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FRACTIONAL NON–AUTONOMOUS EVOLUTION EQUATIONS WITH INTEGRAL IMPULSE CONDITION IN FRÉCHET SPACES

FATIMA MESRI, ABDELKRIM SALIM* AND MOUFFAK BENCHOHRA

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Abstract. The primary focus of this study is to explore the presence of a mild solution within a specific category of fractional non-autonomous differential evolution equations, incorporating integral impulse conditions. The approach employed extends the classical Darbo fixed point theorem for Fréchet spaces, leveraging the notion of a measure of noncompactness along with the principle of K-set contraction. To illustrate our findings, we offer an illustrative example.

1. Introduction

From the modern literature, it is observed that the topic of fractional differential equations received an overwhelming interest from many scholars, due to their importance in understanding the dynamic memory of many real-world phenomena. In this regards, a wide variety of fractional integrals and derivatives have been defined and extensively studied by many researchers. For more details and applications of fractional calculus, the reader is directed to the books of Anastassiou [3], Benchohra *et al.* [6, 7], Cao [12], Dutta [16], Francesco [19], Herrmann [25], Hilfer [26], Kilbas *et al.* [27] and Samko *et al.* [34]. Agrawal [2] introduced some generalizations of fractional integrals and derivatives and present some of their properties. On the other hand, various well-known academics have lately employed a wide range of strategies to solve different problems under the concept of fractional calculus. In the papers [24, 33, 28, 13, 14, 18, 29, 36, 39, 1], the authors presented the existence and uniqueness of solution for some nonlinear fractional differential equations.

Sudden occurrences, prevalent in both natural events and human actions, stem from abrupt shifts in a system's state due to external disruptions. These phenomena can be categorized into two groups based on the duration of the change they bring about. The first group involves alterations that transpire over a short span in comparison to the entire process-a concept termed "instantaneous impulses." On the other hand, the second group entails effects that unfold continuously, commencing from any fixed point

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* Corresponding author.

and persisting for a predetermined duration, referred to as a “non-instantaneous impulses.” The theory of instantaneous impulsive differential equations has undergone significant advancement over time and has assumed a pivotal role in modern applied mathematical models, addressing real-world processes observed in fields such as physics, population dynamics, chemical technology, biotechnology, and economics. Noteworthy progress has been achieved in the study of impulsive evolution equations, as evidenced by recent works including those by [21, 8, 9, 10, 22, 38, 5] and their respective references.

Arthi and Balachandran *et al.* [4] considered the following abstract control system:

$$\begin{cases} \chi''(\delta) = \mathscr{L}\chi(\delta) + Bu(\delta) + \aleph\left(\delta, \chi_{\rho(\delta,\chi_\delta)}\right), \delta \in I = [0,a], \delta \neq \delta_i, \\ \chi_0 = \varphi \in \mathscr{E}, \quad \chi'(0) = \eta \in \mho, \\ \Delta\chi(\delta_i) = I_i\left(\chi_{\delta_i}\right), i = 1,2,\ldots,i, \\ \Delta\chi'(\delta_i) = \Theta_i\left(\chi_{\delta_i}\right), i = 1,2,\ldots,i, \end{cases}$$

where $\mathscr{L}$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(\delta))_{\delta\in\mathbb{R}}$ defined on a Banach space F. The control function $u(\cdot)$ is given in $L^2(I,U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to F$ as a bounded linear operator; the function $\chi_\delta : (-\infty, 0] \to F, \chi_\delta(\vartheta) = \chi(\delta + \vartheta)$, belongs to some abstract phase space $\mathscr{E}$ described axiomatically; $0 < \delta_1 < \cdots < \delta_i < a$ are prefixed numbers; $\aleph : I \times \mathscr{E} \to F, \rho : I \times \mathscr{E} \to (-\infty, a], I_i(\cdot) : \mathscr{E} \to F, \Theta_i(\cdot) : \mathscr{E} \to F$ are appropriate functions and the symbol $\Delta\xi(\delta)$ represents the jump of the function $\xi(\cdot)$ at δ, which is defined by $\Delta\xi(\delta) = \xi(\delta^+) - \xi(\delta^-)$.

In this paper, we examine the possibility that the following nonlinear time fractional non-autonomous evolution equations in Fréchet space have mild solutions to their initial value problem:

$$\begin{cases} {}^cD^{\varsigma}_{0,\delta}\chi(\delta) + \mathscr{L}(\delta)\chi(\delta) = \aleph(\delta, \chi(\delta)), \text{ a.e. } \delta \in \mathbb{R}_+, \ \delta \neq \delta_j, \quad j = 1,2,\ldots. \\ \chi(0) = \chi_0, \\ \mho\chi(\delta_j) = I_j\left(\int_{\delta_j - \varkappa_j}^{\delta_j - \vartheta_j} \widehat{\aleph}(\varepsilon, \chi(\varepsilon))d\varepsilon\right), \quad j = 1,2,\ldots, \end{cases} \tag{1}$$

where $(0 < \varsigma \leqslant 1)$, ${}^cD^{\varsigma}_{0,\delta}$ is the ς-order Caputo derivative operator, $(\Im, \|\cdot\|)$ is a Banach space, and $\{\mathscr{L}(\delta)\}_{\delta>0}$ is a family of linear closed (not necessarily bounded) operators defined on a dense domain $\mathfrak{H}(\mathscr{L})$ in $\Im$ into $\Im$ such that $\mathfrak{H}(\mathscr{L})$ is independent of δ, $\aleph : \mathbb{R}_+ \times \Im \to \Im$ is a given function which will be specified later, $\chi_0 \in \Im$ and $\widehat{\aleph} : \mathbb{R}_+ \times \Im \to \Im$ is a given function; $0 < \delta_0 < \delta_1 < \ldots < \delta_i < \delta_{i+1} < \ldots < \lim_{i\to\infty} \delta_i = \infty$, $I_j \in C(\Im, \Im)$ are bounded functions, $0 \leqslant \vartheta_j \leqslant \varkappa_j \leqslant \delta_j - \delta_{j-1}$ for $j \in \mathbb{N}$, $\mho\chi(\delta_j) = \chi(\delta_j^+) - \chi(\delta_j^-)$ and $\chi(\delta_j^+) = \lim_{\rho\to 0^+} \chi(\delta_j + \rho)$, $\chi(\delta_j^-) = \lim_{\rho\to 0^-} \chi(\delta_j - \rho)$.

The work is organized as follows: In Section 2, we review some essential facts that are used to obtain our main results. In Section 3, we study the existence of mild solution for a the fractional non-autonomous differential evolution equations with integral impulses condition (1). To be more precise, we use a generalization of the classical Darbo fixed point theorem for Fréchet spaces, Darbo fixed point theorem and the concept of measure of noncompactness and K-set contraction principle to obtain our results. Also, we give an appropriate example to illustrate the given result.

2. Preliminaries

This section introduces notation, definitions, and preliminary facts that will be utilized throughout the study.

Let $\mathfrak{R} := \mathbb{R}_+$. Consider the Fréchet space $\digamma = C(\mathbb{R}_+)$ of all continuous functions from $\mathfrak{R}$ into $\mathfrak{I}$ equipped with the family of seminorms

$$\|\chi\|_i = \sup_{\delta\in[0,i]} \|\chi(\delta)\|;\ i \in \mathbb{N},$$

and the distance

$$d(\varpi,\overline{\varpi}) = \sum_{i=1}^{\infty} 2^{-i}\frac{\|\varpi-\overline{\varpi}\|_i}{1+\|\varpi-\overline{\varpi}\|_i};\ \varpi,\overline{\varpi}\in\digamma.$$

Let $\widetilde{\mathfrak{R}} := [0,\kappa];\kappa > 0$. A measurable function $\chi : \widetilde{\mathfrak{R}} \to \mathfrak{I}$ is Bochner integrable if and only if $\|\chi\|$ is Lebesgue integrable [37].

By $\mathfrak{G}(\mathfrak{I})$ we denote the Banach space of all bounded linear operators from $\mathfrak{I}$ into $\mathfrak{I}$ with

$$\|\Xi\|_{\mathfrak{G}(\mathfrak{I})} = \sup_{\|\chi\|=1} \|\Xi(\chi)\|.$$

As usual, $L^1(\widetilde{\mathfrak{R}},\mathfrak{I})$ denotes the Banach space of measurable functions $\chi : \widetilde{\mathfrak{R}} \to \mathfrak{I}$ which are Bochner integrable and normed by

$$\|\chi\|_{L^1} = \int_0^{\kappa} \|\chi(\delta)\|d\delta.$$

DEFINITION 1. ([27,32]) The fractional integral of order $\varsigma > 0$ with the lower limit zero for a function $\aleph \in L^1([0,+\infty),\mathbb{R})$ is defined as

$$I_0^{\varsigma}\aleph(\delta) = \int_0^{\delta} \frac{(\delta-\varepsilon)^{\varsigma-1}}{\Gamma(\varsigma)}\aleph(\varepsilon)d\varepsilon,$$

where Γ is the gamma function.

DEFINITION 2. ([27,32]) The Caputo fractional derivative of order ς with the lower limit zero for a function $\aleph : [0,+\infty) \to \mathbb{R}$, which is at least i-times differentiable can be defined as

$${}^{c}D_{\delta}^{\varsigma}\aleph(\delta) = \frac{1}{\Gamma(i-\varsigma)}\int_0^{\delta} (\delta-\varepsilon)^{i-\varsigma-1}\aleph^{(i)}(\varepsilon)d\varepsilon = I_{\delta}^{i-\varsigma}\aleph^{(i)}(\delta).$$

Here, $i = [\varsigma]+1$ and $[\varsigma]$ denotes the integer part of ς.

Throughout this paper, the linear operator $\mathscr{Z}$ is assumed to satisfy the following assumptions:

(A_1) For any $\overline{h}$ with $Re(\overline{h}) \geqslant 0$, the operator $\overline{h}I + \mathscr{Z}$ has a bounded inverse operator $[(\overline{h}I + \mathscr{Z})(\delta)]^{-1}$ in $\mathfrak{G}(\mathfrak{I})$ and

$$\|[(\overline{h}I + \mathscr{Z})(\delta)]^{-1}\| \leqslant \frac{C}{|\overline{h}|+1},$$

where C is a positive constant independent of both δ and $\overline{h}$;

(A_2) For any $\delta, \varkappa, \varepsilon \in \widetilde{\mathfrak{R}}$, there exists a constant $\gamma \in (0,1]$ such that

$$\|[\mathscr{Z}(\delta) - \mathscr{Z}(\varkappa)]\mathscr{Z}^{-1}(\varepsilon)\| \leqslant C|\delta - \varkappa|^{\gamma},$$

where the constants γ and $C > 0$ are independent of both $\delta, \varkappa$ and ε.

REMARK 1. From Henry [23], Pazy [31] and Temam [35], we know that the assumption (A_1) means that for each $\varepsilon \in \widetilde{\mathfrak{R}}$, the operator $-\mathscr{Z}(\varepsilon)$ generates an analytic semigroup $e^{-\delta\mathscr{Z}(\varepsilon)}$ $(\delta > 0)$, and there exists a positive constant C independent of both δ and ε such that

$$\|\mathscr{Z}^{i}(\varepsilon)e^{-\delta\mathscr{Z}(\varepsilon)}\| \leqslant \frac{C}{\delta^{i}},$$

where $i = 0,\ 1,\ \delta > 0,\ \varepsilon \in \mathfrak{R}$.

According to [17], we can define the operators $\beth(\delta,\varepsilon)$, $\varphi(\delta,\eta)$ and $U(\delta)$ by

$$\beth(\delta,\varepsilon) = \varsigma \int_0^{\infty} \vartheta\delta^{\varsigma-1}\xi_{\varsigma}(\vartheta)e^{-\delta^{\varsigma}\vartheta\mathscr{Z}(\varepsilon)}d\vartheta, \tag{2}$$

$$\varphi(\delta,\eta) = \sum_{j=1}^{\infty}\varphi_j(\delta,\eta), \tag{3}$$

and

$$U(\delta) = -\mathscr{Z}(\delta)\mathscr{Z}^{-1}(0) - \int_0^{\delta}\varphi(\delta,\varepsilon)\mathscr{Z}(\varepsilon)\mathscr{Z}^{-1}(0)d\varepsilon, \tag{4}$$

where ξ_{ς} is a probability density function [20] defined on $[0,+\infty)$ such that it's Laplace transform is given by

$$\int_0^{\infty}\xi_{\varsigma}(\vartheta)e^{-\vartheta\chi}d\vartheta = \sum_{i=1}^{\infty}\frac{(-\chi)^i}{\Gamma(1+\varsigma i)} \quad 0 < \varsigma \leqslant 1,\ \chi > 0,$$

$$\varphi_1(\delta,\eta) = [\mathscr{Z}(\delta) - \mathscr{Z}(\eta)]\beth(\delta-\eta,\eta),$$

and

$$\varphi_{j+1}(\delta,\eta) = \int_{\eta}^{\delta}\varphi_j(\delta,\varepsilon)\varphi_1(\varepsilon,\eta)d\eta, \quad j = 1,2,\ldots.$$

DEFINITION 3. ([17]) We say that a continuous function $\chi(\cdot):\Re\to\Im$ is mild solution of (1), if χ verifies:

$$\begin{aligned}\chi(\delta) &= \chi_0+\int_0^{\delta}\beth(\delta-\eta,\eta)U(\eta)\mathscr{Z}(0)\chi_0 d\eta+\int_0^{\delta}\beth(\delta-\eta,\eta)\aleph(\eta,\chi(\eta))d\eta \\ &+\int_0^{\delta}\int_0^{\eta}\beth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon d\eta, \\ &+\sum_{0<\delta_j<\delta}\beth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon\right), \qquad \text{for each } \delta\in\mathbb{R}_+.\end{aligned}$$

LEMMA 1. ([17]) *The operator-valued functions* $\beth(\delta-\eta,\eta)$ *and* $\mathscr{Z}(\delta)\beth(\delta-\eta,\eta)$ *are continuous in uniform topology about the variables* δ *and* η, *where* $\delta\in\Re$, $0\leqslant\eta\leqslant\delta-\rho$ *for any* $\rho>0$, *and*

$$\|\beth(\delta-\eta,\eta)\|\leqslant C(\delta-\eta)^{\varsigma-1}, \tag{5}$$

where C *is a positive constant independent of both* δ *and* η. *Furthermore,*

$$\|\varphi(\delta,\eta)\|\leqslant C(\delta-\eta)^{\gamma-1} \tag{6}$$

and

$$\|U(\delta)\|\leqslant C(1+\delta^{\gamma}). \tag{7}$$

DEFINITION 4. ([15]) Let $\beth_{\widetilde{F}}$ be the family of all nonempty and bounded subsets of Fréchet space $\widetilde{F}$. A family of functions $\{\wp_i\}_{i\in\mathbb{N}}$ where $\wp_i:\beth_{\widetilde{F}}\to[0,\infty)$ is said to be a family of measures of noncompactness in the Fréchet space $\widetilde{F}$ if it satisfies the following conditions for all $\nabla,\nabla_1,\nabla_2\in\beth_{\widetilde{F}}$:

(a) $\{\wp_i\}_{i\in\mathbb{N}}$ is full, that is: $\wp_i(\nabla)=0$ for $i\in\mathbb{N}$ if and only if ∇ is precompact,

(b) $\wp_i(\nabla_1)\leqslant\wp_i(\nabla_2)$ for $\nabla_1\subset\nabla_2$ and $i\in\mathbb{N}$,

(c) $\wp_i(Conv\nabla)=\wp_i(\nabla)$ for $i\in\mathbb{N}$,

(d) If $\{\nabla_i\}_{i=1,\ldots}$ is sequence of closed sets from $\beth_{\widetilde{F}}$ such that $\nabla_{i+1}\subset\nabla_i$; $i=1,\ldots$ and if $\lim_{i\to\infty}\wp(\nabla_i)=0$, for each $i\in\mathbb{N}$, then the intersection set $\nabla_\infty:=\cap_{i=1}^{\infty}$ is nonempty.

Some Properties:

(e) We can the family of measures of noncompactness if $\{\wp_i\}_{i\in\mathbb{N}}$ to be homogeneous if $\wp_i(\overline{h}\nabla)=|\overline{h}|\wp_i(\nabla)$, for $\overline{h}\in\mathbb{R}$ and $i\in\mathbb{N}$.

(f) If the family $\{\wp_i\}_{\mathbb{N}}$ satisfied the condition $\wp_i(\nabla_1\cup\nabla_2)\leqslant\wp_i(\nabla_1)+\wp_i(\nabla_2)$, for $i\in\mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\wp_i\}_{i\in\mathbb{N}}$ has the maximum property if

$$\wp_i(\nabla_1 \cup \nabla_2) = \max(\wp_i(\nabla_1), \wp_i(\nabla_2)),$$

(i) The family of measure of noncompactness $\{\wp_i\}_{i\in\mathbb{N}}$ is said to be regular if and only if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

DEFINITION 5. ([15]) A nonempty subset $\nabla \subset \digamma$ is said to be bounded if

$$\sup_{\upsilon\in\nabla} \|\upsilon\|_i < \infty \quad \text{for} \quad i \in \mathbb{N}.$$

LEMMA 2. ([11]) *If $\widehat{\digamma}$ is a bounded subset of Banach space $\digamma$, then for each $\rho > 0$, there is a sequence $\{\widehat{\chi}_j\}_{j=1}^{\infty} \subset \widehat{\digamma}$ such that*

$$\wp(\widehat{\digamma}) \leqslant 2\wp(\{\widehat{\chi}_j\}_{j=1}^{\infty}) + \rho,$$

where $\wp$ is a measure of noncompactness.

LEMMA 3. ([30]) *If $\{u_j\}_{j=1}^{\infty} \subset L^1(\widetilde{\mathfrak{R}})$ is uniformly integrable, then $\wp(\{u_j\}_{j=1}^{\infty})$ is measurable and*

$$\wp\left(\left\{\int_0^{\delta} u_j(\varepsilon)d\varepsilon\right\}_{j=1}^{\infty}\right) \leqslant 2\int_0^{\delta} \wp(\{u_j(\varepsilon)\}_{j=1}^{\infty})d\varepsilon,$$

for each $\delta \in [0,i]$, where $\wp$ is a measure of noncompactness.

THEOREM 1. ([15]) *Let $\mho$ be a nonempty, bounded, closed, and convex subset of a Fréchet space $\widetilde{\digamma}$ and let $\mathfrak{T} : \mho \to \mho$ be a continuous mapping. Suppose that $\mathfrak{T}$ is a contraction with respect to a family of measures of noncompactness $\{\wp_i\}_{i\in\mathbb{N}}$. Then $\mathfrak{T}$ has at least one fixed point in the set $\mho$.*

REMARK 2. It is noteworthy that Theorem 1 is a corollary of more general result, namely Theorem 3.11 in [15].

3. Existence of mild solution

In this section, we will state and prove our main results. In order to obtain the existence of mild solutions for the problem (1). We suppose the following assumptions:

(H_1) The function $\aleph$ is Carathéodory on $\mathbb{R}_+ \times \mathfrak{I}$.

(H_2) There exists a continuous function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\aleph(\delta,\varpi)\| \leqslant p(\delta)(1+\|\varpi\|); \text{ for a.e. } \delta \in \mathbb{R}_+, \text{ and each } \varpi \in \mathfrak{I}.$$

(H_3) We have $I_j\left(\int_{\delta_j-\vartheta_j}^{\delta_j-\varkappa_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon\right)\leqslant L_j\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon.$

(H_4) There exist constants $L_{\widehat{\aleph}}, M_\delta$ such that

$$\|\widehat{\aleph}(\delta,u_1)-\widehat{\aleph}(\delta,u_2)\|\leqslant L_{\widehat{\aleph}}\|u_1-u_2\|$$

for a.e. $\delta\in\mathbb{R}_+$, and each $u_1,\ u_2\in\mathfrak{I}$, where

$$M_\delta=\sup_{\varepsilon\in[0,\delta]}\|\widehat{\aleph}(\varepsilon,0)\|.$$

(H_5) There exist constants $L_j; j=1,2,\dots,$ such that

$$\|I_j(v_1)-I_j(v_2)\|\leqslant L_j\|v_1-v_2\|;\ \text{and each } v_1,\ v_2\in\mathfrak{I}$$

(H_6) For each bounded and measurable set $\nabla\subset\mathfrak{I}$ we have

$$\wp(\aleph(\delta,\nabla))\leqslant p(\delta)\wp(\nabla);\ \text{for a.e. } \delta\in\mathbb{R}_+,$$

where $\wp$ is a measure of noncompactness on the Banach space $\mathfrak{I}$.

For $i\in\mathbb{N}$, let

$$p_i^*=\sup_{\delta\in[0,i]}p(\delta).$$

In this section, we denote

$$\beta(\varsigma,\gamma)=\int_0^1\delta^{\varsigma-1}(1-\delta)^{\gamma-1}d\delta$$

be the Beta function.

THEOREM 2. *Assume that hypotheses $(H_1)-(H_3)$ are satisfied, and*

$$\frac{Ci^\varsigma}{\varsigma}p_i^*+C^2\frac{i^{\varsigma+\gamma}}{\varsigma}p_i^*\beta(\varsigma+1,\gamma)+C_i^*i^{\varsigma+1}<1,$$

and

$$\ell_i:=\frac{Ci^\varsigma}{\varsigma}p_i^*+2C^2\frac{i^{\varsigma+\gamma}}{\varsigma}p_i^*\beta(\varsigma+1,\gamma)+2Ci^{\varsigma+1}L_{\widehat{\aleph}}<\frac{1}{4}.$$

Then, the problem (1) has a mild solution.

Proof. Let the operator $\Xi:\digamma\longrightarrow\digamma$ defined by:

$$\begin{aligned}(\Xi\chi)(\delta)&=\chi_0+\int_0^\delta\daleth(\delta-\eta,\eta)U(\eta)\mathscr{Z}(0)\chi_0d\eta+\int_0^\delta\daleth(\delta-\eta,\eta)\aleph(\eta,\chi(\eta))d\eta\\&+\int_0^\delta\int_0^\eta\daleth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon d\eta,\\&+\sum_{0<\delta_j<\delta}\daleth(\delta-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon\right),\quad\text{for each } \delta\in\mathbb{R}_+.\end{aligned}$$

For any $i \in \mathbb{N}$, let R_i be a positive real number with

$$R_i \geqslant \frac{\|\chi_0\| + C^2 i^{\varsigma}\left[\frac{1}{\varsigma} + i^{\gamma}\beta(\varsigma,\gamma+1)\right]\|\mathscr{Z}(0)\chi_0\| + \frac{Ci^{\varsigma}}{\varsigma}p_i^*}{1-\left[\frac{Ci^{\varsigma}}{\varsigma}p_i^* + C^2\frac{i^{\varsigma+\gamma}}{\varsigma}p_i^*\beta(\varsigma+1,\gamma) + C_i^* i^{\varsigma+1}\right]} + \frac{C^2\beta(\varsigma,\gamma)p_i^*\frac{i^{\varsigma+\gamma}}{\varsigma+\gamma} + C_i^* M_i i^{\varsigma+1}}{1-\left[\frac{Ci^{\varsigma}}{\varsigma}p_i^* + C^2\frac{i^{\varsigma+\gamma}}{\varsigma}p_i^*\beta(\varsigma+1,\gamma) + C_i^* i^{\varsigma+1}\right]},$$

and we consider the ball

$$\nabla_{R_i} := B(0,R_i) = \{\omega \in C(R_+) : \|\omega\|_i \leqslant R_i\}.$$

For any $i \in \mathbb{N}$, and each $\chi \in \nabla_{R_i}$ and $\delta \in [0,i]$ we have

$$\begin{aligned}
\|(\Xi\chi)(\delta)\| &\leqslant \|\chi_0\| + \left\|\int_0^{\delta} \daleth(\delta-\eta,\eta)U(\eta)\mathscr{Z}(0)\chi_0 d\eta\right\| \\
&\quad + \left\|\int_0^{\delta} \daleth(\delta-\eta,\eta)\aleph(\eta,\chi(\eta))d\eta\right\| \\
&\quad + \left\|\int_0^{\delta}\int_0^{\eta} \daleth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon d\eta\right\| \\
&\quad + \left\|\sum_{0<\delta_j<\delta} \daleth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon\right)\right\| \\
&\leqslant \|\chi_0\| + C^2\int_0^{\delta}(\delta_j-\eta)^{\varsigma-1}(1+\eta^{\gamma})d\eta\|\mathscr{Z}(0)\chi_0\| \\
&\quad + C\int_0^{\delta}(\delta-\eta)^{\varsigma-1}p(\eta)(1+\|\chi\|_i)d\eta \\
&\quad + C^2\int_0^{\delta}\int_0^{\eta}(\delta-\eta)^{\varsigma-1}(\eta-\varepsilon)^{\gamma-1}p(\varepsilon)(1+\|\chi\|_i)d\varepsilon d\eta \\
&\quad + \sum_{0<\delta_j<\delta} CL_j(\delta_j-\eta)^{\varsigma-1}\left\|\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon\right)\right\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|(\Xi\chi)(\delta)\| &\leqslant \|\chi_0\| + C^2 i^{\varsigma}\left[\frac{1}{\varsigma} + i^{\gamma}\beta(\varsigma,\gamma+1)\right]\|\mathscr{Z}(0)\chi_0\| + \frac{Ci^{\varsigma}}{\varsigma}p_i^*(1+R_i) \\
&\quad + C^2\beta(\varsigma,\gamma)p_i^*\frac{i^{\varsigma+\gamma}}{\varsigma+\gamma}(1+R_i) + \sum_{0<\delta_j<\delta} CL_{\widehat{\aleph}}L_j\delta^{\varsigma}(R_i+M_{\delta})2t \\
&\leqslant \|\chi_0\| + C^2 i^{\varsigma}\left[\frac{1}{\varsigma} + i^{\gamma}\beta(\varsigma,\gamma+1)\right]\|\mathscr{Z}(0)\chi_0\| + \frac{Ci^{\varsigma}}{\varsigma}p_i^*(1+R_i) \\
&\quad + C^2\beta(\varsigma,\gamma)p_i^*\frac{i^{\varsigma+\gamma}}{\varsigma+\gamma}(1+R_i) + \sum_{0<\delta_j<i} 2CL_{\widehat{\aleph}}L_j i^{\varsigma+1}(R_i+M_i)
\end{aligned}$$

$$\leqslant \|\chi_0\| + C^2 i^{\varsigma}\left[\frac{1}{\varsigma} + i^{\gamma}\beta(\varsigma,\gamma+1)\right]\|\mathscr{Z}(0)\chi_0\| + \frac{Ci^{\varsigma}}{\varsigma}p_i^*(1+R_i) + C^2\beta(\varsigma,\gamma)p_i^*\frac{i^{\varsigma+\gamma}}{\varsigma+\gamma}(1+R_i) + C_i^* i^{\varsigma+1}(R_i+M_i) \leqslant R_i.$$

Thus,

$$\|\Xi(\chi)\|_i \leqslant R_i.$$

This proves that Ξ transforms the ball ∇_{R_i} into itself. We shall show that the operator $\Xi : \nabla_{R_i} \to \nabla_{R_i}$ satisfies all the assumptions of Theorem 1. The proof will be given in several steps.

Step 1. $\Xi : \nabla_{R_i} \to \nabla_{R_i}$ *is continuous.*

Let $\{\chi_l\}_{l\in\mathbb{N}}$ be a sequence such that $\chi_j \to \chi$ in ∇_{R_i}. Then, for each $\delta \in [0,i]$, we have

$$\begin{aligned}
&\|(\Xi\chi_l)(\delta) - (\Xi\chi)(\delta)\| \\
&\quad\leqslant \Big\|\int_0^{\delta} \beth(\delta-\eta,\eta)[\aleph(\eta,\chi_l(\eta)) - \aleph(\eta,\chi(\eta))]d\eta\Big\| \\
&\qquad + \Big\|\int_0^{\delta}\int_0^{\eta} \beth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)[\aleph(\varepsilon,\chi_l(\varepsilon)) - \aleph(\varepsilon,\chi(\varepsilon))]d\varepsilon d\eta\Big\| \\
&\qquad + \left\|\sum_{0<\delta_j<\delta} \beth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\left[\widehat{\aleph}(\varepsilon,\chi_l(\varepsilon)) - \widehat{\aleph}(\varepsilon,\chi(\varepsilon))\right]d\varepsilon\right)\right\| \\
&\quad\leqslant C\int_0^{\delta}(\delta-\eta)^{\varsigma-1}\|\aleph(\eta,\chi_l(\eta)) - \aleph(\eta,\chi(\eta))\|d\eta \\
&\qquad + C\int_0^{\delta}\int_0^{\eta}(\delta-\eta)^{\varsigma-1}(\eta-\varepsilon)^{\gamma-1}\|\aleph(\varepsilon,\chi_l(\varepsilon)) - \aleph(\varepsilon,\chi(\varepsilon))\|d\varepsilon d\eta \\
&\qquad + 2tCL_{\widehat{\aleph}}\|\chi_l - \chi\|\sum_{0<\delta_j<\delta}(\delta_j-\eta)^{\varsigma-1}L_j.
\end{aligned}$$

Since $\chi_l \to \chi$ as $l \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|\Xi(\chi_l) - \Xi(\chi)\|_i \to 0 \quad \text{as } l \to \infty.$$

Step 2. $\Xi(\nabla_{R_i})$ *is bounded.*

Since $\Xi(\nabla_{R_i}) \subset \nabla_{R_i}$ and ∇_{R_i} is bounded, then $\Xi(\nabla_{R_i})$ is bounded.

Step 3. *For each equicontinuous subset* $\widehat{\nabla}$ *of* ∇_{R_i}, $\wp_i(\Xi(\widehat{\nabla})) \leqslant l_i\wp_i(\widehat{\nabla})$.

Since $\widehat{\nabla}$ is equicontinuous, we obtain that $\omega_0^i(\widehat{\nabla}) = 0$, from Lemmas 2 and 3, for any $\widehat{\nabla} \subset \nabla_{R_i}$ and any $\rho > 0$, there exists a sequence $\{\chi_l\}_{l=0}^{\infty} \subset \widehat{\nabla}$, such that for all

$\delta \in [0,i]$, we have

$$\begin{aligned}
\wp((\Xi\widehat{\nabla})(\delta)) &= \wp\Bigg(\Bigg\{\chi_0 + \int_0^\delta \daleth(\delta-\eta,\eta)U(\eta)\mathscr{Z}(0)\chi_0 d\eta \\
&\quad + \int_0^\delta \daleth(\delta-\eta,\eta)\aleph(\eta,\chi(\eta))d\eta \\
&\quad + \int_0^\delta\int_0^\eta \daleth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi(\varepsilon))d\varepsilon d\eta \\
&\quad + \sum_{0<\delta_j<\delta} \daleth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi(\varepsilon))d\varepsilon\right);\ \chi\in\widehat{\nabla}\Bigg\}\Bigg) \\
&\leqslant 2\wp\Bigg(\Bigg\{\chi_0 + \int_0^\delta \daleth(\delta-\eta,\eta)U(\eta)\mathscr{Z}(0)\chi_0 d\eta \\
&\quad + \int_0^\delta \daleth(\delta-\eta,\eta)\aleph(\eta,\chi_l(\eta))d\eta \\
&\quad + \int_0^\delta\int_0^\eta \daleth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi_l(\varepsilon))d\varepsilon d\eta \\
&\quad + \sum_{0<\delta_j<\delta} \daleth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi_l(\varepsilon))d\varepsilon\right)\Bigg\}_{l=1}^{\infty}\Bigg) + \rho.
\end{aligned}$$

Then,

$$\begin{aligned}
\wp((\Xi\widehat{\nabla})(\delta)) &\leqslant 2\Bigg(\wp\left\{\int_0^\delta \daleth(\delta-\eta,\eta)U(\eta)\mathscr{Z}(0)\chi_0 d\eta\right\}_{l=1}^{\infty} \\
&\quad + \wp\left\{\int_0^\delta \daleth(\delta-\eta,\eta)\aleph(\eta,\chi_l(\eta))d\eta\right\}_{l=1}^{\infty} \\
&\quad + \wp\left\{\int_0^\delta\int_0^\eta \daleth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi_l(\varepsilon))d\varepsilon d\eta\right\}_{l=1}^{\infty} \\
&\quad + \wp\left\{\sum_{0<\delta_j<\delta} \daleth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi_l(\varepsilon))d\varepsilon\right\}_{l=1}^{\infty}\right\} + \rho \\
&\leqslant 2\Bigg(\wp\left\{\int_0^\delta \daleth(\delta-\eta,\eta)\aleph(\eta,\chi_l(\eta))d\eta\right\}_{l=1}^{\infty} + \\
&\quad + \wp\left\{\int_0^\delta\int_0^\eta \daleth(\delta-\eta,\eta)\varphi(\eta,\varepsilon)\aleph(\varepsilon,\chi_l(\varepsilon))d\varepsilon d\eta\right\}_{l=1}^{\infty} \\
&\quad + \wp\left\{\sum_{0<\delta_j<\delta} \daleth(\delta_j-\eta,\eta)I_j\left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j}\widehat{\aleph}(\varepsilon,\chi_l(\varepsilon))d\varepsilon\right)\right\}_{l=1}^{\infty}\Bigg) + \rho.
\end{aligned}$$

Thus,

$$\begin{aligned}
\wp((\Xi\widehat{\nabla})(\delta)) \leqslant 4\Bigg(& Cp_i^* \wp_i(\widehat{\nabla}) \int_0^{\delta} (\delta-\eta)^{\varsigma-1} d\eta + \\
&+ 2C^2 p_i^* \wp_i(\widehat{\nabla}) \int_0^{\delta}\int_0^{\eta} (\delta-\eta)^{\varsigma-1}(\eta-\varepsilon)^{\gamma-1} d\varepsilon d\eta \\
&+ 2C \sum_{0<\delta_j<\delta} (\delta-\eta)^{\varsigma-1} I_j \left(\int_{\delta_j-\varkappa_j}^{\delta_j-\vartheta_j} L_{\widehat{\aleph}} \wp_i(\widehat{\nabla}) d\varepsilon\right)\Bigg) + \rho \\
\leqslant\ & 4\wp_i(\widehat{\nabla}) \left(Cp_i^* \frac{i^{\varsigma}}{\varsigma} + 2C^2 p_i^* \beta(\varsigma,\gamma) \int_0^{\delta} (\delta-\eta)^{\varsigma+\gamma-1} d\eta + 2Ci^{\varsigma+1} L_{\widehat{\aleph}} \right) \\
&+ \rho \\
\leqslant\ & 4\wp_i(\widehat{\nabla}) \left(Cp_i^* \frac{i^{\varsigma}}{\varsigma} + 2C^2 p_i^* \beta(\varsigma,\gamma) \frac{i^{\varsigma+\gamma}}{\varsigma+\gamma} + 2Ci^{\varsigma+1} L_{\widehat{\aleph}} \right) + \rho \\
\leqslant\ & 4\wp_i(\widehat{\nabla}) \left(Cp_i^* \frac{i^{\varsigma}}{\varsigma} + 2C^2 p_i^* \beta(\varsigma+1,\gamma) \frac{i^{\varsigma+\gamma}}{\varsigma} + 2Ci^{\varsigma+1} L_{\widehat{\aleph}} \right) + \rho \\
\leqslant\ & \ell_i \wp_i(\widehat{\nabla}) + \rho.
\end{aligned}$$

Since $\rho > 0$ is arbitrary, then

$$\wp((\Xi\widehat{\nabla})(\delta)) \leqslant \ell_i \wp_i(\widehat{\nabla}).$$

Thus,

$$\wp_i(\Xi(\widehat{\nabla})) \leqslant \ell_i \wp_i(\widehat{\nabla}).$$

As a consequence of steps 1 to 3 together with Theorem 1, we can conclude that Ξ has at least one fixed point in ∇_{R_i} which is a mild solution of problem (1). □

4. An example

We consider the following problem:

$$\begin{cases}
\dfrac{\partial^{\varsigma}\theta}{\partial\delta^{\varsigma}}(\delta,\chi) + \kappa(\theta,\delta)\dfrac{\partial^2\theta}{\partial\chi^2}(\delta,\chi) = Q(\delta,\theta(\delta,\chi)),;\ \delta\in\mathbb{R}_+\backslash\{\delta_1\},\ \chi\in[0,\pi], \\
\theta(\delta,0) = \theta(\delta,\pi) = 0; \qquad \delta\in\mathbb{R}_+, \\
\theta(0,\chi) = \Phi(\chi); \qquad \chi\in[0,\pi], \\
\mho^*\theta(\delta_1,\chi) = I_1 \displaystyle\int_{\delta_1-\varkappa_1}^{\delta_1-\vartheta_1} \widehat{\aleph}(\theta,\varepsilon,\chi(\theta,\varepsilon)) d\varepsilon,
\end{cases} \tag{8}$$

where $\dfrac{\partial^{\varsigma}}{\partial\delta^{\varsigma}}$ is the Caputo fractional partial derivative of order $0<\varsigma\leqslant 1$, the function $\kappa(\theta,\cdot)$ is continuous for $\theta\in[0,\pi]$ and $\kappa(\cdot,\delta)$ is uniformly Hölder continuous in $\delta\in$

$\mathbb{R}_+$, $\widehat{\aleph}$ is any Lipschitz continous function with Lipschitz constant $L_{\widehat{\aleph}}$. $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $\mathfrak{I} = L^2([0,\pi],\mathbb{R})$ and define $\mathscr{L}$ by $\mathscr{L}(\delta)w = -\kappa(\theta,\delta)w''$ with domain

$$\mathfrak{H}(\mathscr{L}) = H^2(0,\pi) \cap H_0^1(0,\pi)$$

Then $-\mathscr{L}(\varepsilon)$ generates an analytic semigroup $e^{-\delta\mathscr{L}(\varepsilon)}$ in $\mathfrak{I}$, which satisfies the assumptions (A_1) and (A_2).

For $\chi \in [0,\pi]$, we have

$$\widehat{\chi}(\delta)(\chi) = \theta(\delta,\chi); \quad \delta \in \mathbb{R}_+,$$

$$\aleph(\delta,\widehat{\chi}(\delta))(\chi) = Q(\delta,\theta(\delta,\chi)); \quad \delta \in \mathbb{R}_+,$$

$$\widehat{\chi}_0(\chi) = \Phi(\chi),$$

$$\mathscr{L}^{-1}(0) = (\kappa(\cdot,0))^{-1},$$

$$\widehat{\aleph}(\delta,\chi(\delta)) = \widehat{\aleph}(\cdot,\delta,\chi(\cdot,\delta)).$$

Then, the system (8) can be transformed into the abstract form (1), and conditions $(H_1)-(H_6)$ are satisfied. Consequently, Theorem 2 implies that the problem (8) has a mild solution.

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Fatima Mesri
Laboratory of Mathematics
Djillali Liabes University of Sidi
Bel-Abbes, P.O. Box 89, Sidi Bel-Abbes 22000, Algeria
e-mail: f_mesri@yahoo.fr

Abdelkrim Salim
Faculty of Technology
Hassiba Benbouali University of Chlef
P.O. Box 151 Chlef 02000, Algeria
e-mail: salim.abdelkrim@yahoo.com

Mouffak Benchohra
Laboratory of Mathematics
Djillali Liabes University of Sidi
Bel-Abbes, P.O. Box 89, Sidi Bel-Abbes 22000, Algeria
e-mail: benchohra@yahoo.com