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A STUDY OF A HIGH–ORDER TIME–FRACTIONAL PARTIAL DIFFERENTIAL EQUATION WITH PURELY INTEGRAL BOUNDARY CONDITIONS

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Abstract. The aim of this paper is to investigate the existence and uniqueness of the strong solution for the linear time-fractional partial differential equation with purely integral conditions. The aimed investigation is demonstrated based on the so-called energy inequality method and the density of the operator generated by the considered problem. To do so, we first set the position of the problem under consideration coupled with its corresponding equivalent problem, say problem (x). Afterward, we introduce some necessary functional spaces needed for exploring the existence and uniqueness of solution of problem (x). Finally, we investigate the existence and uniqueness of solution of the main operational equation.

1. Introduction

Fractional differential equations (FDEs) are the generalizations of classical differential equations with integer-orders [11]. In recent years, nonlinear FDEss are playing a major role in various fields such as physics, biology, engineering, signal processing, and control theory, finance, fractal dynamics and many other physical processes and diverse applications [27,24,18]. Recently, there has been a significant development in fractional partial differential equations (FPDEs), see the papers [16,23,20,17,10,25,26,30,22,21] and the references therein. Studying or finding approximate and exact solutions to the PFDEs is regarded a very important task. A large amount of literatures were developed concerning the solutions of the PFDEs in linear and nonlinear dynamics [5, 8, 7, 6, 28]. Many powerful and efficient methods have been proposed to obtain the numerical and exact solutions of PFDEs such as optimal homotopy analysis method [3], optimal homotopy asymptotic method [31], variational iteration method [4], Adomian decomposition method [9], and many others.

The theory of existence and uniqueness of solutions of the initial and boundary value problems for FDEs are extensively studied by many authors, see for example [10, 25, 26, 14] and references therein. In particular, the study of existence and uniqueness of solution of FPDEs are investigated by the well-known Lax-Milgram theorem and fixed point theorem, among them we only mention here the papers [21, 19].

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A suitable variational formulation is the starting point of many numerical methods such as finite element methods and spectral methods. The existence and uniqueness of the variational solution is thus essential for these methods to be efficient. The construction of the variational formulation strongly relies on the choice of suitable spaces and norms. Motivated by this, we extend and generalize the study of the problems of FPDEs with integral conditions, and expand the works in classical problems of FPDEs to non standard problems.

It has been demonstrated that many phenomena can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral conditions originating from various engineering disciplines, is of growing interest. That is, a large number of physical phenomena and many problems in modern physics and technology can be described in terms of nonlocal problems such as problems in PDEs with integral conditions. Such problems have been studied and received a great interest extensively by the author Bouziani in several works, among them [32, 13, 15, 12, 14, 2, 29].

2. Preliminaries

In this part, we review some important definitions and lemmas that are needed to explain the problem we shall study in this work. For instance, we recall next the Caputo derivative operator and the fractional integral operator for any positive non integer value $0 < \alpha < 1$ and $\Gamma(\cdot)$ is denoted the well-known gamma function.

DEFINITION 1. The left Caputo derivative operator can be expressed as

$$ {}_0^C\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(\tau)}{(t-\tau)^\alpha} d\tau. \tag{1}$$

DEFINITION 2. The fractional integral operator is defined by

$$ I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{2}$$

COROLLARY 1. *The composition of Caputo derivative operator and integral operator is given by*

$$ \int_0^t {}_0^C\partial_\tau^\alpha u d\tau = I_t^{1-\alpha} u - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} u(0). \tag{3}$$

LEMMA 1. (Poincare type inequality) *For all* $u \in L^2(0,l)$ *and* $m,n \in \mathbb{N}$ *such that* $m \leqslant n$, *we have*

$$ \int_0^1 (\Im_x^n u)^2 dx \leqslant \frac{1}{2^{n-m}} \int_0^1 (\Im_x^m u)^2 dx, \tag{4}$$

where

$$\Im_x^n u = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} u(\xi,t)\,d\xi,\ n \geqslant 1, \tag{5}$$

and

$$\Im_x^0 u = u(x,t). \tag{6}$$

LEMMA 2. *For $0<\alpha<1$ and for all continuous function u on $[0,T]$, we have*

$$({}_0^c\partial_t^\alpha u)\,u \geqslant \frac{1}{2}\,({}_0^c\partial_t^\alpha u)^2. \tag{7}$$

3. Position of the problem

In the rectangular domain $Q=(0,1)\times(0,T)$, with $T<\infty$, and for any positive non integer $0<\alpha<1$ and $m\in\mathbb{N}^*$, we consider the following FPDE:

$$\mathscr{L}v = {}_0^c\partial_t^\alpha v + (-1)^m \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial v}{\partial x}\right) = g(x,t) \tag{8}$$

subject to the initial condition

$$\ell v = v(x,0) = \varphi(x), \qquad x\in(0,1), \tag{9}$$

and the purely integral boundary conditions

$$\int_0^1 \xi^i v(\xi,t)\,d\xi = e_i(t), \qquad i=\overline{0,2m-1}, \tag{10}$$

where $a(x,t)$, $g(x,t)$, $\varphi(x)$, $e_i(t)$ are given functions that satisfy certain conditions which will be specified later and $\frac{\partial^{2m-1}a(x,t)}{\partial x^{2m-1}} \in L^2(Q)$ such that $\left(i=\overline{0,2m-1}\right)$. The function $\varphi(x)$ satisfies some compatibility conditions given as follows

$$\int_0^1 \xi^i \varphi(\xi)\,d\xi = e_i(0), \qquad i=\overline{0,2m-1}. \tag{11}$$

To deal with problem (8)–(10), we first transform it with its in-homogenous condition, to its corresponding equivalent problem with homogenous integral boundary conditions. To do so, we introduce a new unknown function $u(x,t)$ defined by

$$u(x,t) = v(x,t) - U(x,t), \tag{12}$$

where

$$U(x,t) = \sum_{j=0}^{2m-1} b_j(t)x^j, \tag{13}$$

and

$$A = \left(\frac{1}{1+i+j}\right)_{2m\times 2m}, \quad B = (b_j(t))_{2m\times 1}, \quad E = (e_i(t))_{2m\times 1},$$

for $0 \leqslant i,j \leqslant 2m-1$. Consequently, we can have

$$B = A^{-1}E.$$

Thus, problem (8)–(10) is therefore equivalent to

$$\mathscr{L}u = {}_0^c\partial_t^\alpha u + (-1)^m \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \tag{14}$$

subject to the initial condition

$$\ell v = u(x,0) = \psi(x), \tag{15}$$

and the purely integral boundary conditions

$$\int_0^1 \xi^i v(\xi,t)\,d\xi = 0, \qquad i = \overline{0,2m-1}, \tag{16}$$

where

$$f(x,t) = g(x,t) - \mathscr{L}U(x,t),$$

and

$$\psi(x) = \varphi(x) - \ell U.$$

From a compatibility conditions (11), we obtain

$$\int_0^1 \xi^i \psi(\xi)\,d\xi = e_i(0), \qquad i = \overline{0,2m-1}.$$

Hence, the solution of problem (8)–(10) will be obtained immediately by the equations (12)–(13).

4. Uniqueness of the solution of problem (14)–(16)

A priori estimate method is one efficient functional analysis technique, it is also called the energy-integral method by many researchers. This is an important technique for studying PDEs in general and with some purely integral conditions in particular. It has been successfully used when proving the existence, uniqueness, and continuous dependence of the solutions of many PDEs. It is essentially based on the construction of multipliers for each specific problem. In this way a priori estimate is provided, from which it is possible to establish the solvability of the problem. Herein, the proposed proof is based on the energy inequality and the density of the range of the operator generated by the abstract formulation of the problem. Therefore, for the purpose of initiating such proof, we need first to introduce some necessary functional spaces. This would help us later to prove the uniqueness of solution (if it exists), and to explore the existence of the solution of problem (14)–(16). So, let us now observing that problem (14)–(16) is equivalent to the following operational equation:

$$Lu = F,$$

where $L=(\mathscr{L},\ell)$ is considered as an operator from B to H such that B is a Banach space consisting of all function $u\in L^2(Q)$ with the finite norm

$$\|u\|_B^2=\sup_{0\leqslant\tau\leqslant T}\left(I_\tau^{1-\alpha}\left(\int_0^1(\Im_x^m u)^2\,dx\right)+\Im_\tau\left(\int_0^1(\Im_x u)^2\,dx\right)\right),$$

and H is the Hilbert space consisting of all elements $F=(f,\psi)$ with the finite norm

$$\|F\|_H^2=\int_Q f^2dxdt+\int_0^1\psi^2dx.$$

In addition, the domain of definition $D(L)$ is the set of all functions $u\in L^2(Q)$ for which ${}_0^c\partial_t^\alpha u,\ \frac{\partial^{2m}u}{\partial x^{2m}}\in L^2(Q)$ and satisfying integral conditions (16).

In light of the aforesaid discussion, we set the next definition that defines the strong solution of problem (14)–(16). Then, we state and prove an important result that confirms there is a priori estimate that can be satisfied by the solution of problem (14)–(16), followed by a further result that asserts the uniqueness of this solution.

DEFINITION 3. Suppose that the operator L from B into H has a closure $\overline{L}$. Then the solution of the operational equation

$$\overline{L}u=F$$

is called a strong solution of problem (14)–(16).

THEOREM 1. *Suppose that the function $a(x,t)$ satisfies the condition*

$$2\inf_Q a-\frac{1}{2}\sup_Q\frac{\partial a}{\partial x}\geqslant a_0>0, \tag{17}$$

then the solution of problem (14)–(16) satisfies a priori estimate

$$\|u\|_B\leqslant K\|F\|_H,\quad \forall u\in D(L), \tag{18}$$

where K is a positive constant independent of u.

Proof. To prove this result, we first take the scalar product in $L^2(0,1)$ of equation (14) and the integro-differential operator $Mu=(-1)^m\Im_x^{2m}u$. As a consequence, we can have

$$\langle\mathscr{L}u,Mu\rangle_{L^2(0,1)}=\langle f,Mu\rangle_{L^2(0,1)}.$$

Therefore, we obtain

$$(-1)^m\int_0^1({}_0^c\partial_t^\alpha u)\,\Im_x^{2m}udx+\int_0^1\frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m}udx$$
$$=(-1)^m\int_0^1 f\,\Im_x^{2m}udx. \tag{19}$$

Now, integrating by parts the left hand side of equation (19) and using integral conditions (16) give

$$\begin{aligned}(-1)^m \int_0^1 ({}_0^c\partial_t^\alpha u)\,\Im_x^{2m} u dx &= (-1)^{m+1} \int_0^1 ({}_0^c\partial_t^\alpha \Im_x u)\,\Im_x^{2m-1} u dx \\ &= (-1)^{m+2} \int_0^1 \left({}_0^c\partial_t^\alpha \Im_x^2 u\right)\Im_x^{2m-2} u dx \\ &\quad \dots \\ &= (-1)^{2m} \int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx \\ &= \int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx.\end{aligned}$$

Consequently, we have

$$(-1)^m \int_0^1 ({}_0^c\partial_t^\alpha u)\,\Im_x^{2m} u dx = \int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx, \tag{20}$$

and

$$\begin{aligned}\int_0^1 \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m} u dx &= -\int_0^1 \frac{\partial^{2m-2}}{\partial x^{2m-2}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m-1} u dx \\ &= (-1)^2 \int_0^1 \frac{\partial^{2m-3}}{\partial x^{2m-3}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m-2} u dx \\ &\quad \dots \\ &= (-1)^{2m-1} \int_0^1 a(x,t)\frac{\partial u}{\partial x}\Im_x u dx \\ &= \int_0^1 u\left(\frac{\partial a}{\partial x}\Im_x u + a u\right) dx \\ &= \int_0^1 \left(-\frac{1}{2}\frac{\partial a}{\partial x}(\Im_x u)^2 + a u^2\right) dx.\end{aligned}$$

As a result, we obtain

$$\int_0^1 \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m} u dx = \int_0^1 \left(-\frac{1}{2}\frac{\partial a}{\partial x}(\Im_x u)^2 + a u^2\right) dx. \tag{21}$$

Now, substituting equations (20) and (21) in equation (19) yields

$$\begin{aligned}\int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx + \int_0^1 \left(-\frac{1}{2}\frac{\partial a}{\partial x}(\Im_x u)^2\right) dx + \int_0^1 a u^2 dx \\ = (-1)^m \int_0^1 f \Im_x^{2m} u dx.\end{aligned} \tag{22}$$

With the use of Lemma 1, the third term of left hand side of equation (22) might be estimated as

$$\int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\Im_x^m u dx + \left(2\inf_Q a - \frac{1}{2}\sup_Q \frac{\partial a}{\partial x}\right)\int_0^1 (\Im_x u)^2 dx \leqslant (-1)^m \int_0^1 f \Im_x^{2m} u dx., \tag{23}$$

Also, by Lemma 2 and condition (17), we deduce

$$\frac{1}{2}\int_0^1 {}_0^c\partial_t^\alpha (\Im_x^m u)^2 dx + a_0 \int_0^1 u^2 dx \leqslant (-1)^m \int_0^1 f.\Im_x^{2m} u dx. \tag{24}$$

Herein, by using Cauchy inequality coupled with Lemma 1, the right hand side of equation (24) will be consequently in the form

$$\left|(-1)^m \int_0^1 f.\Im_x^{2m} u dx\right| \leqslant \frac{1}{2\varepsilon}\int_0^1 f^2 dx + \frac{\varepsilon}{2^{2m}}\int_0^1 (\Im_x u)^2 dx. \tag{25}$$

Again, substituting inequality (25) in estimate (24) implies

$$\frac{1}{2}\int_0^1 {}_0^c\partial_t^\alpha (\Im_x^m u)^2 dx + \left(a_0 - \frac{\varepsilon}{2^{2m}}\right)\int_0^1 (\Im_x u)^2 dx \leqslant \frac{1}{2\varepsilon}\int_0^1 f^2 dx. \tag{26}$$

As a result, by integrating inequality (26) over $(0,\tau)$ and using Corollary 1, we obtain

$$\frac{1}{2} I_\tau^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2 dx\right) + \left(a_0 - \frac{\varepsilon}{2^{2m}}\right)\Im_\tau\left(\int_0^1 (\Im_x u)^2 dx\right) \tag{27}$$

$$\leqslant \frac{1}{2\varepsilon}\int_0^\tau \int_0^1 f^2 dxdt + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^1 \psi^2 dx. \tag{28}$$

Therefore, we have

$$\frac{1}{2} I_\tau^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2 dx\right) + \left(a_0 - \frac{\varepsilon}{2^{2m}}\right)\Im_\tau\left(\int_0^1 (\Im_x u)^2 dx\right) \tag{29}$$

$$\leqslant \frac{1}{2\varepsilon}\int_0^T \int_0^1 f^2 dxdt + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^1 \psi^2 dx. \tag{30}$$

Now, we may choose ε such that $a_0 - \frac{\varepsilon}{2^{2m}} \geqslant 0$. This would give

$$\frac{1}{2} I_\tau^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2 dx\right) + \Im_\tau\left(\int_0^1 (\Im_x u)^2 dx\right) \leqslant K^2\left(\int_Q f^2 dxdt + \int_0^1 \psi^2 dx\right), \tag{31}$$

where

$$K^2 = \left(\frac{\max\left(\frac{1}{2\varepsilon}, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)}{\min\left(\frac{1}{2}, a_0 - \frac{\varepsilon}{2^{2m}}\right)}\right).$$

Thus, we consequently get the following inequality:

$$\sup\left(I_\tau^{1-\alpha}\left(\int_Q(\Im_x^m u)^2\,dx\right)\right)+\Im_\tau\left(\int_0^1(\Im_x u)^2\,dx\right) \leqslant K^2\left(\int_Q f^2 dxdt+\int_0^1\psi^2 dx\right),$$

where

$$\|u\|_B \leqslant K\,\|Lu\|_H\,. \quad \square \tag{32}$$

In view of the above discussion and due to $\overline{L}$ is the closure of L, we can extend inequality (18) as follows

$$\|u\|_B \leqslant K\left\|\overline{L}u\right\|_H, \tag{33}$$

for all $u\in D(\overline{L})$. Hence, inequality (33) leads to the following corollaries.

COROLLARY 2. *A strong solution of* (14)–(16) *is unique if it exists and depends continuously on* $F=(f,\psi)$.

COROLLARY 3. *The rang of* L *is closed in* H *and* $R(L)=\overline{R(L)}$.

5. Existence of the solution of problem (14)–(16)

Throughout the previous section, we have proved the uniqueness of solution of problem (14)–(16), if it exists. However, we have not demonstrated the existence of such a solution yet. To do so, we will just prove that $R(L)$ is dense in H. This would be confirmed by the following result.

THEOREM 2. *Suppose that the conditions of Theorem 1 are satisfied. Assume that for* $\omega\in L^2(Q)$ *and* $u\in D_0(L)=\{u\in D(L), lu=0\}$, *we have*

$$\int_Q \mathscr{L}u\omega\,dxdt=0, \tag{34}$$

then ω *vanishes almost everywhere in* Q.

Proof. We can rewrite equation (34) as follows:

$$\int_Q({}_0^c\partial_t^\alpha u)\,\omega dxdt+(-1)^m\int_Q\frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\omega dxdt=0. \tag{35}$$

Also, we can express ω in terms of a function u as

$$\omega=\Im_x^{2m}u. \tag{36}$$

As a consequence, Substituting ω in (35) gives

$$\int_Q({}_0^c\partial_t^\alpha u)\,\Im_x^{2m}udxdt+(-1)^m\int_Q\frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m}udxdt=0. \tag{37}$$

By integrating by parts all terms of the left hand side of equation (37) over $(0,1)$ and by using the integral conditions (16), we get

$$\int_Q ({}_0^c\partial_t^\alpha u)\,\Im_x^{2m} u dxdt = (-1)^m \int_Q ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dxdt, \tag{38}$$

and

$$(-1)^m \int_Q \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m} u dxdt$$
$$= (-1)^m \int_Q \left(-\frac{1}{2}\frac{\partial a}{\partial x}(\Im_x u)^2 + au^2\right) dxdt. \tag{39}$$

Consequently, substituting (38) and (39) into (37) implies

$$\int_Q ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dxdt + \int_Q au^2 dxdt - \frac{1}{2}\int_Q \frac{\partial a}{\partial x}(\Im_x u)^2 dxdt = 0. \tag{40}$$

On the other hand, by using Lemma 1, we can have

$$\int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx + \int_0^1 au^2 dx - \frac{1}{2}\int_0^1 \frac{\partial a}{\partial x}(\Im_x u)^2 dx$$
$$\geqslant \int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx + \left(2\inf_Q a - \frac{1}{2}\sup_Q \frac{\partial a}{\partial x}\right)\int_0^1 (\Im_x u)^2 dx. \tag{41}$$

After we use conditions (17) and Lemma 2, we obtain

$$\int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dx + \int_0^1 au^2 dx - \frac{1}{2}\int_0^1 \frac{\partial a}{\partial x}(\Im_x u)^2 dx \tag{42}$$
$$\geqslant \frac{1}{2}\int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)^2 dx + a_0\int_0^1 (\Im_x u)^2 dx. \tag{43}$$

Thus, integrating inequality (42) over $(0,\tau)$ and using Corollary 1 yields

$$\int_0^\tau\int_0^1 ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dxdt + \int_0^\tau\int_0^1 au^2 dxdt - \frac{1}{2}\int_0^\tau\int_0^1 \frac{\partial a}{\partial x}(\Im_x u)^2 dxdt$$
$$\geqslant \frac{1}{2} I_\tau^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2 dx\right) + a_0\int_0^\tau\int_0^1 (\Im_x u)^2 dxdt.$$

Consequently, by replacing τ by T, we conclude

$$\int_Q ({}_0^c\partial_t^\alpha \Im_x^m u)\,\Im_x^m u dxdt + \int_Q au^2 dxdt - \frac{1}{2}\int_Q \frac{\partial a}{\partial x}(\Im_x u)^2 dxdt$$
$$\geqslant \frac{1}{2} I_T^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2 dx\right) + a_0\int_Q (\Im_x u)^2 dxdt.$$

Now, due to

$$\int_Q \left({}_0^c\partial_t^\alpha \Im_x^m u\right) \Im_x^m u dxdt + \int_Q a u^2 dxdt - \frac{1}{2}\int_Q \frac{\partial a}{\partial x} (\Im_x u)^2 dx = 0$$

from where

$$\frac{1}{2} I_T^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2 dx\right) + a_0 \int_Q (\Im_x u)^2 dxdt = 0,$$

we can immediately have

$$(\Im_x u)^2 = 0.$$

Therefore $u = 0$, and hence $\omega \equiv 0$ in $L^2(Q)$. This consequently yields the existence of the solution of problem (14)–(16). □

6. Conclusion

In this paper, we have investigated the existence and uniqueness of the strong solution for the linear time-fractional partial differential equation with purely integral conditions. This has been successfully performed with the help of using of the energy inequality method and the density of the operator generated by the considered problem.

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