_2(1-\ln \mathtt{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right](\ln \mathtt{z})^{q-1}(1-\ln \mathtt{y})^{q-1} \\ -\frac{1}{\Gamma(q)}\left(\ln\frac{\mathtt{z}}{\mathtt{y}}\right)^{q-1} \end{cases}$$

$$\geqslant \begin{cases} (\ln \mathtt{z})^{q-1}\left[\frac{1}{\Delta}\left(b_2(1-\ln \mathtt{y})^{-r} + b_1\frac{\Gamma(q-r)}{\Gamma(q)}\right)(1-\ln \mathtt{y})^{q-1}\right] \\ -\frac{1}{\Gamma(q)}(\ln \mathtt{z})^{q-1} \end{cases}$$

$$\geqslant \left(\frac{1}{4}\right)^{q-1}\mathtt{G}(e,\mathtt{y}). \quad \square$$

3. Main results

An n-tuple $\left(u_1(\mathtt{z}), u_2(\mathtt{z}), \cdots, u_n(\mathtt{z})\right)$ is a solution of the FBVP (1)–(2) if and only if $u_k(\mathtt{z}) \in C^{[q]+1}[1,e]$ satisfies

$$u_1(\mathtt{z}) = \begin{cases} \lambda_1\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)p_1(\mathtt{y}_1)\mathtt{g}_1\left[\lambda_2\int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)p_2(\mathtt{y}_2)\cdots\right. \\ \left.\mathtt{g}_{n-1}\left[\lambda_n\int_1^e \mathtt{G}(\mathtt{y}_{n-1},\mathtt{y}_n)p_n(\mathtt{y}_n)\mathtt{g}_n\left(u_1(\mathtt{y}_n)\right)\frac{d\mathtt{y}_n}{\mathtt{y}_n}\right]\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\right]\frac{d\mathtt{y}_1}{\mathtt{y}_1}, \end{cases}$$

and

$$\begin{cases} u_2(\mathtt{z}) = \lambda_2\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})p_2(\mathtt{y})\mathtt{g}_2\left(u_3(\mathtt{y})\right)\frac{d\mathtt{y}}{\mathtt{y}}, \\ u_3(\mathtt{z}) = \lambda_3\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})p_3(\mathtt{y})\mathtt{g}_3\left(u_4(\mathtt{y})\right)\frac{d\mathtt{y}}{\mathtt{y}}, \\ \qquad\cdots \\ u_n(\mathtt{z}) = \lambda_n\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y})p_n(\mathtt{y})\mathtt{g}_n\left(u_{n+1}(\mathtt{y})\right)\frac{d\mathtt{y}}{\mathtt{y}}, \end{cases}$$

where $u_{n+1}(z) = u_1(z),\ 1 < z < e$. By a positive solution of the FBVP (1)–(2), we mean $\left(u_1(z), u_2(z), \cdots, u_n(z)\right) \in \left(C^{[q]+1}[1,e]\right)^n$ which satisfying the FDE (1) and BCs (2) with $u_k(z) > 0, k = \overline{1,n}\ \forall\, z \in [1,e]$.

Let $\mathscr{B} = \left\{x : x \in C[1,e]\right\}$ be the Banach space endowed with the norm

$$\|x\| = \max_{z \in [1,e]} |x(z)|$$

and $\mathscr{P} \subset \mathscr{B}$ be a cone defined as

$$\mathscr{P} = \left\{ x \in \mathscr{B} \ :\ x(z) \geqslant 0 \text{ on } [1,e] \text{ and } \min_{z \in \left[\sqrt[4]{e}, \sqrt[4]{e^3}\right]} x(z) \geqslant \left(\frac{1}{4}\right)^{q-1} \|x\| \right\}.$$

Define an integral operator $\mathscr{A} : \mathscr{P} \to \mathscr{B}$, for $u_1 \in \mathscr{P}$, by

$$\left.\begin{aligned} \mathscr{A} u_1(z) = \lambda_1 \int_1^e \mathtt{G}(z, y_1) p_1(y_1) g_1 \Bigg[\lambda_2 \int_1^e \mathtt{G}(y_1, y_2) p_2(y_2) \cdots \\ g_{n-1} \left[\lambda_n \int_1^e \mathtt{G}(y_{n-1}, y_n) p_n(y_n) g_n\Big(u_1(y_n)\Big) \frac{dy_n}{y_n} \right] \cdots \frac{dy_2}{y_2} \Bigg] \frac{dy_1}{y_1}. \end{aligned}\right\} \tag{7}$$

Notice from (H_5), (H_6) and Lemma 3 that, for $u_1 \in \mathscr{P}$, $\mathscr{A} u_1(z) \geqslant 0$ on $[1,e]$. In addition, we have

$$\mathscr{A} u_1(z) \leqslant \left\{ \begin{aligned} &\lambda_1 \int_1^e \mathtt{G}(e, y_1) p_1(y_1) g_1 \Bigg[\lambda_2 \int_1^e \mathtt{G}(y_1, y_2) p_2(y_2) \cdots \\ &g_{n-1} \left[\lambda_n \int_1^e \mathtt{G}(y_{n-1}, y_n) p_n(y_n) g_n(u_1(y_n)) \frac{dy_n}{y_n} \right] \cdots \frac{dy_2}{y_2} \Bigg] \frac{dy_1}{y_1} \end{aligned} \right.$$

so that

$$\left.\begin{aligned} \|\mathscr{A} u_1\| \leqslant \lambda_1 \int_1^e \mathtt{G}(e, y_1) p_1(y_1) g_1 \Bigg[\lambda_2 \int_1^e \mathtt{G}(y_1, y_2) p_2(y_2) \cdots \\ g_{n-1} \left[\lambda_n \int_1^e \mathtt{G}(y_{n-1}, y_n) p_n(y_n) g_n\Big(u_1(y_n)\Big) \frac{dy_n}{y_n} \right] \cdots \frac{dy_2}{y_2} \Bigg] \frac{dy_1}{y_1}. \end{aligned}\right\} \tag{8}$$

If $u_1 \in \mathscr{P}$, from Lemma 3 and (8), we deduce that

$$\min_{z\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \mathscr{A}u_1(z) = \begin{cases} \min_{z\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \lambda_1 \int_1^e G(z,y_1)p_1(y_1)g_1\Big[\lambda_2 \int_1^e G(y_1,y_2) \\ \times p_2(y_2)\cdots g_{n-1}\Big[\lambda_n \int_1^e G(y_{n-1},y_n) \\ \times p_n(y_n)g_n(u_1(y_n))\frac{dy_n}{y_n}\Big]\cdots\frac{dy_2}{y_2}\Big]\frac{dy_1}{y_1} \end{cases}$$

$$\geqslant \begin{cases} \lambda_1\left(\frac{1}{4}\right)^{q-1} \int_1^e G(e,y_1)p_1(y_1)g_1\Big[\lambda_2 \int_1^e G(y_1,y_2) \\ \times p_2(y_2)\cdots g_{n-1}\Big[\lambda_n \int_1^e G(y_{n-1},y_n) \\ \times p_n(y_n)g_n(u_1(y_n))\frac{dy_n}{y_n}\Big]\cdots\frac{dy_2}{y_2}\Big]\frac{dy_1}{y_1} \end{cases}$$

$$\geqslant \left(\frac{1}{4}\right)^{q-1} \|\mathscr{A}u_1\|.$$

Therefore $\min_{z\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} \mathscr{A}u_1(z) \geqslant \left(\frac{1}{4}\right)^{q-1} \|\mathscr{A}u_1\|$. Hence $\mathscr{A}u_1 \in \mathscr{P}$ and so $\mathscr{A}: \mathscr{P} \to \mathscr{P}$. Further the operator $\mathscr{A}$ is a completely continuous by an application of the Arzela–Ascoli Theorem.

3.1. Notations

We introduce:

$$\Psi_1 = \max\left\{ \begin{array}{l} \left[\left(\frac{1}{4}\right)^{q-1} \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y)p_1(y)\frac{dy}{y}g_{1\infty}\right]^{-1}, \\ \left[\left(\frac{1}{4}\right)^{q-1} \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y)p_2(y)\frac{dy}{y}g_{2\infty}\right]^{-1}, \\ \cdots \\ \left[\left(\frac{1}{4}\right)^{q-1} \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y)p_n(y)\frac{dy}{y}g_{n\infty}\right]^{-1} \end{array} \right\},$$

$$\Psi_2 = \min\left\{ \begin{array}{l} \left[\int_1^e G(e,y)p_1(y)\frac{dy}{y}g_{10}\right]^{-1}, \\ \left[\int_1^e G(e,y)p_2(y)\frac{dy}{y}g_{20}\right]^{-1}, \\ \cdots \\ \left[\int_1^e G(e,y)p_n(y)\frac{dy}{y}g_{n0}\right]^{-1} \end{array} \right\}.$$

THEOREM 1. *Suppose* (H_1)–(H_4) *hold. Then for each* λ_k, $k=\overline{1,n}$ *satisfying*

$$\Psi_1 < \lambda_k < \Psi_2, \quad k=\overline{1,n}, \tag{9}$$

there exists an n*-tuple* $(u_1,u_2,\cdots,u_n)$ *satisfying the FBVP* (1)–(2) *s.t.* $u_k(z)>0$, $k=\overline{1,n}$ *on* $(1,e)$.

Proof. Let λ_k, $k=\overline{1,n}$ be given as in (9). Now let $\varepsilon>0$ be chosen s.t.

$$\max\left\{\begin{array}{c}\left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,y)p_1(y)\frac{dy}{y}(g_{1\infty}-\varepsilon)\right]^{-1}\\ \left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,y)p_2(y)\frac{dy}{y}(g_{2\infty}-\varepsilon)\right]^{-1}\\ \cdots \\ \left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,y)p_n(y)\frac{dy}{y}(g_{n\infty}-\varepsilon)\right]^{-1}\end{array}\right\} \leqslant \min\left\{\begin{array}{c}\lambda_1,\\ \lambda_2,\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{array}\right\}$$

and

$$\max\left\{\begin{array}{c}\lambda_1,\\ \lambda_2,\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{array}\right\} \leqslant \min\left\{\begin{array}{c}\left[\int_1^e \mathtt{G}(e,y)p_1(y)\frac{dy}{y}(g_{10}+\varepsilon)\right]^{-1}\\ \left[\int_1^e \mathtt{G}(e,y)p_2(y)\frac{dy}{y}(g_{20}+\varepsilon)\right]^{-1}\\ \cdots\\ \left[\int_1^e \mathtt{G}(e,y)p_n(y)\frac{dy}{y}(g_{n0}+\varepsilon)\right]^{-1}\end{array}\right\}.$$

Now from the definitions of g_{k0}, $k=\overline{1,n}$, there exists an $\mathtt{N}_1>0$ s.t., for each $1\leqslant k\leqslant n$, $g_k(x)\leqslant(g_{k0}+\varepsilon)x$, $1<x\leqslant\mathtt{N}_1$.

Let $u_1\in\mathscr{P}$ with $\|u_1\|=\mathtt{N}_1$. By Lemma 3 and the choice of ε, for $1\leqslant y_{n-1}\leqslant e$,

$$\begin{aligned}\lambda_n\int_1^e \mathtt{G}(y_{n-1},y_n)p_n(y_n)g_n\big(u_1(y_n)\big)\frac{dy_n}{y_n} &\leqslant \left\{\begin{array}{l}\lambda_n\int_1^e \mathtt{G}(e,y_n)p_n(y_n)\\ \times\big(g_{n0}+\varepsilon\big)u_1(y_n)\frac{dy_n}{y_n}\end{array}\right.\\ &\leqslant \left\{\begin{array}{l}\lambda_n\int_1^e \mathtt{G}(e,y_n)p_n(y_n)\frac{dy_n}{y_n}\\ \times\big(g_{n0}+\varepsilon\big)\|u_1\|\end{array}\right.\\ &\leqslant\|u_1\|=\mathtt{N}_1.\end{aligned}$$

It follows from Lemma 3, in the same way, for $1 \leqslant \mathtt{y}_{n-2} \leqslant e$,

$$\left.\begin{aligned} &\lambda_{n-1}\int_1^e G(\mathtt{y}_{n-2},\mathtt{y}_{n-1})p_{n-1}(\mathtt{y}_{n-1})\\ &\quad\times \mathtt{g}_{n-1}\bigg(\lambda_n\int_1^e G(\mathtt{y}_{n-1},\mathtt{y}_n)p_n(\mathtt{y}_n)\\ &\qquad\times \mathtt{g}_n\big(u_1(\mathtt{y}_n)\big)\frac{d\mathtt{y}_n}{\mathtt{y}_n}\bigg)\frac{d\mathtt{y}_{n-1}}{\mathtt{y}_{n-1}}\end{aligned}\right\} \leqslant \left\{\begin{aligned} &\lambda_{n-1}\int_1^e G(\mathtt{y}_{n-1},\mathtt{y}_{n-1})\\ &\times p_{n-1}(\mathtt{y}_{n-1})\frac{d\mathtt{y}_{n-1}}{\mathtt{y}_{n-1}}\\ &\times\big(\mathtt{g}_{n-1,0}+\varepsilon\big)\|u_1\|\end{aligned}\right.$$
$$\leqslant \|u_1\| = \mathtt{N}_1.$$

Continuing with this bootstrapping argument, for $1 \leqslant \mathtt{z} \leqslant e$,

$$\left.\begin{aligned} &\lambda_1\int_1^e \mathtt{G}(\mathtt{z},\mathtt{y}_1)p_1(\mathtt{y}_1)\mathtt{g}_1\bigg(\lambda_2\int_1^e \mathtt{G}(\mathtt{y}_1,\mathtt{y}_2)p_2(\mathtt{y}_2)\cdots\\ &\qquad\qquad \mathtt{g}_n\big(u_1(\mathtt{y}_n)\big)\frac{d\mathtt{y}_n}{\mathtt{y}_n}\Big)\cdots\frac{d\mathtt{y}_2}{\mathtt{y}_2}\bigg)\frac{d\mathtt{y}_1}{\mathtt{y}_1}\end{aligned}\right\} \leqslant \mathtt{N}_1,$$

so that, for $1 \leqslant \mathtt{z} \leqslant e$, $\mathscr{A}u_1(\mathtt{z}) \leqslant \mathtt{N}_1$. Hence $\|\mathscr{A}u_1\| \leqslant \mathtt{N}_1 = \|u_1\|$. If we set $\mathscr{E}_1 = \big\{x \in \mathscr{B} : \|x\| < \mathtt{N}_1\big\}$, then

$$\|\mathscr{A}u_1\| \leqslant \|u_1\|, \text{ for } u_1 \in \mathscr{P}\cap\partial\mathscr{E}_1. \tag{10}$$

From the definition of $\mathtt{g}_{k\infty}$, $k = \overline{1,n}$, there exists $\overline{\mathtt{N}}_2 > 0$ s.t., for each $1 \leqslant k \leqslant n$, $\mathtt{g}_k(x) \geqslant (\mathtt{g}_{k\infty}-\varepsilon)x,\ x \geqslant \overline{\mathtt{N}}_2$. Choose $\mathtt{N}_2 = \max\left\{2\mathtt{N}_1, \left(\frac{1}{4}\right)^{1-q}\overline{\mathtt{N}}_2\right\}$. Let $u_1 \in \mathscr{P}$ and $\|u_1\| = \mathtt{N}_2$. Then

$$\min_{\mathtt{z}\in\left[\sqrt[4]{e},\sqrt[4]{e^3}\right]} u_1(\mathtt{z}) \geqslant \left(\frac{1}{4}\right)^{q-1}\|u_1\| \geqslant \overline{\mathtt{N}}_2.$$

Based on Lemma 3 and choice of ε, for $1 \leqslant \mathtt{y}_{n-1} \leqslant e$, we have that

$$\begin{aligned} \lambda_n\int_1^e \mathtt{G}(\mathtt{y}_{n-1},\mathtt{y}_n)p_n(\mathtt{y}_n)\mathtt{g}_n\big(u_1(\mathtt{y}_n)\big)\frac{d\mathtt{y}_n}{\mathtt{y}_n} &\geqslant \left\{\begin{aligned}&\lambda_n\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,\mathtt{y}_n)p_n(\mathtt{y}_n)\\ &\times\mathtt{g}_n\big(u_1(\mathtt{y}_n)\big)\frac{d\mathtt{y}_n}{\mathtt{y}_n}\end{aligned}\right.\\ &\geqslant \left\{\begin{aligned}&\left(\frac{1}{4}\right)^{q-1}\lambda_n\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,\mathtt{y}_n)\\ &\times p_n(\mathtt{y}_n)\big(\mathtt{g}_{n\infty}-\varepsilon\big)u_1(\mathtt{y}_n)\frac{d\mathtt{y}_n}{\mathtt{y}_n}\end{aligned}\right.\\ &\geqslant \left\{\begin{aligned}&\left(\frac{1}{4}\right)^{q-1}\lambda_n\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,\mathtt{y}_n)\\ &\times p_n(\mathtt{y}_n)\frac{d\mathtt{y}_n}{\mathtt{y}_n}\big(\mathtt{g}_{n\infty}-\varepsilon\big)\|u_1\|\end{aligned}\right.\\ &\geqslant \|u_1\| = \mathtt{N}_2.\end{aligned}$$

It stems in the same way from Lemma 3 and choice of ε, for $1 \leqslant y_{n-2} \leqslant e$,

$$\left.\begin{aligned} &\lambda_{n-1}\int_1^e G(y_{n-2},y_{n-1})p_{n-1}(y_{n-1}) \\ &\quad\times g_{n-1}\Bigg(\lambda_n\int_1^e G(y_{n-1},y_n)p_n(y_n) \\ &\qquad\times g_n\big(u_1(y_n)\big)\frac{dy_n}{y_n}\Bigg)\frac{dy_{n-1}}{y_{n-1}}\end{aligned}\right\} \geqslant \left\{\begin{aligned} &\left(\frac{1}{4}\right)^{q-1}\lambda_{n-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y_{n-1}) \\ &\times p_{n-1}(y_{n-1})\frac{dy_{n-1}}{y_{n-1}} \\ &\times\big(g_{n-1,\infty}-\varepsilon\big)\|u_1\|\end{aligned}\right.$$

$$\geqslant \|u_1\| = N_2.$$

Once again using a bootstrapping argument, we discover

$$\left.\begin{aligned} &\lambda_1\int_1^e G(z,y_1)p_1(y_1)g_1\Big(\lambda_2\int_1^e G(y_1,y_2)p_2(y_2)\cdots \\ &\qquad g_n\big(u_1(y_n)\big)\frac{dy_n}{y_n}\Big)\cdots\frac{dy_2}{y_2}\Big)\frac{dy_1}{y_1}\end{aligned}\right\} \geqslant N_2,$$

so that $\mathscr{A}u_1(z) \geqslant N_2 = \|u_1\|$. Hence $\|\mathscr{A}u_1\| \geqslant \|u_1\|$. So if we set $\mathscr{E}_2 = \big\{x \in \mathscr{B} : \|x\| < N_2\big\}$, then

$$\|\mathscr{A}u_1\| \geqslant \|u_1\|, \text{ for } u_1 \in \mathscr{P}\cap\partial\mathscr{E}_2. \tag{11}$$

By utilizing (10), (11) and Guo–Krasnosel'skii fixed point theorem (see [5, 11]), we conclude that $\mathscr{A}$ has a fixed point $u_1 \in \mathscr{P}\cap\big(\overline{\mathscr{E}}_2\backslash\mathscr{E}_1\big)$. Setting $u_1 = u_{n+1}$, we obtain a positive solution $\big(u_1,u_2,\cdots,u_n\big)$ of the FBVP (1)–(2) iteratively indicated by

$$u_k(z) = \lambda_k\int_1^e G(z,y)p_k(y)g_k\big(u_{k+1}(y)\big)dy, \quad k = n, n-1, \cdots, 1. \quad \square$$

3.2. Notations

$$\Psi_3 = \max\left\{\begin{aligned} &\left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y)p_1(y)\frac{dy}{y}g_{10}\right]^{-1}, \\ &\left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y)p_2(y)\frac{dy}{y}g_{20}\right]^{-1}, \\ &\cdots \\ &\left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} G(e,y)p_n(y)\frac{dy}{y}g_{n0}\right]^{-1}\end{aligned}\right\}, \text{ and}$$

$$\Psi_4 = \min\left\{\begin{aligned} &\left[\int_1^e G(e,y)p_1(y)\frac{dy}{y}g_{1\infty}\right]^{-1}, \\ &\left[\int_1^e G(e,y)p_2(y)\frac{dy}{y}g_{2\infty}\right]^{-1}, \\ &\cdots \\ &\left[\int_1^e G(e,y)p_n(y)\frac{dy}{y}g_{n\infty}\right]^{-1}\end{aligned}\right\}.$$

THEOREM 2. *Suppose* (H_1)–(H_3) *hold, then for each* $\lambda_k,\ k=\overline{1,n}$ *satisfying*

$$\Psi_3 < \lambda_k < \Psi_4,\ k=\overline{1,n}, \tag{12}$$

there exists an n*-tuple* $\big(u_1,u_2,\cdots,u_n\big)$ *satisfying the FBVP* (1)–(2) *s.t.* $u_k(z)>0$, $k=\overline{1,n}$ *on* $(1,e)$.

Proof. Let $\lambda_k,\ k=\overline{1,n}$ be given as in (12). Now let $\varepsilon>0$ be chosen s.t.

$$\max\left\{\begin{array}{c}\left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,y)p_1(y)\frac{dy}{y}\big(g_{10}-\varepsilon\big)\right]^{-1},\\ \left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,y)p_2(y)\frac{dy}{y}\big(g_{20}-\varepsilon\big)\right]^{-1},\\ \cdots\\ \left[\left(\frac{1}{4}\right)^{q-1}\int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,y)p_n(y)\frac{dy}{y}\big(g_{n0}-\varepsilon\big)\right]^{-1}\end{array}\right\} \leqslant \min\left\{\begin{array}{c}\lambda_1,\\ \lambda_2,\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{array}\right\}$$

and

$$\max\left\{\begin{array}{c}\lambda_1,\\ \lambda_2,\\ \cdot\\ \cdot\\ \cdot\\ \lambda_n\end{array}\right\} \leqslant \min\left\{\begin{array}{c}\left[\int_1^e \mathtt{G}(e,y)p_1(y)\frac{dy}{y}(g_{1\infty}+\varepsilon)\right]^{-1}\\ \left[\int_1^e \mathtt{G}(e,y)p_2(y)\frac{dy}{y}(g_{2\infty}+\varepsilon)\right]^{-1}\\ \cdots\\ \left[\int_1^e \mathtt{G}(e,y)p_n(y)\frac{dy}{y}(g_{n\infty}+\varepsilon)\right]^{-1}\end{array}\right\}.$$

From the definitions of $g_{k0},\ 1\leqslant k\leqslant n$ there exists $\overline{\mathtt{N}}_3>0$ s.t., for each $1\leqslant k\leqslant n$,

$$g_k(x) \geqslant \big(g_{k0}-\varepsilon\big)x,\ 1<x\leqslant \overline{\mathtt{N}}_3.$$

According to the definitions of g_{k0}, it follows that $g_{k0}(1)=0,\ 1\leqslant k\leqslant n$ and so there exist $1<\Theta_n<\Theta_{n-1}<\cdots<\Theta_2<\overline{\mathtt{N}}_3$ s.t.

$$\left.\begin{array}{r}\lambda_k g_k(z) \leqslant \dfrac{\Theta_{k-1}}{\int_1^e \mathtt{G}(e,y)p_k(y)\frac{dy}{y}},\ z\in\big[1,\Theta_k\big],\ 3\leqslant k\leqslant n, \text{ and}\\ \lambda_2 g_2(z) \leqslant \dfrac{\overline{\mathtt{N}}_3}{\int_1^e \mathtt{G}(e,y)p_2(y)\frac{dy}{y}},\ z\in\big[1,\Theta_2\big].\end{array}\right\}$$

Let $u_1\in\mathscr{P}$ with $\|u_1\|=\Theta_n$. Then,

$$\begin{aligned}\lambda_n\int_1^e \mathtt{G}(y_{n-1},y_n)a_n(y_n)g_n\big(u_1(y_n)\big)\frac{dy_n}{y_n} &\leqslant \lambda_n\int_1^e \mathtt{G}(e,y_n)p_n(y_n)g_n\big(u_1(y_n)\big)\frac{dy_n}{y_n}\\ &\leqslant \frac{\int_1^e \mathtt{G}(e,y_n)p_n(y_n)\Theta_{n-1}\frac{dy_n}{y_n}}{\int_1^e \mathtt{G}(e,y_n)p_n(y_n)\frac{dy_n}{y_n}}\\ &\leqslant \Theta_{n-1}.\end{aligned}$$

Utilizing this bootstrapping technique, it implies that

$$\left.\begin{array}{r}\lambda_2 \int_1^e \mathtt{G}(e, y_2) p_2(y_2) g_2 \Big(\lambda_3 \int_1^e \mathtt{G}(y_2, y_3) p_3(y_3) \cdots \\ g_n\big(u_1(y_n)\big) \frac{dy_n}{y_n} \Big) \cdots \frac{dy_3}{y_3} \Big) \frac{dy_2}{y_2}\end{array}\right\} \leqslant \overline{\mathtt{N}}_3.$$

Then

$$\mathscr{A} u_1(z) = \left\{\begin{array}{r}\lambda_1 \int_1^e \mathtt{G}(z, y_1) p_1(y_1) g_1 \Big(\lambda_2 \int_1^e \mathtt{G}(y_1, y_2) \\ \times p_2(y_2) \cdots g_n\big(u_1(y_n)\big) \frac{dy_n}{y_n} \Big) \cdots \frac{dy_2}{y_2} \Big) \frac{dy_1}{y_1}\end{array}\right.$$

$$\geqslant \left(\frac{1}{4}\right)^{q-1} \lambda_1 \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e, y_1) p_1(y_1) \big(g_{10} - \varepsilon\big) \|u_1\| \frac{dy_1}{y_1} \geqslant \|u_1\|.$$

So $\|\mathscr{A} u_1\| \geqslant \|u_1\|$. If we set $\mathscr{E}_1 = \Big\{ x \in \mathscr{B} : \|x\| < \Theta_n \Big\}$, then

$$\|\mathscr{A} u_1\| \geqslant \|u_1\|, \text{ for } u_1 \in \mathscr{P} \cap \partial \mathscr{E}_1. \tag{13}$$

Since each $g_{k\infty}$ is taken to be a positive real number, it follows that g_k, $1 \leqslant k \leqslant n$ is unbounded at ∞. For each $1 \leqslant k \leqslant n$, set

$$g_k^*(x) = \sup_{y \in [1,x]} g_k(y).$$

By definition of $g_{k\infty}$, $1 \leqslant k \leqslant n$, there exists $\overline{\mathtt{N}}_4$ s.t., for each $1 \leqslant k \leqslant n$,

$$g_k^*(x) \leqslant \big(g_{k\infty} + \varepsilon\big) x, \; x \geqslant \overline{\mathtt{N}}_4.$$

It follows that there exists $\mathtt{N}_4 = \max\Big\{ 2\overline{\mathtt{N}}_3, \overline{\mathtt{N}}_4 \Big\}$ s.t., for each $1 \leqslant k \leqslant n$,

$$g_k^*(x) \leqslant g_k^*\big(\mathtt{N}_4\big), \; 1 < x \leqslant \mathtt{N}_4.$$

Choose $u_1 \in \mathscr{P}$ with $\|u_1\| = \mathtt{N}_4$. Then, by using bootstrapping argument, we have

$$\begin{aligned}\mathscr{A} u_1(z) &= \lambda_1 \int_1^e \mathtt{G}(z, y_1) p_1(y_1) g_1(\lambda_2 \cdots) \frac{dy_1}{y_1} \\ &\leqslant \lambda_1 \int_1^e \mathtt{G}(z, y_1) p_1(y_1) g_1^*(\lambda_2 \cdots) \frac{dy_1}{y_1} \\ &\leqslant \lambda_1 \int_1^e \mathtt{G}(e, y_1) p_1(y_1) g_1^*(\mathtt{N}_4) \frac{dy_1}{y_1} \\ &\leqslant \lambda_1 \int_1^e \mathtt{G}(e, y_1) p_1(y_1) \frac{dy_1}{y_1} \big(g_{1\infty} + \varepsilon\big) \mathtt{N}_4 \\ &\leqslant \mathtt{N}_4 = \|u_1\|.\end{aligned}$$

And therefore $\|\mathscr{A} u_1\| \leqslant \|u_1\|$. So, if we let $\mathscr{E}_2 = \left\{x \in \mathscr{B} : \|x\| < \mathtt{N}_4\right\}$, then

$$\|\mathscr{A} u_1\| \leqslant \|u_1\|, \text{ for } u_1 \in \mathscr{P} \cap \partial \mathscr{E}_2. \tag{14}$$

By utilizing (13), (14) and Guo–Krasnosel'skii fixed point theorem (see [5, 11]), we get that $\mathscr{A}$ has a fixed point $u_1 \in \mathscr{P} \cap (\overline{\mathscr{E}}_2 \backslash \mathscr{E}_1)$, which in turn with $u_1 = u_{n+1}$ yields an n-tuple $\left(u_1, u_2, \cdots, u_n\right)$ satisfying the FBVP (1)–(2) for the chosen values of λ_k, $k = \overline{1,n}$. $\square$

4. Example

Let $q = 2.6$, $r = 1.4$, $b_1 = 4$, $b_2 = 8$. Consider the FBVP for $\mathrm{z} \in (1, e)$:

$$\left.\begin{aligned} {}^{H}\mathfrak{D}_{1^+}^{2.6} u_1(\mathrm{z}) + \lambda_1 p_1(\mathrm{z}) \mathrm{g}_1\left(u_2(\mathrm{z})\right) = 0, \\ {}^{H}\mathfrak{D}_{1^+}^{2.6} u_2(\mathrm{z}) + \lambda_2 p_2(\mathrm{z}) \mathrm{g}_2\left(u_3(\mathrm{z})\right) = 0, \\ {}^{H}\mathfrak{D}_{1^+}^{2.6} u_3(\mathrm{z}) + \lambda_3 p_3(\mathrm{z}) \mathrm{g}_3\left(u_1(\mathrm{z})\right) = 0, \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} u_1(1) = 0, \quad u_1'(1) = 0, \quad 4u_1(e) + 8\,{}^{H}\mathfrak{D}_{1^+}^{1.4} u_1(e) = 0, \\ u_2(1) = 0, \quad u_2'(1) = 0, \quad 4u_2(e) + 8\,{}^{H}\mathfrak{D}_{1^+}^{1.4} u_2(e) = 0, \\ u_3(1) = 0, \quad u_3'(1) = 0, \quad 4u_3(e) + 8\,{}^{H}\mathfrak{D}_{1^+}^{1.4} u_3(e) = 0, \end{aligned}\right\} \tag{16}$$

where

$$\begin{cases} p_1(\mathrm{z}) = p_2(\mathrm{z}) = p_3(\mathrm{z}) = 1, \\ \mathrm{g}_1\left(u_2\right) = \dfrac{1}{67}\left(3473e^{-u_2} - 3472\right)\left(2143e^{-3u_2} - 2032\right)u_2, \\ \mathrm{g}_2\left(u_3\right) = \dfrac{6}{25}\left(205e^{-2u_3} - 214\right)\left(3883e^{-4u_3} - 3915\right)u_3, \\ \mathrm{g}_3\left(u_1\right) = \dfrac{7}{29}\left(653e^{-3u_1} - 678\right)\left(2188e^{-u_1} - 2243\right)u_1. \end{cases}$$

In view of the data given, we get $\Delta \approx 15.06103540$, $\mathrm{g}_{10} \approx 1.656716417$, $\mathrm{g}_{1\infty} \approx 105300.05970149$, $\mathrm{g}_{20} \approx 69.12$, $\mathrm{g}_{2\infty} \approx 201074.4$, $\mathrm{g}_{30} \approx 331.896551724$, $\mathrm{g}_{3\infty} \approx 367078.5517241379$,

$$\begin{aligned} \Psi_1 = \max &\left\{\begin{aligned} &\left[\left(\frac{1}{4}\right)^{1.6} \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,\mathrm{y}) p_1(\mathrm{y}) \frac{d\mathrm{y}}{\mathrm{y}} \mathrm{g}_{1\infty}\right]^{-1}, \\ &\left[\left(\frac{1}{4}\right)^{1.6} \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,\mathrm{y}) p_2(\mathrm{y}) \frac{d\mathrm{y}}{\mathrm{y}} \mathrm{g}_{2\infty}\right]^{-1}, \\ &\left[\left(\frac{1}{4}\right)^{1.6} \int_{\sqrt[4]{e}}^{\sqrt[4]{e^3}} \mathtt{G}(e,\mathrm{y}) p_3(\mathrm{y}) \frac{d\mathrm{y}}{\mathrm{y}} \mathrm{g}_{3\infty}\right]^{-1} \end{aligned}\right\} \\ = \max &\left\{0.000206082, 0.000107922, 0.000059116\right\} \approx 0.000206082 \end{aligned}$$

and

$$\Psi_2 = \min \left\{ \begin{array}{c} \left[\int_1^e \mathtt{G}(e,\mathrm{y}) p_1(\mathrm{y}) \frac{d\mathrm{y}}{\mathrm{y}} \mathrm{g}_{10} \right]^{-1}, \\ \left[\int_1^e \mathtt{G}(e,\mathrm{y}) p_2(\mathrm{y}) \frac{d\mathrm{y}}{\mathrm{y}} \mathrm{g}_{20} \right]^{-1}, \\ \left[\int_1^e \mathtt{G}(e,\mathrm{y}) p_3(\mathrm{y}) \frac{d\mathrm{y}}{\mathrm{y}} \mathrm{g}_{30} \right]^{-1} \end{array} \right\}$$
$$= \min\left\{0.744041422, 0.017833704, 0.003714005\right\} \approx 0.003714005.$$

Then all the conditions of Theorem 1 are fulfilled. Therefore, by Theorem 1, we get an optimal eigenvalue interval $0.000206082 < \lambda_k < 0.003714005$, for $k = 1,2,3$ in which the FBVP (15)–(16) has at least one positive solution.

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REFERENCES

[1] B. AHMAD, A. ALSAEDI, S. K. NTOUYAS AND J. TARIBOON, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017.

[2] G. ALOTTA, E. BOLOGNA, G. FAILLA AND M. ZINGALES, *A Fractional Approach to Non-Newtonian Blood Rheology in Capillary Vessels*, J. Peridyn Nonlocal Model, **1** (2019), 88–96.

[3] P. L. BUTZER, A. A. KILBAS AND J. J. TRUJILLO, *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl., **269** (2002), 1–27.

[4] P. L. BUTZER, A. A. KILBAS AND J. J. TRUJILLO, *Mellin transform analysis and integration by parts for Hadamard-type fractional integrals*, J. Math. Anal. Appl., **270** (2002), 1–15.

[5] D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, 1988.

[6] E. F. D. GOUFO, *A biomathematical view on the fractional dynamics of cellulose degradation*, Fract. Calc. Appl. Anal., **18** (3) (2015), 554–564.

[7] J. HADAMARD, *Essai sur l'étude des fonctions données par leur développement de Taylor*, J. Mat. Pure et Appl., Sér. 4, **8** (1892), 101–186.

[8] A. A. KILBAS, *Hadamard-type fractional calculus*, J. Korean Math. Soc., **38** (2001), 1191–1204.

[9] A. A. KILBAS AND J. J. TRUJILLO, *Hadamard-type integrals as G-transforms*, Integral Transforms Spec. Funct., **14** (2003), 413–427.

[10] A. A. KILBAS, H. M. SRIVASTHAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[11] M. A. KRASNOSEL'SKII, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[12] I. PODULBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[13] M. SEGHIER, A. OUAHAB AND J. HENDERSON, *Random solutions to a system of fractional differential equations via the Hadamard fractional derivative*, Eur. Phys. J. Spec. Top., **226** (2017), 3525–3549.

[14] H. H. SHERIEF AND M. A. EL-HAGARY, *Fractional order theory of thermo-viscoelasticity and application*, Mech Time-Depend Mater., **24** (2020), 179–195.

[15] H. SUN, W. CHEN, C. LI AND Y. Q. CHEN, *Fractional differential models for anomalous diffusion*, Physica A: Statistical mechanics and its applications, **389** (2010), 2719–2724.

[16] K. R. PRASAD AND B. M. B. KRUSHNA, *Multiple positive solutions for a coupled system of Riemann-Liouville fractional order two-point boundary value problems*, Nonlinear Studies, **20** (2013), 501–511.

[17] K. R. PRASAD AND B. M. B. KRUSHNA, *Eigenvalues for iterative systems of Sturm–Liouville fractional order two-point boundary value problems*, Fract. Calc. Appl. Anal., **17** (2014), 638–653.

[18] P. THIRAMANUS, S. K. NTOUYAS AND J. TARIBOON, *Positive solutions for Hadamard fractional differential equations on infinite domain*, Adv. Difference Equ., **2016** (2016), 1–18.

[19] K. PEI, G. T. WANG AND Y. Y. SUN, *Successive iterations and positive extremal solutions for a Hadamard type fractional integro-differential equations on infinite domain*, Appl. Math. Comput., **312** (2017), 158–168.

[20] J. TARIBOON, S. K. NTOUYAS, S. ASAWASAMRIT AND C. PROMSAKON, *Positive solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain*, Open Math., **15** (2017), 645–666.

[21] G. T. WANG, K. PEI, R. P. AGARWAL, L. H. ZHANG AND B. AHMAD, *Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line*, J. Comput. Appl. Math., **343** (2018), 230–239.

[22] W. YANG, *Positive solutions for singular Hadamard fractional differential system with four-point coupled boundary conditions*, J. Appl. Math. Comput., **49** (2015), 357–381.

[23] C. ZHAI, W. WANG AND H. LI, *A uniqueness method to a new Hadamard fractional differential system with four-point boundary conditions*, J. Inequal. Appl. 2018, 207 (2018).

[24] W. ZHANG AND W. B. LIU, *Existence, uniqueness, and multiplicity results on positive solutions for a class of Hadamard-type fractional boundary value problem on an infinite interval*, Math. Methods Appl. Sci., **43** (2020), 2251–2275.

[25] W. ZHANG AND J. NI, *New multiple positive solutions for Hadamard-type fractional differential equations with nonlocal conditions on an infinite interval*, Appl. Math. Lett., **118** (2021), 1–10.

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