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ON THE GENERALIZED IYENGAR TYPE INEQUALITIES

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Abstract. In the paper, we establish two inequalities for differentiable bounded functions and Lipschitzian function, which are connected with Iyengar integral inequalities, and we present some new results.

1. Introduction

The following result is known in the literature as Iyengar's inequality. In 1938, Iyengar proved the following theorem obtaining bounds for a trapezoidal quadrature rule for functions whose derivative $|f(x)| < M$ for $x \in (a,b)$ (see for example [6]):

THEOREM 1. *Let f be a differentiable function on (a,b) and assume that there is a constant $M > 0$ such that $|f(x)| \leqslant M$, $\forall x \in (a,b)$. Then, the following inequality holds:*

$$\left| \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) \right| \leqslant \frac{M(b-a)^2}{4} - \frac{1}{4M}(f(b)-f(a))^2. \tag{1}$$

Clearly, detailed analysis shows that the famous Iyengar inequality actually says that the Trapezoidal formula is a central algorithm for approximating integrals over an appropriate interval for the class of functions whose derivatives are bounded by a positive number M. Since 1938, considerable efforts have contributed to extensions and generalizations of (1). During the past few years many researchers have given considerable attention to the above inequality and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1]–[9] and the references cited therein.

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see [10]. Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type. The main aim of this work is to establish Iyengar type inequalities involving the Riemann-Liouville integrals via the Taylor theorem and Lipschitzian functions. Let us begin by introducing some definitions.

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2. Preliminaries

Let $C^{n+1}[a,b]$ be the set of real valued functions h defined on $[a,b]$, n times differentiable at every point and with $h^{(n+1)}(x)$ that exists everywhere on $[a,b]$ except possibly finitely many points such that $h^{(n+1)}(x)$ is Riemann integrable over $[a,b]$, hence bounded.

THEOREM 2. (Taylor's Theorem) *For $n \in N$, $h(x)$ be a function satisfying the following conditions:*

i) $h^{(k)}(x)$ for $0 \leqslant k \leqslant n$ are continuous on the closed interval $[a,b]$;

ii) $h^{(n+1)}(x)$ exists in the open interval (a,b).

Then for any given $x \in (a,b]$ there exists at least one point $\xi \in (a,x)$ such that

$$h(x) = h(a) + \sum_{k=1}^{n} \frac{h^{(k)}(a)}{k!}(x-a)^k + \frac{h^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

DEFINITION 1. Let $h \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}h$ and $J_{b-}^{\alpha}h$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J_{a+}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} h(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} h(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J_{a+}^{0}h(x) = J_{b-}^{0}h(x) = h(x)$.

3. Main results

We start the main results as the following theorem:

THEOREM 3. *Let h be a differentiable function of $C^n[a,b]$ such that $m \leqslant h^{(n)}(t) \leqslant M$ for all $t \in [a,b]$. Then, when n is even, for $t \in [a,b]$, $\alpha > 0$, the following inequalities hold:*

$$\begin{aligned} & \frac{m\left[(b-t)^{\alpha+n} + (t-a)^{\alpha+n}\right]}{\Gamma(\alpha+n+1)} \\ \leqslant & \left[J_{b-}^{\alpha}h(t) + J_{a+}^{\alpha}h(t)\right] - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)(t-a)^{\alpha+k} + h^{(k)}(b)(b-t)^{\alpha}(t-b)^k}{\Gamma(\alpha+k+1)} \\ \leqslant & \frac{M\left[(b-t)^{\alpha+n} + (t-a)^{\alpha+n}\right]}{\Gamma(\alpha+n+1)}, \end{aligned} \tag{2}$$

and when n *is odd, for* $t \in [a,b]$, $\alpha > 0$, *the following inequalities hold:*

$$\begin{aligned}
&\frac{m(t-a)^{\alpha+n} - M(b-t)^{\alpha+n}}{\Gamma(\alpha+n+1)} \\
&\leqslant \left[J_{b^-}^{\alpha} h(t) + J_{a^+}^{\alpha} h(t)\right] - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)(t-a)^{\alpha+k} + h^{(k)}(b)(b-t)^{\alpha}(t-b)^{k}}{\Gamma(\alpha+k+1)} \\
&\leqslant \frac{M(t-a)^{\alpha+n} - m(b-t)^{\alpha+n}}{\Gamma(\alpha+n+1)}.
\end{aligned}$$

Proof. By Taylor's theorem, we write

$$h(x) = \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{k!}(x-a)^k + \frac{h^{(n)}(\eta)}{n!}(x-a)^n, \quad a < \eta < x \tag{3}$$

$$h(x) = \sum_{k=0}^{n-1} \frac{h^{(k)}(b)}{k!}(x-b)^k + \frac{h^{(n)}(\mu)}{n!}(x-b)^n, \quad x < \mu < b. \tag{4}$$

Multiplying both sides of (3) by $\frac{1}{\Gamma(\alpha)}(t-x)^{\alpha-1}$, then integrating the resulting inequality with respect to x from a to t, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} h(x)dx \\
&= \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{k!\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1}(x-a)^k dx + \frac{h^{(n)}(\eta)}{n!\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1}(x-a)^n dx.
\end{aligned}$$

By calculating the above integrals, we have

$$J_{a^+}^{\alpha} h(t) = \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(\alpha+k+1)}(t-a)^{\alpha+k} + \frac{h^{(n)}(\eta)}{\Gamma(\alpha+n+1)}(t-a)^{\alpha+n}. \tag{5}$$

Note that using the change of the variable we have

$$\begin{aligned}
&\int_a^t (t-x)^{\alpha-1}(x-a)^k dx \\
&= (t-a)^{\alpha+k} \int_0^1 u^{\alpha-1}(1-u)^k du \\
&= (t-a)^{\alpha+k} \frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)}.
\end{aligned}$$

Since $m \leqslant h^{(n)}(x) \leqslant M$, then we get

$$\begin{aligned} &\frac{m}{\Gamma(\alpha+n+1)}(t-a)^{\alpha+n} \\ &\leqslant J_{a^+}^{\alpha} h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{\Gamma(\alpha+k+1)}(t-a)^{\alpha+k} \leqslant \frac{M}{\Gamma(\alpha+n+1)}(t-a)^{\alpha+n}. \end{aligned} \tag{6}$$

Similarly, by multiplying both sides of (4) by $\frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$, then integrating the resulting inequality with respect to x from t to b, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1} h(x)dx \\ &= \sum_{k=0}^{n-1} \frac{h^{(k)}(b)}{k!\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}(x-b)^k dx + \frac{h^{(n)}(\mu)}{n!\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}(x-b)^n dx. \end{aligned}$$

By calculating the above integrals, we have

$$\begin{aligned} &\frac{m}{\Gamma(\alpha+n+1)}(b-t)^{\alpha}(t-b)^n \\ &\leqslant J_{b^-}^{\alpha} h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(b)}{\Gamma(\alpha+k+1)}(b-t)^{\alpha}(t-b)^k \leqslant \frac{M}{\Gamma(\alpha+n+1)}(b-t)^{\alpha}(t-b)^n. \end{aligned} \tag{7}$$

When n is even from (7), it follows that

$$\begin{aligned} &\frac{m}{\Gamma(\alpha+n+1)}(b-t)^{\alpha+n} \\ &\leqslant J_{b^-}^{\alpha} h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(b)}{\Gamma(\alpha+k+1)}(b-t)^{\alpha}(t-b)^k \leqslant \frac{M}{\Gamma(\alpha+n+1)}(b-t)^{\alpha+n}. \end{aligned} \tag{8}$$

When n is odd, the reversed inequalities of (8) hold. Therefore, from (6) and (8), when n is even we get the desired result (2). □

COROLLARY 1. *With the assumptions of Theorem 3, if we take $t = \frac{a+b}{2}$, we have i) when n is even, the following inequalities hold:*

$$\begin{aligned} &m\frac{(b-a)^{\alpha+n}}{2^{\alpha+n-1}\Gamma(\alpha+n+1)} \\ &\leqslant \left[J_{b^-}^{\alpha} h\left(\frac{a+b}{2}\right) + J_{a^+}^{\alpha} h\left(\frac{a+b}{2}\right)\right] - \sum_{k=0}^{n-1} \frac{(b-a)^{\alpha+k}}{2^{\alpha+k-1}\Gamma(\alpha+k+1)} \frac{h^{(k)}(a)+(-1)^k h^{(k)}(b)}{2} \\ &\leqslant M\frac{(b-a)^{\alpha+n}}{2^{\alpha+n-1}\Gamma(\alpha+n+1)}. \end{aligned}$$

ii) when n is odd, the following inequalities hold:

$$\begin{aligned}
&\frac{(m-M)(b-a)^{\alpha+n}}{2^{\alpha+n}\Gamma(\alpha+n+1)} \\
&\leqslant \left[J_{b^-}^{\alpha}h\left(\frac{a+b}{2}\right)+J_{a^+}^{\alpha}h\left(\frac{a+b}{2}\right)\right]-\sum_{k=0}^{n-1}\frac{(b-a)^{\alpha+k}}{2^{\alpha+k-1}\Gamma(\alpha+k+1)}\frac{h^{(k)}(a)+(-1)^k h^{(k)}(b)}{2} \\
&\leqslant \frac{(M-m)(b-a)^{\alpha+n}}{2^{\alpha+n}\Gamma(\alpha+n+1)}.
\end{aligned}$$

REMARK 1. If we choose $\alpha = 1$ and $m = -M$ in Corollary 1, we have the following inequalities hold:

$$\left|\int_a^b h(x)dx-\sum_{k=0}^{n-1}\frac{(b-a)^{1+k}}{2^k\Gamma(k+2)}\frac{h^{(k)}(a)+(-1)^k h^{(k)}(b)}{2}\right| \leqslant M\frac{(b-a)^{n+1}}{2^n\Gamma(n+2)}.$$

If we take $n = 1$ in the above inequality, it follows that

$$\left|\int_a^b h(x)dx-(b-a)\frac{h(a)+h(b)}{2}\right| \leqslant M\frac{(b-a)^2}{4}.$$

THEOREM 4. *Let h be a function and for all $x \in [a,b]$ and $M > 0$ the following conditions hold*

$$|h(x)-h(a)| \leqslant M(x-a) \ \text{ and } \ |h(x)-h(b)| \leqslant M(b-x). \tag{9}$$

Then, for $t \in [a,b]$, $\alpha > 0$, the following inequalities hold:

$$\begin{aligned}
&\left|\left[J_{b^-}^{\alpha}h(t)+J_{a^+}^{\alpha}h(t)\right]-\frac{h(a)(t-a)^{\alpha+1}+h(b)(b-t)^{\alpha+1}}{\Gamma(\alpha+1)}\right| \\
&\leqslant M\frac{(t-a)^{\alpha+1}+(b-t)^{\alpha+1}}{\Gamma(\alpha+2)}.
\end{aligned} \tag{10}$$

Proof. By using (9), we have

$$h(a)-M(x-a) \leqslant h(x) \leqslant h(a)+M(x-a) \tag{11}$$

and

$$h(b)-M(b-x) \leqslant h(x) \leqslant h(b)+M(b-x). \tag{12}$$

Multiplying both sides of (11) and (12) by $\frac{1}{\Gamma(\alpha)}(t-x)^{\alpha-1}$ and $\frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}$, respectively, then integrating the resulting inequality with respect to x from a to t, and from

t to b, respectively, we obtain

$$\begin{aligned}
&\frac{h(a)}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}dx-\frac{M}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}(x-a)\,dx\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}h(x)dx\\
&\leqslant \frac{h(a)}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}dx+\frac{M}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}(x-a)\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\frac{h(b)}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}dx-\frac{M}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}(b-x)\,dx\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}h(x)dx\\
&\leqslant \frac{h(b)}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}dx+\frac{M}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}(b-x)\,dx.
\end{aligned}$$

By calculating the above integrals, we have

$$-\frac{M(t-a)^{\alpha+1}}{\Gamma(\alpha+2)}\leqslant J_{a^+}^{\alpha}h(t)-\frac{h(a)(t-a)^{\alpha}}{\Gamma(\alpha+1)}\leqslant\frac{M(t-a)^{\alpha+1}}{\Gamma(\alpha+2)} \tag{13}$$

and similarly

$$-\frac{M(b-t)^{\alpha+1}}{\Gamma(\alpha+2)}\leqslant J_{b^-}^{\alpha}h(t)-\frac{h(b)(b-t)^{\alpha}}{\Gamma(\alpha+1)}\leqslant\frac{M(b-t)^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{14}$$

By adding (13) and (14) and using the properties of the modulus we get the desired result (10). This completes the proof. □

COROLLARY 2. *With the assumptions of Theorem 4, if we take* $t=\frac{a+b}{2}$, *we have*

$$\left|\left[J_{b^-}^{\alpha}h\left(\frac{a+b}{2}\right)+J_{a^+}^{\alpha}h\left(\frac{a+b}{2}\right)\right]-\frac{(b-a)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{h(a)+h(b)}{2}\right|\leqslant M\frac{(b-a)^{\alpha+1}}{2^{\alpha}\Gamma(\alpha+2)}.$$

REMARK 2. If we choose $\alpha=1$ in Corollary 2, we have

$$\left|\int_a^b h(x)dx-(b-a)\frac{h(a)+h(b)}{2}\right|\leqslant M\frac{(b-a)^2}{4}.$$

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