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ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO–FABRIZIO DERIVATIVE

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Abstract. The main objective of this paper is to prove the existence and uniqueness of mild solution for abstract differential equations by using the resolvent operators and fixed point theorem. Moreover, we studied some examples on partial differential equation with Caputo-Fabrizio derivative.

1. Introduction

A generalisation of conventional calculus known as fractional calculus produces ideas and methods that are comparable to those of normal differential calculus but have a far broader range of applications. Fractional differentiation has made significant contributions over the past two decades to a number of disciplines, including mechanics, control theory, chemistry, electricity, biology, economics, signal and image processing [10, 11, 12, 16, 25]. S. Abbas [1] investigated some existence and $Ulam's$ type stability concepts for functional abstract fractional differential inclusions with not instantaneous impulses in Banach spaces. In [17] C. Li and Z. Li investigated the stability and logarithmic decline of the solutions to fractional differential equations (FDEs) for both linear and nonlinear cases. In [2], authors studied existence of solutions of nonlinear fractional integro-differential equations of Sobolev type with nonlocal condition in Banach spaces. During last two decades, many researcher [7, 8, 9, 13, 15, 18, 21, 22, 24] developed theory of abstract impulsive, abstract fractional differential equation with non local condition by employing resolvent operator and other properties of fractional differential equations. In [3, 4, 5, 6, 19], researchers have been studied new definition of and its properties of Caputo-Fabrizio fractional derivative and integral operator. Motivated from the above works cited, in this paper we study the following problem

$$\begin{cases} \mathscr{D}^{\delta}_{0,t}\left[Bu(t)\right] = Au(t) + f(t, \mathscr{I}^{\sigma}_{0,t}u(t), u(t)),\ t \in [0,T], \\ u(0) = u_0 + g(u), \end{cases} \tag{1}$$

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with $\mathscr{D}_{0,t}^{\delta}$ is the Caputo-Fabrizio fractional derivative and $\mathscr{I}_{0,t}^{\sigma}$ is the Caputo-Fabrizio fractional integral $0<\delta,\sigma<1$, A and B are the closed linear unbounded operators, $f\in C([0,T],X)$ and the operators are with domain contained in a Banach space X and ranges contained in a Banach space Y, $u_0\in X$ and $f:[0,T]\times X^2\to X$, $g: C([0,T],X)\to X$ are continuous.

2. Preliminaries

In the part, we introduce several definitions and results which will be used later. The operators $A: D(A)\subset X\to Y$ and $B: D(B)\subset X\to Y$ satisfy the following hypotheses:

(H1) A and B are closed linear operators with $D(A)\subset D(B)$.

(H2) B is bijective and $B^{-1}: Y\to D(B)$.

(H3) The resolvent operator $S(t)$, $t\geqslant 0$ is analytic and there exists a function φ_M in $L^1_{loc}([0,\infty);\mathbb{R}^+)$ such that:

$$\left\|S'(t)x\right\|\leqslant \varphi_M(t)\left\|x\right\|_{D(M)},\ \text{for all } t>0 \text{ with } M=B^{-1}A.$$

We assume in all this work the following conditions:

(H4) The function $f:[0,T]\times X^2\to X$ is complement continuous, there exists a constant $L_1>0$ such that

$$\left\|f(t,x_1,y_1)-f(t,x_2,y_2)\right\|_{D(M)}\leqslant L_1\left(\|x_1-x_2\|+\|y_1-y_2\|\right),$$
$$\forall(t,x_i,y_i)\in[0,T]\times X^2,\ i=1,2,$$

and

$$N=\max\left\{f(t,0,0),\ t\in[0,T]\right\}.$$

(H5) There exists a constant $L_2>0$, such that

$$\left\|g(x_1)-g(x_2)\right\|_{D(M)}\leqslant L_2\left\|x_1-x_2\right\|,\forall x_i\in X,\ i=1,2.$$

(H6) $\Upsilon=\left(L_2+\frac{2-\sigma}{(1-\delta)(1-\sigma)}RL_1\right)(1+\|\varphi_M\|_{L^1})<1$ with $R=\left\|B^{-1}\right\|$.

In this work, $D(M)$ is the domain of the operator M endowed with the graph norm $\|x\|_{D(M)}=\|x\|+\|Mx\|$ and $\|x\|_{C([0,T),X)}=\max\limits_{t\in[0,T]}\|x(t)\|$.

REMARK 1. Since (H1). If $x\in D(M)=D(B^{-1}A)$ then $x\in D(A)$ and $x\in D(B)$.

Here, we give the following definitions.

DEFINITION 1. ([3], [4], [6]) Let $\delta,a\in\mathbb{R}$ such that $0<\delta<1$, $\phi\in C^1([0,T])$. The Caputo-Fabrizio fractional derivative of order δ of a function ϕ is define by

$$\mathscr{D}_{a,t}^{\delta}\left[\phi(t)\right]=\frac{1}{1-\delta}\int_a^t e^{-\left(\frac{\delta}{1-\delta}\right)(t-s)}\phi'(s)ds.$$

DEFINITION 2. ([3], [4], [6]) Let $\delta \in \mathbb{R}$, such that $0 < \delta \leqslant 1$. The Caputo-Fabrizio fractional integral of order δ of a function ϕ is define by

$$\mathscr{I}_{0,t}^{\delta}[\phi(t)] = \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}\phi(s)ds.$$

For basic facts about fractional integrals and fractional derivatives and the properties of the operators $\mathscr{D}_{a,t}^{\delta}$ and $\mathscr{I}_{0,t}^{\delta}$ one can refer to the articles [3], [4], [5], [6], [20].

Suppose the following fractional problem of Sobolev type

$$\begin{cases} \mathscr{D}_{0,t}^{\delta}[Bu(t)] = Au(t) + f(t),\ t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{2}$$

Equation (2) is equivalent to the following:

$$u(t) = u_0 + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}Mu(s)ds + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}B^{-1}f(s)ds.$$

The above equation can also be written as the integral equation of the form

$$u(t) = h(t) + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}Mu(s)ds,\ t \geqslant 0, \tag{3}$$

where $h(t) = u_0 + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}B^{-1}f(s)ds$. Let us assume that the integral equation (3) has an associated resolvent operator $S(t),\ t \geqslant 0$ on X.

We define the resolvent operator for the integral equation (3).

DEFINITION 3. ([26]) A one parameter family of bounded linear operators $\{S(t)\}_{t\geqslant 0}$ on X is called resolvent operator for (3) if the following conditions hold:

i) $S(.)x \in C([0,\infty), X)$ and $S(0)x = x$ for all $x \in X$.

ii) $S(t)D(M) \subset D(M)$ and $MS(t)x = S(t)Mx$ for all $x \in D(M)$ and $t \geqslant 0$.

iii) For every $x \in D(M)$ and $t \geqslant 0$

$$S(t)x = x + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}MS(s)xds.$$

We have the following concept of solution

DEFINITION 4. A function $u \in C([0,T],X)$ is called a mild solution of the integral equation (3) on $[0,T]$ if $\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s)ds \in D(M)$ for all $t \in [0,T]$, $h(t) \in C([0,T],X)$ and

$$u(t) = \frac{M}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s)ds + h(t),\ t \in [0,T]. \tag{4}$$

The next result follows from ([23], [26]).

LEMMA 1. *Under the above conditions of definition 4 the following properties are valid*

1. *If u is a mild solution of (3) on $[0,T]$, then the function $t \to \int_0^t S(t-s)h(s)ds$ is continuously differentiable on $[0,T]$ and*

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)h(s)ds,\ t \in [0,T].$$

2. *If $(S(t))_{t\geqslant 0}$ is analytic, $h \in C^{\alpha}([0,T],X)$ for some $\alpha \in (0,1)$, then the function defined by*

$$u(t) = S(t)h(t) + \int_0^t S'(t-s)\left(h(s)-h(t)\right)ds,\ t \in [0,T],$$

is a mild solution of (3) on $[0,T]$.

3. *If $(S(t))_{t\geqslant 0}$ is differentiable and $h \in C([0,T],D(M))$, then the function $u:[0,T] \to X$ define by*

$$u(t) = \int_0^t S'(t-s)h(s)ds + h(t),\ t \in [0,T],$$

is a mild solution of (3) on $[0,T]$.

The assertion (1) follows directly from [26], Proposition 1.2. The condition in (2) is proved in [26], Theorem 2.4. We note that (3) can be found in [26], Proposition 1.3.

The paper is organized as follows. In section 2, we discuss the existence of mild solutions and we prove the theorems for the existence and uniqueness mild solution of the problem (1). In section 3, we study an example of partial differential equations with Caputo-Fabrizio derivatives.

3. Main results

In this section, we discuss the existence of mild solutions for the problem (1) and here we assume that the resolvent $(S(t))_{t\geqslant 0}$ is differentiable. The problem (1) is equivalent to the following integral equation

$$\begin{aligned} u(t) &= u_0 + g(u) + \frac{M}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s)ds \\ &\quad + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}B^{-1}f(t,\mathscr{I}_{0,s}^{\sigma}u(s),u(s))ds,\ \ t \in [0,T]. \end{aligned} \tag{5}$$

DEFINITION 5. A function $u \in C([0,T],X)$ is said to be a mild solution of (1), on $[0,T]$ if $\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s)ds \in D(M)$ for all $t \in [0,T]$ and satisfies the integral equation (5).

Suppose there exists a resolvent operator $S(t)$, $t \geqslant 0$ which is differentiable and the function f,g are continuous in X then

$$\begin{aligned} u(t) = {} & u_0 + g(u) + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} B^{-1} f(t, \mathscr{I}_{0,t}^{\sigma} u(t), u(t)) ds \\ & + \int_0^t S'(t-s)\left(u_0 + g(u) + \frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau)) d\tau\right) dt, \\ & t \in [0,T]. \end{aligned} \tag{6}$$

REMARK 2.

- We have $\left\|\mathscr{I}_{0,t}^{\sigma} u(\tau) - \mathscr{I}_{0,t}^{\sigma} v(\tau)\right\| \leqslant \frac{1}{1-\delta}\|u-v\|$, since

$$\begin{aligned} \left\|\mathscr{I}_{0,t}^{\sigma} u(\tau) - \mathscr{I}_{0,t}^{\sigma} v(\tau)\right\| &= \left\|\frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} u(s) ds - \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} v(s) ds\right\| \\ &\leqslant \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} \|u(s)-v(s)\| ds \\ &\leqslant \frac{1}{1-\delta}\left(1 - e^{-\left(\frac{1-\delta}{\delta}\right)t}\right)\|u(s)-v(s)\| \\ &\leqslant \frac{1}{1-\delta}\|u-v\|. \end{aligned}$$

Now, we prove the theorems for the existence and uniqueness mild solution of the problem (1).

THEOREM 1. *Assume* (H1)–(H6) *and let* $u_0 \in D(M)$. *Then there exists a unique mild solution of* (1) *in* $[0,T]$.

Proof. By considering Lemma 1, property 3, we define the map $\Phi : C([0,T],X) \to C([0,T],X)$ by

$$\begin{aligned} \Phi u(t) = {} & u_0 + g(u) + \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} B^{-1} f(s, \mathscr{I}_{0,s}^{\sigma} u(s), u(s)) ds \\ & + \int_0^t S'(t-s)\left(u_0 + g(u) + \frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau)) d\tau\right) ds. \end{aligned} \tag{7}$$

Next, we proof that Φ is a contraction. Let $u \in C([0,T],X)$. From the assumption on $f(.)$ and $g(.)$, we see that:

$$\begin{aligned} & \left\|\int_0^t S'(t-s)\left(u_0 + g(u) + \frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau)) d\tau\right) ds\right\| \\ & \leqslant \int_0^t \|S'(t-s)\| \|u_0 + g(u)\| ds \\ & \quad + \int_0^t \|S'(t-s)\| \frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} \|B^{-1}\| \left\|f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau))\right\| d\tau ds \\ & \leqslant \left(\|u_0 + g(u)\| + \frac{1}{1-\delta}\|B^{-1}\| \sup_{s\in[0,T]} \left\|f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau))\right\|\right) \|\varphi_E\|_{L^1}. \end{aligned}$$

from which we infer that the function

$$s \to S'(t-s)\left(u_0+g(u)+\frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)}B^{-1}f(\tau,\mathscr{I}_{0,t}^{\sigma}u(\tau),u(\tau))d\tau\right),$$

is integrable on $[0,t]$ for all $t\in[0,T]$. This implies that $\Phi u \in C([0,T],X)$ and Φ is well defined. Moreover for $u,v\in C([0,T],X)$ and $t\in[0,T]$ we get

$$\begin{aligned}
&\|\Phi u-\Phi v\|\\
&\leqslant \|g(u)-g(v)\|+\frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}\left\|B^{-1}\right\|\\
&\quad\times\left\|f(s,\mathscr{I}_{0,s}^{\sigma}u(s),u(s))-f(s,\mathscr{I}_{0,s}^{\sigma}v(s),v(s))\right\|ds\\
&\quad+\int_0^t \varphi_M(t-s)(\|g(u)-g(v)\|)ds+\int_0^t \varphi_M(t-s)\Big(\frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)}\left\|B^{-1}\right\|\\
&\quad\times\left\|f(\tau,\mathscr{I}_{0,\tau}^{\sigma}u(\tau),u(\tau))-f(\tau,\mathscr{I}_{0,\tau}^{\sigma}v(\tau),v(\tau))\right\|d\tau\Big)ds\\
&\leqslant L_2\|u-v\|+\frac{1}{1-\delta}\left\|B^{-1}\right\|L_1\left(\left\|\mathscr{I}_{0,\tau}^{\sigma}u(\tau)-\mathscr{I}_{0,\tau}^{\sigma}v(\tau)\right\|+\|u-v\|\right)\\
&\quad+\|\varphi_M\|_{L_1}G\|u-v\|+\left\|B^{-1}\right\|L_1\left(\left\|I_{0,\tau}^{\sigma}u(\tau)-\mathscr{I}_{0,\tau}^{\sigma}v(\tau)\right\|+\|u-v\|\right)\frac{1}{1-\delta}\|\varphi_M\|\\
&\leqslant L_2\|u-v\|\left(1+\|\varphi_M\|_{L_1}\right)+\frac{1}{1-\delta}\left\|B^{-1}\right\|L_1\left(\left\|\mathscr{I}_{0,\tau}^{\sigma}u(\tau)-\mathscr{I}_{0,\tau}^{\sigma}v(\tau)\right\|+\|u-v\|\right)\\
&\quad\times(1+\|\varphi_M\|)\\
&\leqslant\left[L_2+\frac{(2-\sigma)RL_1}{(1-\delta)(1-\sigma)}\right]\left(1+\|\varphi_M\|_{L_1}\right)\|u-v\|,
\end{aligned}$$

since (H6), Φ is a contraction. So there exists a unique mild solution of (1) and the proof is complete. □

THEOREM 2. *Assume* (H1)–(H6) *and let* $u_0\in D(M)$. *Then there exists a mild solution of* (1) *in* $[0,T]$.

Proof. We transform the existence of solution of (1) into a fixed point problem. By considering Lemma 1, we define the map $\Phi: C([0,T],X)\to C([0,T],X)$ by (7) in the theorem president

$$\begin{aligned}
\Phi u(t) &= u_0+g(u)+\frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}B^{-1}f(s,\mathscr{I}_{0,x}^{\sigma}u(s),u(s))ds \qquad (8)\\
&\quad+\int_0^t S'(t-s)\left(u_0+g(u)+\frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)}B^{-1}f(\tau,\mathscr{I}_{0,x}^{\sigma}u(\tau),u(\tau))d\tau\right)ds.
\end{aligned}$$

Now we decompose Φ as $\Phi_1+\Phi_2$ on $B_r(0,E)$ the closed ball with center at 0 and radius r in $E=C([0,T],X)$, where

$$\Phi_1 u(t)=u_0+g(u)+\int_0^t S'(t-s)(u_0+g(u))ds,$$

and

$$\Phi_2 u(t) = \frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} B^{-1} f(s, \mathscr{I}_{0,s}^{\sigma} u(s), u(s)) ds$$
$$+ \int_0^t S'(t-s) \frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,s}^{\sigma} u(\tau), u(\tau)) d\tau ds.$$

Obviously, we have

$$u_0 + g(u) + \frac{1}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau)) d\tau \in C([0,T], D(M)).$$

Let $B_r(0,E) = \{z \in E = C([0,T],X) : \|z\| \leqslant r\}$. For $u, v \in C([0,T],X)$, Choose:

$$2\left(1 + \|\varphi_M\|_{L^1}\right)\left(\|u_0\| + \|g(0)\| + \frac{RN}{1-\delta}\right) < r.$$

then, for $u, v \in C([0,T],X)$, we have

$$\begin{aligned}
&\|\Phi_1 u(t) + \Phi_2 v(t)\| \\
\leqslant\ & \|u_0\| + \|g(u) - g(0)\| + \|g(0)\| \\
&+ \frac{\|B^{-1}\|}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} \left\|f(s, \mathscr{I}_{0,t}^{\sigma} v(s), v(s)) - f(s,0,0)\right\| ds \\
&+ \frac{\|B^{-1}\|}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} \|f(s,0,0)\| \\
&+ \int_0^t \left\|S'(t-s)\right\| \Bigg[\|u_0\| + \|g(u) - g(0)\| + \|g(0)\| \\
&+ \frac{\|B^{-1}\|}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(t-\tau)} \left\|f(\tau, \mathscr{I}_{0,\tau}^{\sigma} v(\tau), v(\tau)) - f(\tau,0,0)\right\| d\tau \\
&+ \frac{\|B^{-1}\|}{\delta}\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(t-\tau)} \|f(\tau,0,0)\| d\tau \Bigg] ds,
\end{aligned}$$

so

$$\begin{aligned}
&\|\Phi_1 u(t) + \Phi_2 v(t)\| \\
\leqslant\ & \|u_0\| + L_2 r + \|g(0)\| + \frac{RN}{1-\delta} + \frac{\|B^{-1}\|}{1-\delta} L_1 \left(\|v(s)\| + \frac{1}{1-\sigma}\|v(s)\|\right) \\
&+ \int_0^t \left\|S'(t-s)\right\| \Bigg[\|u_0\| + Gr + \|g(0)\| \\
&+ \frac{RL_1}{1-\delta}\left(\|v(s)\| + \frac{1}{1-\sigma}\|v(s)\|\right) + \frac{RN}{1-\delta} \Bigg] ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|u_0\| + L_2 r + \|g(0)\| + \frac{RN}{1-\delta} + \frac{R}{1-\delta} L_1 \left(\frac{2-\sigma}{1-\sigma}\right) r \\
&\quad + \|\varphi_M\|_{L^1} \left[\|u_0\| + L_2 r + \|g(0)\| + \frac{RL_1}{1-\delta} \left(\frac{2-\sigma}{1-\sigma}\right) r + \frac{RN}{1-\delta} \right] \\
&\leqslant (1 + \|\varphi_M\|_{L^1}) \left(\|u_0\| + L_2 r + \|g(0)\| + \frac{RL_1}{1-\delta} \left(\frac{2-\sigma}{1-\sigma}\right) r + \frac{RN}{1-\delta} \right) \\
&\leqslant (1 + \|\varphi_M\|_{L^1}) \left(\|u_0\| + \|g(0)\| + \frac{RN}{1-\delta} \right) \\
&\quad + (1 + \|\varphi_M\|_{L^1}) \left(L_2 + \frac{RL_1}{1-\delta} \left(\frac{2-\sigma}{1-\sigma}\right) \right) r. \\
&\leqslant r.
\end{aligned}
$$

Then Φ maps $B_r(0,E)$ into self and so $\Phi_1 u + \Phi_2 v \in B_r$. From the assumptions (H3) and (H5), we see that, for any $u \in C([0,T],X)$

$$
\left\| \int_0^t S'(t-s)(u_0 + g(u))\, ds \right\| \leqslant \|\varphi_M\|_{L^1} (\|u_0\| + L_2 r + \|g(0)\|),
$$

from which we infer that the function $s \to S'(t-s)(u_0 + g(u))$ is integrable on $[0,T]$, for all $t \in [0,T]$ and $\Phi_1 u \in C([0,T],X)$. Moreover for $u, v \in C([0,T],X)$ and $t \in [0,T]$ we get

$$
\begin{aligned}
\|\Phi_1 u(t) - \Phi_1 v(t)\| &= \|g(u) - g(v)\| + \int_0^t \|S'(t-s)\| \|g(u) - g(v)\|\, ds \\
&\leqslant G\|u - v\| + \|\varphi_M\|_{L^1} G \|u - v\| \\
&\leqslant G(1 + \|\varphi_M\|_{L^1}) \|u - v\|.
\end{aligned}
$$

Use hypothesis (H6), Φ_1 is a contraction on $B_r(0,E)$. Now we show that the operator Φ_2 is completely continuous, note that the function

$$
s \to \int_0^t S'(t-s) \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau))\, d\tau ds,
$$

is integrable from the assumptions (H3), (H4) as show above. First we show that Φ_2 is uniformly bounded. Now for $t \in [0,T]$

$$
\begin{aligned}
\|\Phi_2 u(t)\| &= \frac{1}{\delta} \|B^{-1}\| \int_0^x e^{-\left(\frac{1-\delta}{\delta}\right)(x-s)} \|f(t, \mathscr{I}_{0,x}^{\sigma} u(t), u(t))\|\, ds \\
&\quad + \int_0^t \|S'(t-s)\| \frac{1}{\delta} \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} \|B^{-1}\| \|f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau))\|\, d\tau ds \\
&\leqslant (1 + \|\varphi_M\|_{L^1}) \left(\frac{RN}{1-\delta} + \frac{RL_1}{1-\delta} \left(\frac{2-\sigma}{1-\sigma}\right) \right) r,
\end{aligned}
$$

so Φ_2 is uniformly bounded. Let (u_n) be a sequence in $B_r(0,E)$ such that $u_n \to u$ in $B_r(0,E)$. Since the function f is continuous

$$
f(s, \mathscr{I}_{0,s}^{\sigma} u_n(s), u_n(s)) \to f(s, \mathscr{I}_{0,s}^{\sigma} u(s), u(s)) \text{ as } n \to \infty.
$$

Now for each $t \in [0,T]$, we have

$$\begin{aligned}
&\|\Phi_2 u_n(t) - \Phi_2 u(t)\| \\
\leqslant\ & \frac{1}{\delta}\left\|B^{-1}\right\| \int_0^x e^{-\left(\frac{1-\delta}{\delta}\right)(x-s)} \left\|f(s, \mathscr{I}_{0,s}^{\sigma} u_n(s), u_n(s)) - f(t, \mathscr{I}_{0,s}^{\sigma} u(t), u(t))\right\| ds \\
&+ \int_0^t \left\|S'(t-s)\right\| \frac{1}{\delta}\left\|B^{-1}\right\| \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} \\
&\times \left\|f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u_n(\tau), u_n(\tau)) - f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau))\right\| d\tau ds \to 0 \text{ as } n \to \infty,
\end{aligned}$$

then Φ_2 is continuous. We need to prove that the set $\{\Phi_2 u(t) : u \in B_r(0,E)\}$ is relativeness compact in X for all $t \in [0,T]$. Obviously $\{\Phi_2 u(t) : u \in B_r(0,E)\}$ is a compact, fix $t \in (0,T]$ and $u \in B_r(0,E)$, define the operator Φ_2^{ε}

$$\begin{aligned}
\Phi_2^{\varepsilon} u(t) = & \frac{1}{\delta} \int_0^{t-\varepsilon} e^{-\left(\frac{1-\delta}{\delta}\right)(x-s)} B^{-1} f(t, \mathscr{I}_{0,\tau}^{\sigma} u(t), u(t)) ds \\
&+ \int_0^{t-\varepsilon} S'(t-s) \frac{1}{\delta} \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,\tau}^{\sigma} u(\tau), u(\tau)) d\tau ds,
\end{aligned}$$

since by (H4) f is completely continuous, the set

$$X_{\varepsilon} = \{\Phi_2^{\varepsilon} u(t) : u \in B_r(0,E)\},$$

is precompact in X, for every $\varepsilon > 0, 0 < \varepsilon < t$, Moreover for every $u \in B_r(0,E)$ we have

$$\begin{aligned}
&\|\Phi_2 u_n(t) - \Phi_2^{\varepsilon} u(t)\| \\
\leqslant\ & \frac{1}{\delta}\left\|B^{-1}\right\| \int_{t-\varepsilon}^t e^{-\left(\frac{1-\delta}{\delta}\right)(x-s)} \left\|f(s, \mathscr{I}_{0,s}^{\sigma} u_n(s), u_n(s))\right\| ds \\
&+ \int_{t-\varepsilon}^t \left\|S'(t-s)\right\| \frac{1}{\delta}\left\|B^{-1}\right\| \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)} \left\|f(\tau, \mathscr{I}_{0,s}^{\sigma} u_n(\tau), u_n(\tau))\right\| d\tau ds.
\end{aligned}$$

This means that the precompact sets X_{ε}, $0 < \varepsilon < t$, are close to the set $\{\Phi_2 u(t) : u \in B_r(0,E)\}$. Hence the set $\{\Phi_2 u(t) : u \in B_r(0,E)\}$ is precompact in X. Next, let us prove that $\Phi_2(B_r(0,E))$ is equicontinuous. The functions $\Phi_2 u, u \in B_r(0,E)$ are equicontinuous at $t = 0$. If $t < t+h \leqslant T, h > 0$ we have:

$$\begin{aligned}
&\|\Phi_2 u(t+h) - \Phi_2 u(t)\| \\
\leqslant\ & \frac{1}{\delta}\left\| \int_0^{t+h} e^{-\left(\frac{1-\delta}{\delta}\right)(t+h-s)} B^{-1} f(s, \mathscr{I}_{0,s}^{\sigma} u_n(s), u_n(s)) ds \right. \\
&\left. - \int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)} B^{-1} f(s, \mathscr{I}_{0,s}^{\sigma} u_n(s), u_n(s)) ds \right\| \\
&+ \frac{1}{\delta}\left\| \int_0^{t+h} S'(t+h-s) \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(t+h-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,s}^{\sigma} u_n(\tau), u_n(\tau)) d\tau ds \right. \\
&\left. - \int_0^t S'(t-s) \int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(t-\tau)} B^{-1} f(\tau, \mathscr{I}_{0,s}^{\sigma} u_n(\tau), u_n(\tau)) d\tau ds \right\| \qquad (9)
\end{aligned}$$

$$\leqslant \frac{1}{\delta}\int_0^t \left(e^{-\left(\frac{1-\delta}{\delta}\right)(t+h-s)} - e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}\right)\left\|B^{-1}\right\|\left\|f(s,\mathscr{I}_{0,s}^{\sigma}u_n(s),u_n(s))\right\|ds$$
$$+\frac{1}{\delta}\left\|\int_t^{t+h} e^{-\left(\frac{1-\delta}{\delta}\right)(t+h-s)}B^{-1}f(s,\mathscr{I}_{0,s}^{\sigma}u_n(s),u_n(s))ds\right\|$$
$$+\frac{1}{\delta}\int_0^h \left\|S'(t+h-s)\right\|\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(t+h-\tau)}\left\|B^{-1}\right\|\left\|f(\tau,\mathscr{I}_{0,\tau}^{\sigma}u_n(\tau),u_n(\tau))\right\|d\tau ds$$
$$+\frac{1}{\delta}\int_0^t \left\|S'(t-s)\right\|\left\|B^{-1}\right\|\left\|\int_0^{s+h} e^{-\left(\frac{1-\delta}{\delta}\right)(s+h-\tau)}f(\tau,\mathscr{I}_{0,\tau}^{\sigma}u_n(\tau),u_n(\tau))d\tau\right.$$
$$\left.-\int_0^s e^{-\left(\frac{1-\delta}{\delta}\right)(s-\tau)}f(\tau,\mathscr{I}_{0,\tau}^{\sigma}u_n(\tau),u_n(\tau))d\tau\right\|ds,$$

using (9), then

$$\lim_{h\to 0}\left\|\Phi_2 u(t+h)-\Phi_2 u(t)\right\| = 0.$$

By (H4), f is completely continuous, the set

$$\left\{\Phi_2 u(t) : u \in B_r(0,E)\right\},$$

is equicontinuous. Thus we have proved $\Phi_2(B_r(0,E))$ is relatively compact for $t \in [0,T]$. By Arzela-Ascoli's theorem Φ_2 is compact.

Hence by the Krasnoseskii fixed point theorem [27], there exists a fixed point $u \in C([0,T],X)$ such that $\Phi u = u$ which is a mild solution to the problem (1).

COROLLARY 1. *Let $B = I$ and $u_0 \in D(A)$. Assume* (H1), (H3), (H4), (H5) *and if*

$$\left(L_2 + \frac{2-\sigma}{(1-\delta)(1-\sigma)}L_1\right)\left(1+\|\varphi_A\|_{L^1}\right) < 1,$$

then there exists a mild solution of the problem

$$\begin{cases} \mathscr{D}_{0,t}^{\delta}(u(t)) = Au(t) + f(t,\mathscr{I}_{0,t}^{\sigma}u(t),u(t)),\ t\in[0,T], \\ u(0) = u_0 + g(u). \end{cases}$$

4. Application

In this section, we study the existence of a mild solution for the partial differential system with Caputo-Fabrizio derivatives as follows:

EXAMPLE 1. Consider the following partial differential equation with Caputo-Fabrizio derivative in the space $X = L^2([0,\pi])$:

$$\left.\begin{cases} \frac{\partial^{\delta}}{\partial t^{\delta}}\left(u(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x)\right) = \frac{\partial^2 u}{\partial x^2}(t,x) + \frac{1}{\omega\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s,x)ds, \\ u(t,0) = u(t,\pi) = 0, \\ u(0,x) = G(x) + \frac{1}{\omega\Gamma(\alpha)}\sum_{i=0}^{3}\int_0^{t_i}(t_i-s)^{\alpha-1}u(s,x)ds,\ t_i\in[0,T],\ x\in[0,\pi]. \end{cases}\right\}. \quad (10)$$

where $0<\delta<1,\ 0<\alpha<1,\ \omega$ parameter large enough, $G\in L^2([0,\pi])$. Let $Au=u''$ and $Bu=u-u''$ with domain

$$D(B)=D(A)=\left\{u\in X, u''\in X,\ u(0)=u(\pi)=0\right\}.$$

Then A and B can be written as

$$Au=\sum_{n=1}^{\infty} n^2\langle u,e_n\rangle e_n, u\in D(A),$$
$$Bu=\sum_{n=1}^{\infty}\left(1+n^2\right)\langle u,e_n\rangle e_n, u\in D(B),$$

(see [14]), where

$$e_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx);\ x\in[0,\pi],\ n=1,2,\ldots$$

we have, for $u\in X$

$$B^{-1}u=\sum_{n=1}^{\infty}\frac{1}{(1+n^2)}\langle u,e_n\rangle e_n,$$
$$Mu=B^{-1}Au=\sum_{n=1}^{\infty}\frac{-n^2}{(1+n^2)}\langle u,e_n\rangle e_n,$$

and, It is well known that A is the infinitesimal generator of an analytic semigroup on X (see [14]).

$$S(t)u=\sum_{n=1}^{\infty}e^{-\frac{n^2}{1+n^2}t}\langle u,e_n\rangle e_n.$$

We have also $\left\|B^{-1}\right\|\leqslant\frac{1}{4}$ and $\|S(t)\|\leqslant e^{-t}$. Then, since $B^{-1}A$ is a bounded operator, we have $\|S'(t)\|\leqslant M$, for all $t\geqslant 0$.

Hence the assumption (H1), (H2) and (H3) are satified.

Now we have for ω large enough,

$$\begin{aligned}
&\left\|f(t,I_{0,x}^{\sigma}u(t),u(t))-f(t,I_{0,x}^{\sigma}v(t),v(t))\right\|\\
&=\left\|\frac{1}{\omega\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s,x)ds-\frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}v(s,x)ds\right\|\\
&\leqslant\left\|\frac{1}{\omega\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}u(s,x)ds-\frac{1}{\delta}\int_0^t e^{-\left(\frac{1-\delta}{\delta}\right)(t-s)}v(s,x)ds\right\|\\
&\leqslant\frac{1}{\omega}\left(\frac{1}{1-\delta}\right)\|u-v\|
\end{aligned}$$

thus

$$\left\|f(t,I_{0,x}^{\sigma}u(t),u(t))-f(t,I_{0,x}^{\sigma}v(t),v(t))\right\|\leqslant L_1\|u-v\|,\ \text{with } L_1=\frac{1}{\omega}\left(\frac{1}{1-\delta}\right).$$

and

$$\begin{aligned}\|g(u)-g(v)\| &= \left\| \sum_{i=0}^{3} \frac{1}{\omega\Gamma(\alpha)} \int_0^{t_i} (t_i-s)^{\alpha-1} u(s,x)ds \right. \\ &\quad \left. - \sum_{i=0}^{3} \frac{1}{\omega\Gamma(\alpha)} \int_0^{t_i} (t_i-s)^{\alpha-1} v(s,x)ds \right\| \\ &= \frac{1}{\omega\Gamma(\alpha)} \left\| \sum_{i=0}^{3} \int_0^{t_i} (t_i-s)^{\alpha-1} u(s,x)ds - \sum_{i=0}^{3} \int_0^{t_i} (t_i-s)^{\alpha-1} v(s,x)ds \right\| \\ &\leqslant \frac{1}{\omega\Gamma(\alpha)} \sum_{i=0}^{3} \int_0^{t_i} (t_i-s)^{\alpha-1} ds \, \|u-v\| \\ &\leqslant \frac{3T^{\alpha}}{\omega\alpha\Gamma(\alpha)} \|u-v\|,\end{aligned}$$

thus

$$\|g(u)-g(v)\| \leqslant L_2 \|u-v\|, \text{ with } L_2 = \frac{3T^{\alpha}}{\omega\alpha\Gamma(\alpha)}.$$

So (H4) and (H5) are satisfied.

On the other hund, we have

$$\begin{aligned}\Upsilon &= \left(L_2 + \frac{2-\delta}{(1-\delta)^2} RL_1 \right) (1+\|\varphi_M\|_{L^1}) \\ &= \left(\frac{3T^{\alpha}}{\omega\alpha\Gamma(\alpha)} + \frac{2-\delta}{(1-\delta)^2} \frac{1}{4\omega} \left(\frac{1}{1-\delta} \right) \right) (1+\|\varphi_M\|_{L^1}) \\ &= \frac{1}{\omega} \left(\frac{3T^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{1}{4} \frac{2-\delta}{(1-\delta)^3} \right) (1+\|\varphi_M\|_{L^1}),\end{aligned}$$

then $\exists \omega^* > 0$, such that $\forall \omega \geqslant \omega^*$

$$\Upsilon = \frac{1}{\omega} \left(\frac{3T^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{1}{4} \frac{2-\delta}{(1-\delta)^3} \right) (1+\|\varphi_M\|_{L^1}) < 1.$$

Finally, the conditions (H1)–(H6) are satisfied, then the problem (10) has a mild solution $u \in C\left([0,T], L^2([0,\pi])\right)$.

5. Conclusion

In this work, with Caputo-Fabrizio derivative, we have studied the uniqueness and existence of mild solutions for the abstract fractional differential equations with nonlocal boundary conditions.We have establish the uniqueness results applying the Banach contraction principle and the existence results by applying the Krasnoseskii fixed point theorem. For justification, we give an application to which our results apply. We will find to investigate stability of similar problem in our future research work.

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