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ANALYTICAL AND NUMERICAL STUDY OF A LINEAR COUPLED SYSTEM INVOLVING CAPUTO–FABRIZIO FRACTIONAL DERIVATIVE WITH BOUNDARY CONDITIONS

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Abstract. This article is concerned with a coupled system of linear fractional differential equations of Caputo-Fabrizio type conformable fractional derivation with boundary conditions. In order to prove the existence and uniqueness of the solution, the problem is transformed into an equivalent linear Volterra-Fredholm integral equations of the second kind, and by using the Banach's fixed point theory the existence and uniqueness of solutions are obtained. Finally, the analytical results are supported by numerical results to illustrate the obtained results.

1. Introduction

Fractional differential equations appear in many engineering and scientific disciplines because fractional derivatives are involved in mathematical modeling of systems and processes in the fields such as physics, chemistry, biology, aerodynamics, electrodynamics of complex media, and more. Fractional differential equations are also excellent tools for describing the genetic properties of different materials and processes. Therefore, the topic of fractional differential equations has received more attention, see [3, 4, 8, 9, 2, 21, 24]. On the other hand, it is also important to study coupled systems with fractional differential equations, since such systems appear in problems of various applied nature, for instance, see [1, 6, 7, 15, 18, 20, 26]. The purpose of this paper is to provide a state of the art that can be easily used as a basis to familiarize oneself with couple system of linear fractional order of Caputo-Fabrizio type. Solely, the most recent advances, are included in this review.

The Adomian decomposition method (ADM) was first introduced by George Adomian in the beginning of 1980's and developed in [1]. The method is a kind of algorithm and it is an advantageous method for solving a linear and nonlinear differential equations of fractional order, which gives the approximate solution and even exact solution. This method has many advantages, such as: it's quite straight forward to write computer

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codes, it also avoids the cumber some integration of the Picard method, and it can solve some nonlinear problems which can not be solved by other numerical method (iterative, etc).

Another interesting field of recent research is the coupled system of fractional differential equations. It has been proven to be more accurate, realistic and has many applications in real-world problems such as: [6, 7].

In this paper, we study a coupled system of linear fractional differential equations, as follows:

$$\begin{cases} \mathscr{D}^{(\rho)} u(x) = c_1 u(x) + c_2 v(x) + f(x), \\ \mathscr{D}^{(\rho)} v(x) = c_3 u(x) + c_4 v(x) + g(x), \end{cases} \tag{1}$$

for $x \in I$ under condition $u(s_i) = v(s_i) = 0$, $i = 1,2$ where $1 < \rho < 2$ is a real number, $\mathscr{D}^{(\rho)}$ is the new fractional derivative of Caputo-Fabrizio, $f, g \in C(I)$ and c_i, $i = 1,2,3,4$ real constants. And we present some basic materials needed to prove our main results, then, obtain the equivalent linear Volterra-Fredholm integral equation of the second kind by using the fractional integral in problem (1). Finally, the existence and uniqueness of solution is established by applying Banach's contraction mapping principle, Adomin method and algorithm are introduced to solve the numerical solution of this class of problem.

2. Preliminaries

For the convenience of the reader, we give the necessary definitions and lemmas from fractional calculus theory. These definitions can be found in the recent literature.

DEFINITION 1. [14] Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left-sided CF-FD ${}^{CF}\mathscr{D}^{\alpha}_{a^+}h$ of order $\alpha \in [0,1[$ of a function h is defined as follows:

$$\mathscr{D}^{(\alpha)} h(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x h'(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s, \tag{2}$$

where $\alpha \in [0,1]$ and $a \in]-\infty, x)$, $h \in H^1(a,b)$, $b > a$, and $M(\alpha)$ is a normalization function, such that $M(0) = M(1) = 1$.

DEFINITION 2. [14] Let $n \geqslant 1$, and $\alpha \in [0,1]$. The fractional derivative $\mathscr{D}^{(\alpha+n)}h$ of order $(n+\alpha)$ is defined by

$$\mathscr{D}^{(\alpha+n)} h(x) := \mathscr{D}_x^{(\alpha)}\left(\mathscr{D}^{(n)} h(x)\right) = \frac{M(\alpha)}{1-\alpha} \int_a^x h^{(n+1)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s.$$

Such that

$$\mathscr{D}^{(\alpha+n)} h(t) = \frac{M(\alpha)}{1-\alpha} \int_a^x h^{(n+1)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s. \tag{3}$$

DEFINITION 3. [10] (A new fractional integral) Let $n \geqslant 1$, $\alpha \in [0,1]$, and $h \in \mathscr{C}^1[a,b]$. The formula:

$$I_a^{n+\alpha} h(x) = \frac{1}{M(\alpha)\cdot n!}\int_a^x (x-s)^{n-1}[\alpha(x-s)+n(1-\alpha)]h(s)\,\mathrm{d}s,$$

where $M(\alpha)$, is a normalization function such that $M(0) = M(1) = 1$, is a new fractional integral of order $(n+\alpha)$, and it's as an inverse of the conformable fractional derivative of Caputo of order $(n+\alpha)$.

LEMMA 1. [10] *Let $\rho \in (n,n+1)$, $n = [\rho] \geqslant 0$. Assume that $h \in \mathscr{C}^n[a,b]$, then, those statements hold:*

1. *if $h(a) = 0$, then $\mathscr{D}^{(\rho)}\,(I_a^{\rho} h(x)) = h(x)$.*
2. $I_a^{\rho}\left(\mathscr{D}^{(\rho)} h(x)\right) = h(x) + \sum_{i=0}^{n} a_i x^i$, $a_i \in \mathbb{R}$ $i = 0,1,\ldots,n$.

LEMMA 2. *Let $(E,\|\,.\,\|)$ be Banach space, $T : E \longrightarrow E$ be a contraction on E. Then, T has a unique fixed point $x \in E$ (such that $T(x) = x$).*

3. Analytic study

In the following, we suppose the function $M(\alpha) = 1$.

LEMMA 3. *Let $1 < \rho < 2$, $u,v \in \mathscr{C}^1[0,1]$, $f,g : [0,1] \to \mathbb{R}$ are continuous functions, and c_i real constants and $i = 1,2,3,4$. Then, the solution of*

$$\begin{cases} \mathscr{D}^{(\rho)} u(x) = c_1 u(x) + c_2 v(x) + f(x), \\ \mathscr{D}^{(\rho)} v(x) = c_3 u(x) + c_4 v(x) + g(x), \end{cases} \tag{4}$$

for $x \in I$ under condition $u(s_i) = v(s_i) = 0$, $i = 1,2$ where $1 < \rho < 2$ is a real number, $f,g \in C(I)$ and c_i, $i = 1,2,3,4$ real constants. satisfies the following linear Volterra integral equations of the second kind

$$u(x) = \int_0^x L(x,s)(c_1 u(s) + c_2 v(s))\,\mathrm{d}s + \int_0^1 F(x,s)(c_1 u(s) + c_2 v(s))\,\mathrm{d}s + K(x), \tag{5}$$

$$v(x) = \int_0^x L(x,s)(c_3 u(s) + c_4 v(s))\,\mathrm{d}s + \int_0^1 F(x,s)(c_3 u(s) + c_4 v(s))\,\mathrm{d}s + G(x). \tag{6}$$

Where

$$\begin{aligned} K(x) &= \int_0^x (\alpha(x-s)+1-\alpha))f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1)f(s)\,\mathrm{d}s, \\ G(x) &= \int_0^x (\alpha(x-s)+1-\alpha))g(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1)g(s)\,\mathrm{d}s, \\ L(x,s) &= \alpha(x-s)+1-\alpha, \\ F(x,s) &= x(\alpha s - 1). \end{aligned}$$

Proof. We may apply Lemma 1 to reduce Eq. (4) to an equivalent integral equation

$$I_0^\rho\left(\mathscr{D}^{(\rho)}u(x)\right) = I_0^\rho\left(c_1u(x)+c_2v(x)+f(x)\right),$$
$$I_0^\rho\left(\mathscr{D}^{(\rho)}v(x)\right) = I_0^\rho\left(c_3u(x)+c_4v(x)+g(x)\right),$$

we obtain,

$$u(x)+a_1x+b_1 = \int_0^x (\alpha(x-s)+1-\alpha)(c_1u(s)+c_2v(s)+f(s))\,\mathrm{d}s, \tag{7}$$

$$v(x)+a_2x+b_2 = \int_0^x (\alpha(x-s)+1-\alpha)(c_3u(s)+c_4v(s)+g(s))\,\mathrm{d}s, \tag{8}$$

where $a_1,a_2,b_1,b_2\in\mathbb{R}$. Using conditions $u(0)=0$ and $v(0)=0$ of the problem (4) in (7) and (8) respectively yields, we obtain $b_1=b_2=0$. Thus (7) and (8) reduces to

$$u(x)+a_1x = \int_0^x (\alpha(x-s)+1-\alpha)(c_1u(s)+c_2v(s)+f(s))\,\mathrm{d}s, \tag{9}$$

$$v(x)+a_2x = \int_0^x (\alpha(x-s)+1-\alpha)(c_3u(s)+c_4v(s)+g(s))\,\mathrm{d}s, \tag{10}$$

Considering boundary conditions $u(1)=0$ and $v(1)=0$, we get

$$a_1 = \int_0^1 (1-\alpha s)\left(c_1u(s)+c_2v(s)+f(s)\right)\mathrm{d}s,$$

and,

$$a_2 = \int_0^1 (1-\alpha s)\left(c_3u(s)+c_4v(s)+g(s)\right)\mathrm{d}s.$$

The substitution of the values of a_1 and a_2 in (9) and (10), leads to the next solution

$$\begin{aligned}u(x) = &\int_0^x (\alpha(x-s)+1-\alpha)(c_1u(s)+c_2v(s)+f(s))\,\mathrm{d}s\\ &+\int_0^1 x(\alpha s-1)(c_1u(s)+c_2v(s)+f(s))\,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}v(x) = &\int_0^x (\alpha(x-s)+1-\alpha)(c_3u(s)+c_4v(s)+g(s))\,\mathrm{d}s\\ &+\int_0^1 x(\alpha s-1)(c_3u(s)+c_4v(s)+g(s))\,\mathrm{d}s.\end{aligned}$$

So,

$$\begin{aligned}u(x) = &\int_0^x (\alpha(x-s)+1-\alpha)(c_1u(s)+c_2v(s))\,\mathrm{d}s+\int_0^1 x(\alpha s-1)(c_1u(s)+c_2v(s))\,\mathrm{d}s\\ &+\int_0^x (\alpha(x-s)+1-\alpha)f(s)\,\mathrm{d}s+\int_0^1 x(\alpha s-1)f(s)\,\mathrm{d}s,\end{aligned}$$

$$v(x)=\int_0^x(\alpha(x-s)+1-\alpha)(c_3u(s)+c_4v(s))\,\mathrm{d}s+\int_0^1 x(\alpha s-1)(c_3u(s)+c_4v(s))\,\mathrm{d}s$$
$$+\int_0^x(\alpha(x-s)+1-\alpha)g(s)\,\mathrm{d}s+\int_0^1 x(\alpha s-1)g(s)\,\mathrm{d}s.$$

Hence, the unique solution of problem (4) is

$$u(x)=\int_0^x L(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s+\int_0^1 F(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s+K(x),$$

$$v(x)=\int_0^x L(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s+\int_0^1 F(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s+G(x).$$

The proof is complete. □

4. Existence and uniqueness of the solution

Let us introduce the space $\mathscr{C}(I)$ endowed with the norm $\| u \|=\sup_{x\in[0,1]}|u(x)|$.

Obviously, $(\mathscr{C}(I),\| \,.\, \|)$ is a Banach space. Denote by $\Lambda=\mathscr{C}(I)\times\mathscr{C}(I)$. Then, the product space $(\Lambda,\| \,.\, \|)$ is also a Banach space endowed with the norm $\| (u,v) \|=\| u \|+\| v \|=\sup_{x\in[0,1]}|u(x)|+\sup_{x\in[0,1]}|v(x)|$, for $(u,v)\in\Lambda$.

In view of Lemma 3, we introduce an operator $T:\Lambda\longrightarrow\Lambda$ associated with the problem (1) as follows

$$T(u,v)(x):=(T_1(u,v)(x),T_2(u,v)(x)), \tag{11}$$

$$T_1(u,v)(x)=\int_0^x L(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s+\int_0^1 F(x,s)(c_1u(s)+c_2v(s))\,\mathrm{d}s+K(x), \tag{12}$$

and,

$$T_2(u,v)(x)=\int_0^x L(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s+\int_0^1 F(x,s)(c_3u(s)+c_4v(s))\,\mathrm{d}s+G(x). \tag{13}$$

Here we establish the existence of the solutions for the boundary value problem (4) by using Banach's contraction mapping principle.

THEOREM 1. *Let $f,g:I\to\mathbb{R}$ be a jointly continuous functions. Then, the problem (4) has a unique solution on I, if*

$$\lambda+\varepsilon<\frac{1}{2-\alpha}. \tag{14}$$

where, $\lambda=\max(|c_1|,|c_2|)$, $\varepsilon=\max(|c_3|,|c_4|)$.

Proof. As f and g are continuous functions on $I=[0,1]$ which is compact, then, f and g are bounded, so there exist real constants $m_1>0$, $m_2>0$, such that $|f(x)|\leqslant m_1$ and $|g(x)|\leqslant m_2$ for all $x\in I$.

Consider the operator $T:\Lambda \longrightarrow \Lambda$ defined by (11) and fix

$$k > \frac{(2-\alpha)(m_1+m_2)}{1-(\lambda+\varepsilon)(2-\alpha)}.$$

Then, we show that $TB_k \subset B_k$, where $B_k = \{(u,v)\in\Lambda : \| (u,v) \| \leqslant k\}$.

By our assumption, for $(u,v)\in B_k,\ x\in I$, we have

$$\begin{aligned}
|T_1(u,v)(x)| \leqslant{} & \int_0^x (\alpha(x-s)+1-\alpha)|c_1u(s)+c_2v(s)|\,\mathrm{d}s \\
& + \int_0^1 x|\alpha s-1||c_1u(s)+c_2v(s)|\,\mathrm{d}s + \int_0^x (\alpha(x-s)+1-\alpha)|f(s)|\,\mathrm{d}s \\
& + \int_0^1 x|\alpha s-1||f(s)|\,\mathrm{d}s, \\
\leqslant{} & \max(|c_1|,|c_2|)(|u|+|v|)\left(\int_0^x (\alpha(x-s)+1-\alpha)\,\mathrm{d}s + \int_0^1 x(1-\alpha s)\,\mathrm{d}s\right) \\
& + m_1\left(\int_0^x (\alpha(x-s)+1-\alpha)\,\mathrm{d}s + \int_0^1 x(1-\alpha s)\,\mathrm{d}s\right), \\
\leqslant{} & (\lambda(|u|+|v|)+m_1)\left(\int_0^x (\alpha(x-s)+1-\alpha)\,\mathrm{d}s + \int_0^1 x(1-\alpha s)\,\mathrm{d}s\right), \\
\leqslant{} & (2-\alpha)(\lambda(|u|+|v|)+m_1).
\end{aligned}$$

Which, once taking the norm for $x\in I$, leads to

$$\| T_1(u,v)(x) \| \leqslant (2-\alpha)(\lambda(\| u \| + \| v \|)+m_1).$$

In the same way, for $(u,v)\in B_k$, one can obtain

$$\| T_2(u,v)(x) \| \leqslant (2-\alpha)(\varepsilon(\| u \| + \| v \|)+m_2).$$

Therefore, for any $(u,v)\in B_k$, we have

$$\begin{aligned}
\| T(u,v)(x) \| &= \| T_1(u,v)(x) \| + \| T_2(u,v)(x) \| \\
&\leqslant (2-\alpha)((\lambda+\varepsilon)(\| u \| + \| v \|)+m_1+m_2) \\
&\leqslant (2-\alpha)((\lambda+\varepsilon)k+m_1+m_2) \\
&< k,
\end{aligned}$$

which show that T maps B_k into itself.

In order to show that the operator T is a contraction, let $(u_1,v_1),\ (u_2,v_2)\in\Lambda,\ \forall x\in$

I. Then, we have

$$\begin{aligned}
&|T_1(u_1,v_1)(x)-T_1(u_2,v_2)(x)|\\
\leqslant &\int_0^x(\alpha(x-s)+1-\alpha)|c_1(u_1(s)-u_2(s))+c_2(v_1(s)-v_2(s))|\,\mathrm{d}s\\
&+\int_0^1 x|\alpha s-1||c_1(u_1(s)-u_2(s))+c_2(v_1(s)-v_2(s))|\,\mathrm{d}s,\\
\leqslant &\int_0^x(\alpha(x-s)+1-\alpha)(|c_1(u_1(s)-u_2(s))|+|c_2(v_1(s)-v_2(s))|)\,\mathrm{d}s\\
&+\int_0^1 x|\alpha s-1|(|c_1(u_1(s)-u_2(s))|+|c_2(v_1(s)-v_2(s))|)\,\mathrm{d}s,\\
\leqslant &\max(|c_1|,|c_2|)(|u_1(x)-u_2(x)|+|v_1(x)-v_2(x)|)\left(\int_0^x(\alpha(x-s)\right.\\
&\left.+1-\alpha)\,\mathrm{d}s+\int_0^1 x(1-\alpha s)\,\mathrm{d}s\right),\\
\leqslant &\lambda(|u_1(x)-u_2(x)|+|v_1(x)-v_2(x)|)\left(\int_0^x(\alpha(x-s)+1-\alpha)\,\mathrm{d}s\right.\\
&\left.+\int_0^1 x(1-\alpha s)\,\mathrm{d}s\right),\\
\leqslant &(2-\alpha)\lambda(|u_1(x)-u_2(x)|+|v_1(x)-v_2(x)|).
\end{aligned}$$

Which, once taking the norm for $x\in I$, leads to

$$\| T_1(u_1,v_1)-T_1(u_2,v_2)\| \leqslant (2-\alpha)\lambda(\| u_1-u_2\| + \| v_1-v_2\|).$$

And likewise, for (u_1,v_1) $(u_2,v_2)\in\Lambda$, $x\in[0,1]$, one can obtain

$$\| T_2(u_1,v_1)-T_2(u_2,v_2)\| \leqslant (2-\alpha)\varepsilon(\| u_1-u_2\| + \| v_1-v_2\|).$$

Obviously, it follows from the foregoing inequalities that

$$\begin{aligned}
\| T(u_1,v_1)-T(u_2,v_2)\| &= \| T_1(u_1,v_1)-T_1(u_2,v_2)\| + \| T_2(u_1,v_1)-T_2(u_2,v_2)\|,\\
&\leqslant (2-\alpha)(\lambda+\varepsilon)(\| u_1-u_2\| + \| v_1-v_2\|),\\
&=(2-\alpha)(\lambda+\varepsilon)\| (u_1,v_1)-(u_2,v_2)\|,\\
&=N\| (u_1-u_2,v_1-v_2)\|.
\end{aligned}$$

Since $N=(2-\alpha)(\lambda+\varepsilon)<1$. Which, in view of (14), implies that T is a contraction mapping. Hence, by Banach's fixed-point theorem, the operator T has a unique fixed point. This, in turn, shows that the problem 4 has a unique solution on I. The proof is completed. □

5. Numerical study

In this section, we introduce an algorithm for finding a numerical solution of linear Volterra integral equations of the second kind, the methods based the Adomian Decomposition.

In the decomposition method we usually express the solution $u(x)$ and $v(x)$ of the integral equation (5) and (6) in a series form defined by

$$u(x)=\sum_{i=0}^{\infty}u_i(x),\qquad v(x)=\sum_{i=0}^{\infty}v_i(x). \tag{15}$$

The substitution of the decomposition (15) into both sides of (5) and (6) yields

$$\begin{aligned}
\sum_{i=0}^{\infty}u_i(x)&=K(x)+\int_0^x L(x,s)\left(c_1\sum_{i=0}^{\infty}u_i(s)+c_2\sum_{i=0}^{\infty}v_i(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_1\sum_{i=0}^{\infty}u_i(s)+c_2\sum_{i=0}^{\infty}v_i(s)\right)\mathrm{d}s,\\
\sum_{i=0}^{\infty}v_i(x)&=G(x)+\int_0^x L(x,s)\left(c_3\sum_{i=0}^{\infty}u_i(s)+c_4\sum_{i=0}^{\infty}v_i(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_3\sum_{i=0}^{\infty}u_i(s)+c_4\sum_{i=0}^{\infty}v_i(s)\right)\mathrm{d}s,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
u_0(x)+u_1(x)+u_2(x)+\cdots&=K(x)+\int_0^x L(x,s)\left(c_1u_0(s)+c_2v_0(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_1u_0(s)+c_2v_0(s)\right)\mathrm{d}s\\
&\quad+\int_0^x L(x,s)\left(c_1u_1(s)+c_2v_1(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_1u_1(s)+c_2v_1(s)\right)\mathrm{d}s\\
&\quad+\int_0^x L(x,s)\left(c_1u_2(s)+c_2v_2(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_1u_2(s)+c_2v_2(s)\right)\mathrm{d}s+\cdots\\
v_0(x)+v_1(x)+v_2(x)+\cdots&=G(x)+\int_0^x L(x,s)\left(c_3u_0(s)+c_4v_0(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_3u_0(s)+c_4v_0(s)\right)\mathrm{d}s\\
&\quad+\int_0^x L(x,s)\left(c_3u_1(s)+c_4v_1(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_3u_1(s)+c_4v_1(s)\right)\mathrm{d}s\\
&\quad+\int_0^x L(x,s)\left(c_3u_2(s)+c_4v_2(s)\right)\mathrm{d}s\\
&\quad+\int_0^1 F(x,s)\left(c_3u_2(s)+c_4v_2(s)\right)\mathrm{d}s+\cdots
\end{aligned}$$

The components $u_0(x), u_1(x), u_2(x), \cdots$ and $v_0(x), v_1(x), v_2(x), \cdots$ of the unknown functions $u(x)$ and $v(x)$ respectively are completely determined in a recurrent manner if we set

$$u_0(x) = K(x), \qquad v_0(x) = G(x),$$

$$u_1(x) = \int_0^x L(x,s)\left(c_1 u_0(s) + c_2 v_0(s)\right) + \int_0^1 F(x,s)\left(c_1 u_0(s) + c_2 v_0(s)\right) \mathrm{d}s,$$

$$v_1(x) = \int_0^x L(x,s)\left(c_3 u_0(s) + c_4 v_0(s)\right) + \int_0^1 F(x,s)\left(c_3 u_0(s) + c_4 v_0(s)\right) \mathrm{d}s,$$

$$u_2(x) = \int_0^x L(x,s)\left(c_1 u_1(s) + c_2 v_1(s)\right) + \int_0^1 F(x,s)\left(c_1 u_1(s) + c_2 v_1(s)\right) \mathrm{d}s,$$

$$v_2(x) = \int_0^x L(x,s)\left(c_3 u_1(s) + c_4 v_1(s)\right) + \int_0^1 F(x,s)\left(c_3 u_1(s) + c_4 v_1(s)\right) \mathrm{d}s,$$

$$u_3(x) = \int_0^x L(x,s)\left(c_1 u_2(s) + c_2 v_2(s)\right) + \int_0^1 F(x,s)\left(c_1 u_2(s) + c_2 v_2(s)\right) \mathrm{d}s,$$

$$v_3(x) = \int_0^x L(x,s)\left(c_3 u_2(s) + c_4 v_2(s)\right) + \int_0^1 F(x,s)\left(c_3 u_2(s) + c_4 v_2(s)\right) \mathrm{d}s,$$

and so on. So the above discussed scheme for the determination of the components $u_0(x), u_1(x),\ u_2(x), \cdots$ and $v_0(x), v_1(x),\ v_2(x), \cdots$ of the solution $u(x)$ and $v(x)$ of Eq.5-6 respectively can be written in a recursive manner by

$$\begin{aligned}
u_0(x) &= K(x),\\
u_{n+1}(x) &= \int_0^x L(x,s)\left(c_1 u_n(s) + c_2 v_n(s)\right) + \int_0^1 F(x,s)\left(c_1 u_n(s) + c_2 v_n(s)\right) \mathrm{d}s, \quad n \geqslant 0.\\
v_0(x) &= G(x),\\
v_{n+1}(x) &= \int_0^x L(x,s)\left(c_3 u_n(s) + c_4 v_n(s)\right) + \int_0^1 F(x,s)\left(c_3 u_n(s) + c_4 v_n(s)\right) \mathrm{d}s, \quad n \geqslant 0.
\end{aligned}$$

6. Numerical result

In this section, we give two numerical examples to illustrate the above methods to solve the linear Volterra integral equations of the second kind.

The exact solution is known and used to justify the numerical solution obtained with our method is correct. We used MATLAB to solve these examples.

EXAMPLE 1. Consider the following fractional boundary value problem:

$$\begin{cases}
\mathscr{D}^{(\rho)} u(x) = c_1 u(x) + c_2 v(x) + f(x), & x \in I := [0,1],\\
\mathscr{D}^{(\rho)} v(x) = c_3 u(x) + c_4 v(x) + g(x), & x \in I := [0,1],\\
u(0) = u(1) = 0,\ v(0) = v(1) = 0,
\end{cases} \tag{16}$$

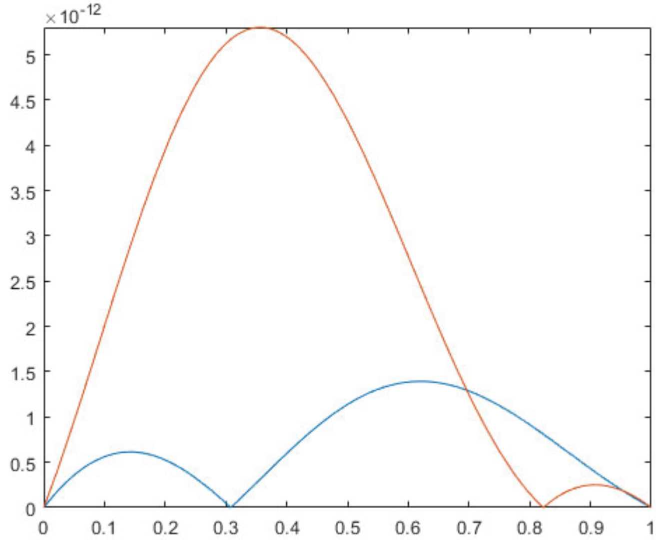

Figure 1: *The Absolute Error of test Example (1) with* $N = 5$.

where,

$$\rho = 1.5,\ \ c_1 = \frac{1}{6},\ \ c_2 = \frac{1}{4},\ \ c_3 = -\frac{1}{3},\ \ c_4 = -\frac{1}{5},$$

$$f(x) = 3\cos(x) - 3e^{-x} - \frac{\sin(x)(x-1)}{4} - \frac{(\cos(x)-1)(x-1)}{6} - x\cos(x) - x\sin(x),$$

$$g(x) = 3\sin(x) + \frac{\sin(x)(x-1)}{5} + \frac{(\cos(x)-1)(x-1)}{3} + x\cos(x) - x\sin(x),$$

with the exact solution $u(x) = (x-1)(\cos(x)-1)$ and $v(x) = (x-1)\sin(x)$.

EXAMPLE 2. Consider the following fractional boundary value problem:

$$\begin{cases} \mathscr{D}^{(\rho)}u(x) = c_1u(x) + c_2v(x) + f(x), & x \in I := [0,1], \\ \mathscr{D}^{(\rho)}v(x) = c_3u(x) + c_4v(x) + g(x), & x \in I := [0,1], \\ u(0) = u(1) = 0,\ v(0) = v(1) = 0, \end{cases} \tag{17}$$

where

$$\rho = 1.75,\ \ c_1 = -\frac{1}{5},\ \ c_2 = \frac{1}{6},\ \ c_3 = \frac{1}{5},\ \ c_4 = -\frac{1}{3},$$

$$f(x) = 8 - 8e^{-3x} + \frac{3x(x-1)}{5} - \frac{x(e^x - e^1)}{42},$$

$$g(x) = \frac{e^x - e^{-3x}}{4} + \frac{x(4e^x - e^1)}{21} - \frac{3x(x-1)}{5},$$

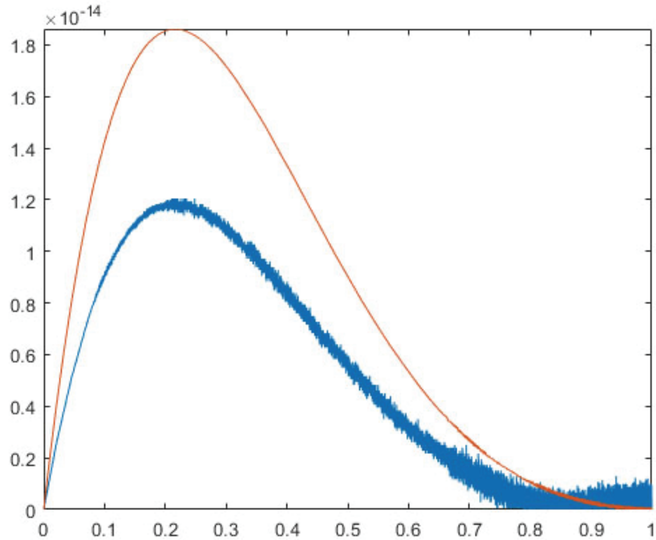

Figure 2: *The Absolute Error of test Example (2) with* $N = 7$.

with the exact solution $u(x) = 3x(x-1)$ and $v(x) = \frac{x}{7}(e^x - e^1)$.

Conclusion

In this paper, the existence and uniqueness of solutions for a coupled system of linear fractional differential equations with boundary conditions were established and discussed, which was done using the fixed point theory, (see [16, 22, 13, 19, 5, 12, 11, 25, 17, 23]).

Furthermore, a powerful numerical method represented by the Adomian decomposition method was suggested in order to implement the proposed model in an appropriate, precise manner, and program it with Matlab.

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