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ON GENERALIZED CAPUTO'S FRACTIONAL ORDER FUZZY ANTI PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. In this article, we consider an anti periodic fuzzy boundary value problem with order α, where $1<\alpha<2$, under newly defined generalized Caputo's fractional derivative (called as OBC) and study the existence and uniqueness of the solution for the considered problem via fixed point technique. Also, we illustrate the results with some examples involving the developed numerical technique based on fractional Euler's method of integration.

1. Introduction

Due to its various applications in many fields, fractional calculus has a famous analogy with an adjustable wrench for its varying order. Fractional calculus has grown so wide that there are very few areas of science and engineering which are untouched by it. For a brief history of the subject, [37] is referred. The standard literature related to the topic can be found in [27, 31, 33, 35]. The motivation to study fractional calculus comes from, of course, its application part. Along with the wide areas of application like viscoelasticity, dynamics of turbulence, electro-chemistry, control and many more, some specific problems which were encountered using fractional calculus are in Schrödinger's equation [28], wave equation [42], study of damage and fatigue [18], heart valve vibrations [21], public economics [17] and more, which shows the diverseness of the applications of the fractional calculus. Recently a modified definition of a generalized Caputo derivative is introduced in a paper [32] by Z. Odibat and D. Baleanu in 2020 which we are going to call the OBC fractional derivative from now on.

Human involvement is necessary, to some extent, for making and studying a model describing a physical phenomenon (e.g. to collect data for the model). This makes the models inevitably uncertain. Fuzzy sets are perfect tools to handle this kind of uncertainty. L. A. Zadeh made it all possible by introducing the concept of fuzzy sets in his paper [40] in 1965. A healthy literature for fuzzy sets can be found in the book [41] by H. J. Zimmermann. For the literature of fuzzy differential equations one can look into the book [26] by V. Lakshmikantham and R. N. Mohapatra. Further readings can be done in the paper [27] and the references therein. A few papers [11, 23] published

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in 2005 and 2009 respectively have the existence and uniqueness of fuzzy differential equation of second order.

It seemed that these two topics were destined to be merged and the year was 2010 when this happened. The credit goes to Agarwal et al. [5]. Since the birth of fuzzy fractional differential equations in 2010 it has grown with a significant amount of speed which is summarized in a literature survey [3] by Agarwal et al. A book containing a very broad survey of the topic is the book [9] by T. Allahviranloo in 2021. For very new trends in this area one can look into these very recent papers [8, 10, 14, 20, 24, 38].

In 2008, following fuzzy boundary value problem was discussed in [19] by M. Chen et al.

$$\begin{aligned}&\mathscr{X}'' = \mathsf{f}(t,\mathscr{X},\mathscr{X}'),\ \ t\in[a,b]\\&\mathscr{X}(a) = A,\ \mathscr{X}(b) = B.\end{aligned}$$

In this article, an equivalent integral equation was constructed and an existence result was proved using Schauder fixed point theorem.

In 2009, Alsaedi [12] studied the following integro-differential equation,

$$\begin{aligned}&{}^{C}_{0}D^{\alpha}_{t}\mathscr{X}(t) = \mathsf{f}\left(t,\mathscr{X}(t),\int_0^t \gamma(t,s)\mathscr{X}(s)ds\right),\ \ t\in[0,T],\ \ 1<\alpha<2\\&\mathscr{X}(0) = -\mathscr{X}(T),\ \mathscr{X}'(0) = -\mathscr{X}'(T).\end{aligned}$$

He employed two fixed point theorems to prove the existence results for the problem under consideration.

In 2010, Ahmad and Nieto [6] independently studied following fractional order anti periodic boundary value problem,

$$\begin{aligned}&{}^{C}_{0}D^{\alpha}_{t}\mathscr{X}(t) = \mathsf{f}(t,\mathscr{X}(t)),\ \ t\in[0,T],\ \ 1<\alpha<2\\&\mathscr{X}(0) = -\mathscr{X}(T),\ \mathscr{X}'(0) = -\mathscr{X}'(T).\end{aligned}$$

They exploited Leray-Schauder degree theory to prove the existence of solution for thr above anti-periodic boundary value problem.

In 2017, the literature for fractional anti periodic boundary value problems got so rich that a survey was presented by Agarwal et. al. [1]. They also studied the following problem,

$$\begin{aligned}&{}^{C}_{0}D^{\alpha}_{t}\mathscr{X}(t) = \mathsf{f}(t,\mathscr{X}(t)),\ \ t\in[0,T],\ \ 5<\alpha<6\\&\mathscr{X}(0) = -\mathscr{X}(T),\ \mathscr{X}'(0) = -\mathscr{X}'(T)\\&\mathscr{X}''(0) = -\mathscr{X}''(T),\ \mathscr{X}'''(0) = -\mathscr{X}'''(T)\\&\mathscr{X}^{(4)}(0) = -\mathscr{X}^{(4)}(T),\ \mathscr{X}^{(5)}(0) = -\mathscr{X}^{(5)}(T).\end{aligned}$$

They proved the existence results of the considered problem using four different fixed point theorems.

For more works on the fractional anti periodic BVPs one is referred to [2, 4, 7, 15, 16, 22, 30, 34, 39] and references therein.

Now, we turn our attention to the newly defined OBC fractional derivative [32] in order to prove some existence and uniqueness results in fuzzy sense. The OBC fractional derivative is somewhat similar to the Caputo-Katugampola (CK) [25] fractional derivative, indeed it is the same for $0 < \alpha < 1$, but for $\alpha > 1$ it is different and has a little computational advantage over the CK fractional derivative. We refer a work [36] on OBC-fractional stochastic differential equations for some insights.

There does not seem to exist any work on the fuzzy fractional anti periodic boundary value problem involving OBC-fractional derivative in the literature. The motivation for us to study this OBC-fractional derivative is its efficiency which makes it useful in computational aspects. We consider the following class of fuzzy anti periodic boundary value problem,

$$\begin{aligned} &{}^{OBC}_{t_0}D^{\alpha,\rho}_t \mathscr{X}(t) = \mathsf{f}(t, \mathscr{X}(t)), \ t \in [t_0, T], \\ &\mathscr{X}(t_0) = -1 \odot \mathscr{X}(T), \ \mathscr{X}'(t_0) = -1 \odot \mathscr{X}'(T), \\ &\mathscr{X}(t_0), \mathscr{X}(T), \mathscr{X}'(t_0), \mathscr{X}'(T) \in \mathbb{F}_{\mathbb{R}}, \end{aligned} \tag{1}$$

where, ${}^{OBC}D$ is the OBC fractional derivative, t_0, t are terminals of the derivative with $t \in [t_0, T]$, the order $1 < \alpha < 2$, ρ is a parameter with $\rho > 0$, $\mathscr{X}$ is a fuzzy number valued function defined on $[t_0, T]$ and $\mathsf{f} : [t_0, T] \times \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$, where $\mathbb{F}_{\mathbb{R}}$ is the set of all fuzzy numbers with universe $\mathbb{R}$.

We have the following novelty and point wise highlights with of our manuscript.

(i) We have introduced new generalized Caputo's fractional derivative and integral based on the concept of fuzzy theory.

(ii) Introduced OBC generalized Caputo's derivative is the single base point derivative having physical meaning.

(iii) We have derived a corresponding solution as integral equations of fractional fuzzy anti periodic boundary value problems.

(iv) We have proved existence and uniqueness of solutions for a class of fractional fuzzy anti periodic boundary value problems of order $1 < \alpha < 2$.

(v) We considered some application of a class of fractional fuzzy anti periodic boundary value problems.

(vi) We verified analytical solution and numerical solution via fixed point and Euler's method of integration respectively.

The rest of the paper is organized in the following sections, Section 2 has some useful definitions which we require to obtain our results. Section 3 discusses the motivation of the present work along with a needed detailed background. Section 4 consists our main results which is divided into three parts and hence making up three theorems. Section 5 is the application section which is devoted to three examples containing a numerically analyzed example and the paper is concluded in Section 6.

2. Preliminary supplements

In an attempt to make this paper self-sufficient, we provide the following definitions and some well known results. The sources of the contents in this section are listed in cited books and papers.

DEFINITION 1. [9, 14] A fuzzy set P is called a fuzzy number if its membership function $\mu_P : \mathbb{R} \to [0,1]$ satisfies the following properties:

1. P is normal. i.e, there exists a real member y_0 such that $\mu_P(y_0) = 1$.

2. μ_P is fuzzy convex. i.e,
   for two arbitrary real numbers y_1, y_2 and $\lambda \in [0,1]$ we have,
$$\mu_P(\lambda y_1 + (1-\lambda)y_2) \geqslant Min\{\mu_P(y_1), \mu_P(y_2)\}.$$

3. μ_P is upper semi-continuous.

4. The closure of $Supp(P) = \{y \in \mathbb{R} : \mu_P(y) > 0\}$ is compact.

DEFINITION 2. [9, 14] Any $P \in \mathbb{F}_{\mathbb{R}}$ has the parametric form $r_P = [P_l(r), P_u(r)]$ for any $0 \leqslant r \leqslant 1$, if and only if,

1. $P_l(r) \leqslant P_u(r)$.

2. $P_l(r)$ is an increasing and left continuous function on $[0,1]$ and right continuous on 0 with respect to r.

3. $P_u(r)$ is a decreasing and left continuous function on $[0,1]$ and right continuous on 0 with respect to r.

4. $r_P = [P_l(r), P_u(r)]$ is a compact interval for any $0 \leqslant r \leqslant 1$.

DEFINITION 3. [9, 14] Let $P, Q \in \mathbb{F}_{\mathbb{R}}$ in level-wise form. The generalized Hukuhara difference of P and Q is defined as

$$P \ominus_g Q = R \Leftrightarrow \begin{cases} (i) \;\; P = Q \oplus R \\ \quad\quad \text{or} \\ (ii) \;\; Q = P \oplus (-1)R. \end{cases}$$

DEFINITION 4. [9, 14] The Hausdorff distance $D_H : \mathbb{F}_{\mathbb{R}} \times \mathbb{F}_{\mathbb{R}} \to \mathbb{R}$, between $P, Q \in \mathbb{F}_{\mathbb{R}}$ is given by

$$D_H(P,Q) = \sup_{r \in [0,1]} \max\{|P_l(r) - Q_l(r)|, |P_u(r) - Q_u(r)|\}.$$

DEFINITION 5. [9] The generalized Hukuhara derivative of a fuzzy number valued function $f:[0,T]\to\mathbb{F}_{\mathbb{R}}$ at $t_0\in[0,T]$ is defined as

$$f'(t_0)=\lim_{h\to 0}\frac{f(t_0+h)\ominus_g f(t_0)}{h},$$

provided that the difference $f(t_0+h)\ominus_g f(t_0)$ and the limit exists then the function f is called gH-differentaible. The level-wise form of gH-differentaible function f can be defined in following two cases:

Case 1: $f'(t,r)=[f_l'(t,r),f_u'(t,r)]$, if f is $i-gH$ differentiable at t.
Case 2: $f'(t,r)=[f_u'(t,r),f_l'(t,r)]$, if f is $ii-gH$ differentiable at t.

DEFINITION 6. [9] The Riemann-Liouville fuzzy fractional integration of $\mathscr{X}\in L_{1,loc}([t_0,t],\mathbb{F}_{\mathbb{R}})$ of order $\alpha>0$ is defined as

$${}_{t_0}J_t^{\alpha}\mathscr{X}(t)=\frac{1}{\Gamma(\alpha)}\odot\int_{t_0}^{t}(t-\tau)^{\alpha-1}\odot\mathscr{X}(\tau)d\tau.$$

DEFINITION 7. [9, 14] The fuzzy fractional Caputo derivative of order $m-1<\alpha<m$ of a fuzzy number valued function $\mathscr{X}$, such that $\mathscr{X}$ is m- times gH differentiable and $\mathscr{X}^{(m)}\in L_{1,loc}([t_0,t],\mathbb{F}_{\mathbb{R}})$, is given by:

$${}_{t_0}^{C}D_t^{\alpha}\mathscr{X}(t)=\frac{1}{\Gamma(m-\alpha)}\odot\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}\odot\mathscr{X}^{(m)}(\tau)d\tau.$$

THEOREM 1. (Contraction Theorem) [29] *Consider a complete metric space* $\mathcal{X}=(\mathcal{X},d)$, *where* $\mathcal{X}\neq\Phi$. *Let* $\mathfrak{T}:\mathcal{X}\to\mathcal{X}$ *be a contraction on* $\mathcal{X}$. *Then* $\mathfrak{T}$ *has precisely one fixed point.*

THEOREM 2. (Schauder Fixed Point Theorem) *Let* D *be a non-empty closed and bounded subset of a Banach space* $\mathcal{X}$. *Let* $\mathfrak{T}:D\to D$ *be a completely continuous mapping, then* $\mathfrak{T}$ *has a fixed point.*

3. New OBC fractional derivative

In 2011, Katugampola [25] defined a new fractional order integral as a generalization of usual fractional integral by updating the kernel of the transformation. He added a new parameter $\rho>0$. For this, the generalized fractional integral ${}_{t_0}^{K}J_t^{\alpha}$ of $f\in L_{1,loc}([t_0,t],\mathbb{R})$ of order $\alpha>0$ is defined as:

$${}_{t_0}^{K}J_t^{\alpha,\rho}f(t)=[{}_{t_0^{\rho}/\rho}J_t^{\alpha}(fog)](t^{\rho}/\rho)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1}f(\tau)d\tau, \tag{2}$$

where, $g(t)=(\rho t)^{1/\rho}$. Using this, the Riemann-Katugampola (RK) derivative is defined as,

$$\begin{aligned}{}_{t_0}^{RK}D_t^{\alpha,\rho}f(t)&=[D^m\,{}_{t_0^{\rho}/\rho}J_t^{m-\alpha}(fog)](t^{\rho}/\rho)\\&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^m\int_{t_0}^{t}\tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{m-\alpha-1}f(\tau)d\tau.\end{aligned}\tag{3}$$

Then, motivated by the above definition and the following identity for usual Caputo fractional derivative,

$$ {}_{t_0}J_t^{\alpha}\, {}^{C}_{t_0}D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k, \tag{4}$$

the Caputo-Katugampola (CK) derivative is defined as,

$$ {}^{CK}_{t_0}D_t^{\alpha,\rho} f(t) = {}^{RK}_{t_0} D_t^{\alpha,\rho} \left(f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k \right). \tag{5}$$

But this derivative has a drawback, unlike the usual Caputo derivative, it doesn't satisfy the generalized form of the identity (4) which makes the calculation of the fractional derivatives easy. In an attempt to remove this drawback, Odibat-Baleanu-Caputo (OBC) derivative was defined in [32] as follows,

$$ \begin{aligned} {}^{OBC}_{t_0}D_t^{\alpha,\rho} f(t) &= [{}_{t_0^{\rho}/\rho}J_t^{m-\alpha} D^m(fog)](t^{\rho}/\rho) \\ &= \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_{t_0}^{t} \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{m-\alpha-1} \left(\tau^{1-\rho}\frac{d}{d\tau}\right)^m f(\tau)d\tau, \end{aligned} \tag{6}$$

this derivative satisfies the following generalization of the identity (4),

$$ {}^{K}_{t_0}J_t^{\alpha,\rho}\, {}^{OBC}_{t_0}D_t^{\alpha,\rho} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{(t^{\rho}-t_0^{\rho})^k}{\rho^k k!} \left[\left(t^{1-\rho}\frac{d}{dt}\right)^k f(t) \right]_{t=t_0}. \tag{7}$$

Now, we are going to define the new representation of generalized fractional integral and OBC fractional derivative in fuzzy sense, in the following manner,

DEFINITION 8. The generalized fractional integral of order $\alpha > 0$ of a fuzzy number valued function $\mathscr{X}$, such that $\mathscr{X} \in L_{1,loc}([t_0,t],\mathbb{F}_{\mathbb{R}})$, is given by:

$$ {}^{K}_{t_0}J_t^{\alpha,\rho} \mathscr{X}(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \odot \int_{t_0}^{t} \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} \odot \mathscr{X}(\tau)d\tau.$$

DEFINITION 9. The fuzzy OBC fractional derivative of order $m-1 < \alpha < m$ of a fuzzy number valued function $\mathscr{X}$, such that $\mathscr{X}$ is m- times gH differentiable and $\mathscr{X}^{(m)} \in L_{1,loc}([t_0,t],\mathbb{F}_{\mathbb{R}})$, is given by:

$$ {}^{OBC}_{t_0}D_t^{\alpha,\rho} \mathscr{X}(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \odot \int_{t_0}^{t} \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{m-\alpha-1} \left(\tau^{1-\rho}\frac{d}{d\tau}\right)^m \odot \mathscr{X}(\tau)d\tau,$$

where gH-derivative of $\mathscr{X}^m$ is defined as

$$ \mathscr{X}^m(t) = \lim_{h\to 0} \frac{\mathscr{X}^{m-1}(t_0+h) \ominus_g \mathscr{X}^{m-1}(t_0)}{h}.$$

4. Existence and uniqueness results for the solution

In this section, we shall prove the existence and uniqueness of solution for problem (1). To establish our results, we first give the following lemma.

LEMMA 1. *The fractional counterpart corresponding to BVP (1) is equivalent to the following integral equation*

$$\begin{aligned}\mathscr{X}(t) =&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau\\&-\frac{\rho^{1-\alpha}}{2\Gamma(\alpha)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau\\&+\frac{t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau.\end{aligned}\tag{8}$$

Proof. Since we are looking at the fractional counterpart of the BVP (1), we have

$$\begin{aligned}&{}_{t_0}^{OBC}D_t^{\alpha,\rho}\mathscr{X}(t)=\mathsf{f}(t,\mathscr{X}(t)),\ t\in[t_0,T],\\&\mathscr{X}(t_0)=-\mathscr{X}(T),\ \mathscr{X}'(t_0)=-\mathscr{X}'(T),\ \ \mathscr{X}(t_0),\mathscr{X}(T),\mathscr{X}'(t_0),\mathscr{X}'(T)\in\mathbb{R}.\end{aligned}\tag{9}$$

Using equations (2) and (7) and applying the generalized fractional integral both the sides of (9), we get for $m=2$,

$$\begin{aligned}&\mathscr{X}(t)-\mathscr{X}(t_0)-\frac{(t^\rho-t_0^\rho)}{\rho}\left[t^{1-\rho}\frac{d}{dt}\mathscr{X}(t)\right]_{t=t_0}\\&=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau,\end{aligned}$$

which gives,

$$\mathscr{X}(t)=\mathscr{X}(t_0)+\frac{(t^\rho-t_0^\rho)}{\rho}\left[t_0^{1-\rho}\mathscr{X}'(t_0)\right]+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau.\tag{10}$$

Now, using the boundary condition $\mathscr{X}(t_0)=-\mathscr{X}(T)$ and equation (10), we get,

$$\begin{aligned}\mathscr{X}(t_0)=&-\mathscr{X}(t_0)-\frac{(T^\rho-t_0^\rho)}{\rho}\left[t_0^{1-\rho}\mathscr{X}'(t_0)\right]\\&-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau,\end{aligned}$$

which implies,

$$2\mathscr{X}(t_0)+\frac{(T^\rho-t_0^\rho)}{\rho}\left[t_0^{1-\rho}\mathscr{X}'(t_0)\right]=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau.\tag{11}$$

Also, (10) gives,

$$\mathscr{X}'(t) = t^{\rho-1}t_0^{1-\rho}\mathscr{X}'(t_0) + \frac{t^{\rho-1}\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_0}^{t}\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau. \tag{12}$$

Using (12) and the remaining boundary condition $\mathscr{X}'(t_0) = -\mathscr{X}'(T)$ gives,

$$\mathscr{X}'(t_0) = -T^{\rho-1}t_0^{1-\rho}\mathscr{X}'(t_0) - \frac{T^{\rho-1}\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau,$$

which gives,

$$\mathscr{X}'(t_0) = -\frac{\rho^{2-\alpha}}{(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau. \tag{13}$$

Using (13) in (11), we get,

$$\begin{aligned}\mathscr{X}(t_0) =& \frac{t_0^{1-\rho}(T^\rho-t_0^\rho)}{2\rho}\frac{\rho^{2-\alpha}}{(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\\ &\times\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau\\ &-\frac{\rho^{1-\alpha}}{2\Gamma(\alpha)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau.\end{aligned} \tag{14}$$

Using these values from (13) and (14) in (10), we get the following integral equation,

$$\begin{aligned}\mathscr{X}(t) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau\\ &-\frac{\rho^{1-\alpha}}{2\Gamma(\alpha)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau\\ &+\frac{t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau.\end{aligned} \tag{15}$$

Now, we will show that this integral equation (8) can be converted into the BVP (1) in the fractional sense only. Putting $t = t_0$ in (8), we get,

$$\begin{aligned}\mathscr{X}(t_0) =& -\frac{\rho^{1-\alpha}}{2\Gamma(\alpha)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau\\ &+\frac{t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t_0^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\int_{t_0}^{T}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2}\mathsf{f}(\tau,\mathscr{X}(\tau))d\tau.\end{aligned} \tag{16}$$

and $t = T$ gives,

$$\begin{aligned}
\mathscr{X}(T) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
& - \frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
& + \frac{t_0^{1-\rho} \rho^{1-\alpha}(T^\rho + t_0^\rho - 2T^\rho)}{2(T^{1-\rho} + t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau. \\
=& \frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
& - \frac{t_0^{1-\rho} \rho^{1-\alpha}(T^\rho + t_0^\rho - 2t_0^\rho)}{2(T^{1-\rho} + t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau. \\
=& - \mathscr{X}(t_0).
\end{aligned}$$

Differentiating (8) with respect to t, we get,

$$\begin{aligned}
\mathscr{X}'(t) =& \frac{\rho^{2-\alpha} t^{\rho-1}}{\Gamma(\alpha-1)} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
& + \frac{t_0^{1-\rho} \rho^{2-\alpha} t^{\rho-1}}{(T^{1-\rho} + t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau
\end{aligned} \tag{17}$$

here, $t = t_0$ gives,

$$\mathscr{X}'(t_0) = - \frac{\rho^{2-\alpha}}{(T^{1-\rho} + t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \tag{18}$$

and, $t = T$ yields,

$$\begin{aligned}
\mathscr{X}'(T) =& \frac{\rho^{2-\alpha} T^{\rho-1}}{\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
& + \frac{t_0^{1-\rho} \rho^{2-\alpha} T^{\rho-1}}{(T^{1-\rho} + t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
=& \frac{\rho^{2-\alpha}}{(T^{1-\rho} + t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho - \tau^\rho)^{\alpha-2} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau \\
=& - \mathscr{X}'(t_0).
\end{aligned} \tag{19}$$

This shows that the anti periodic boundary conditions are satisfied. Now, using there values of $\mathscr{X}(t_0), \mathscr{X}(T), \mathscr{X}'(t_0)$ and $\mathscr{X}'(T)$ and a similar calculation done in the first part of the Lemma, we get,

$$\mathscr{X}(t) = \mathscr{X}(t_0) + \frac{(t^\rho - t_0^\rho)}{\rho} \left[t_0^{1-\rho} \mathscr{X}'(t_0) \right] + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho - \tau^\rho)^{\alpha-1} \mathsf{f}(\tau, \mathscr{X}(\tau)) d\tau. \tag{20}$$

Which can be written as,

$$ {}^{K}_{t_0}J_t^{\alpha,\rho}\ {}^{OBC}_{t_0}D_t^{\alpha,\rho}\mathscr{X}(t) = {}^{K}_{t_0}J_t^{\alpha,\rho}\mathsf{f}(t,\mathscr{X}(t)) \tag{21} $$

applying ${}^{OBC}_{t_0}D_t^{\alpha,\rho}$ both the sides and using a result from [32], we get,

$$ {}^{OBC}_{t_0}D_t^{\alpha,\rho}\mathscr{X}(t) = \mathsf{f}(t,\mathscr{X}(t)) \tag{22} $$

which is required boundary value problem. □

Having the integral equation (8), we define the following required integral equation in fuzzy sense with an assumption that both $\mathscr{X}$ and $\mathscr{X}'$ are i-gH differentiable.

$$ \begin{aligned} \mathscr{X}(t) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \odot \int_{t_0}^{t} \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \ominus_g \frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \\ & \odot \int_{t_0}^{T} \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\alpha-1} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \\ & \oplus \frac{t_0^{1-\rho}\rho^{1-\alpha}(T^{\rho}+t_0^{\rho}-2t^{\rho})}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)} \odot \int_{t_0}^{T} \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\alpha-2} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \end{aligned} \tag{23} $$

To prove the existence and uniqueness results, we'll first exploit Banach fixed point theorem on the complete metric space $(C([t_0,T],\mathbb{F}_{\mathbb{R}}),\mathfrak{D})$, where $\mathfrak{D}(\mathscr{X},\mathscr{Y}) = \sup_{t\in[t_0,T]}(D_H(\mathscr{X}(t),\mathscr{Y}(t)))$. The following theorem gives the existence and uniqueness results for the above BVP (1).

THEOREM 3. *Let* f *be continuous function and let there be* $L>0$ *such that*

$$ D_H(\mathsf{f}(t,\mathscr{X}_1(t)),\mathsf{f}(t,\mathscr{X}_2(t))) \leqslant L D_H(\mathscr{X}_1(t),\mathscr{X}_2(t)) \tag{24} $$

$\forall t\in[t_0,T]$ *and* $\mathscr{X}_1(t),\mathscr{X}_2(t)\in\mathbb{F}_{\mathbb{R}}$. *Then the BVP (1) has a unique solution if* $\gamma<1$. *Where* $\gamma = \dfrac{L(T^{\rho}-t_0^{\rho})^{\alpha}}{2\rho^{\alpha}\Gamma(\alpha+1)}\left[3+\dfrac{t_0^{1-\rho}\alpha}{(T^{1-\rho}+t_0^{1-\rho})}\right]$.

Proof. Having equation (23) in mind, we define the following operator on $C([t_0,T],\mathbb{F}_{\mathbb{R}})$ as,

$$ \begin{aligned} (\mathfrak{T}\mathscr{X})(t) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \odot \int_{t_0}^{t} \tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\alpha-1} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \\ & \ominus_g \frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \odot \int_{t_0}^{T} \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\alpha-1} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \\ & \oplus \frac{t_0^{1-\rho}\rho^{1-\alpha}(T^{\rho}+t_0^{\rho}-2t^{\rho})}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)} \odot \int_{t_0}^{T} \tau^{\rho-1}(T^{\rho}-\tau^{\rho})^{\alpha-2} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \end{aligned} \tag{25} $$

The operator is well defined as the right hand side completely exists for all t. We'll show that $\mathfrak{T}$ is a contraction. For $\mathscr{X},\mathscr{W} \in C([t_0,T],\mathbb{F}_{\mathbb{R}})$, consider the following,

$$\begin{aligned}
\mathfrak{D}(\mathfrak{T}\mathscr{X},\mathfrak{T}\mathscr{W}) = &\sup_{t\in[t_0,T]} (D_H(\mathfrak{T}\mathscr{X}(t),\mathfrak{T}\mathscr{W}(t)))\\
\leqslant &\sup_{t\in[t_0,T]} \left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left(\int_{t_0}^{t} \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathsf{f}(\tau,\mathscr{W}(\tau)))d\tau \right)\right.\\
&+\frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \left(\int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathsf{f}(\tau,\mathscr{W}(\tau)))d\tau \right)\\
&+\frac{t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\\
&\left.\times\left(\int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathsf{f}(\tau,\mathscr{W}(\tau)))d\tau \right)\right\}\\
\leqslant &L\mathfrak{D}(\mathscr{X},\mathscr{W}) \sup_{t\in[t_0,T]} \left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left(\int_{t_0}^{t} \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1} d\tau \right)\right.\\
&+\frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \left(\int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1} d\tau \right)\\
&\left.+\frac{t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}\left(\int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1} d\tau \right)\right\}\\
= &L\mathfrak{D}(\mathscr{X},\mathscr{W})\left[\frac{(T^\rho-t_0^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+\frac{(T^\rho-t_0^\rho)^\alpha}{2\rho^\alpha\Gamma(\alpha+1)}+\frac{t_0^{1-\rho}(T^\rho-t_0^\rho)^\alpha}{2\rho^\alpha(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha)}\right]\\
= &\frac{L(T^\rho-t_0^\rho)^\alpha}{2\rho^\alpha\Gamma(\alpha+1)}\left[3+\frac{t_0^{1-\rho}\alpha}{(T^{1-\rho}+t_0^{1-\rho})}\right]\mathfrak{D}(\mathscr{X},\mathscr{W})=\gamma\mathfrak{D}(\mathscr{X},\mathscr{W}),
\end{aligned}$$

now, for $\mathfrak{T}$ to be a contraction, we want the quantity $\frac{L(T^\rho-t_0^\rho)^\alpha}{2\rho^\alpha\Gamma(\alpha+1)}\left[3+\frac{t_0^{1-\rho}\alpha}{(T^{1-\rho}+t_0^{1-\rho})}\right]$ <1 which requirement is fulfilled as $\gamma<1$. With this condition in hand, we can ensure that $\mathfrak{T}$ is a contraction and has a unique fixed point. □

Now, we'll use Schauder's fixed point theorem to prove the existence of the solution of the same BVP (1) under different assumptions on f.

THEOREM 4. *Let* f *be continuous and let there be* $L_1>0$ *such that*

$$D_H(\mathsf{f}(t,\mathscr{X}(t)),\mathbf{0})\leqslant L_1 \tag{26}$$

$\forall t\in[t_0,T]$ *and* $\mathscr{X}(t)\in\mathbb{F}_{\mathbb{R}}$. *Then the BVP (1) has a solution.*

Proof. Taking the same operator $\mathfrak{T}$ defined in equation (25). We need to show that $\mathfrak{T}$ is completely continuous. Clearly $\mathfrak{T}$ is continuous as f is continuous. Now, let

$\Omega \subset C[[t_0,T],\mathbb{F}_{\mathbb{R}}]$ be bounded. Then for any $\mathscr{X} \in \Omega$ we have,

$$\begin{aligned} D_H((\mathfrak{T}\mathscr{X})(t),\mathbf{0}) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-1} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathbf{0})d\tau \\ &+ \frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathbf{0})d\tau \\ &+ \frac{t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)} \\ &\times \int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathbf{0})d\tau \\ \leqslant& \frac{L_1(T^\rho-t_0^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{L_1(T^\rho-t_0^\rho)^\alpha}{2\rho^\alpha\Gamma(\alpha+1)} \\ &+ \frac{L_1 t_0^{1-\rho}\rho^{1-\alpha}(T^\rho+t_0^\rho-2t^\rho)}{2(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)}(T^\rho-t_0^\rho)^{\alpha-1} \\ \leqslant& \frac{L_1(T^\rho-t_0^\rho)^\alpha}{2\rho^\alpha\Gamma(\alpha+1)}\left[3+\frac{t_0^{1-\rho}\alpha}{(T^{1-\rho}+t_0^{1-\rho})}\right] = L_2 \ (let). \end{aligned} \tag{27}$$

Also, using equation (25), we get,

$$\begin{aligned} (\mathfrak{T}\mathscr{X})'(t) =& \frac{t^{\rho-1}\rho^{2-\alpha}}{\Gamma(\alpha-1)} \odot \int_{t_0}^{t} \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-2} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \\ &\oplus \frac{t_0^{1-\rho}\rho^{2-\alpha}t^{\rho-1}}{(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)} \odot \int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2} \odot \mathsf{f}(\tau,\mathscr{X}(\tau))d\tau \end{aligned}$$

which gives,

$$\begin{aligned} & D_H((\mathfrak{T}\mathscr{X})'(t),\mathbf{0}) \\ =& \frac{|t|^{\rho-1}\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_{t_0}^{t} \tau^{\rho-1}(t^\rho-\tau^\rho)^{\alpha-2} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathbf{0})d\tau \\ &+ \frac{t_0^{1-\rho}\rho^{2-\alpha}|t|^{\rho-1}}{(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)} \int_{t_0}^{T} \tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-2} D_H(\mathsf{f}(\tau,\mathscr{X}(\tau)),\mathbf{0})d\tau \\ \leqslant& \frac{L_1 T^{\rho-1}\rho^{2-\alpha}}{\rho\Gamma(\alpha-1)} \frac{(t^\rho-t_0^\rho)^{\alpha-1}}{\alpha-1} + \frac{L_1 t_0^{1-\rho}\rho^{2-\alpha}T^{\rho-1}}{\rho(T^{1-\rho}+t_0^{1-\rho})\Gamma(\alpha-1)} \frac{(t^\rho-t_0^\rho)^{\alpha-1}}{\alpha-1} \\ \leqslant& \frac{L_1 T^{\rho-1}\rho^{1-\alpha}(t^\rho-t_0^\rho)^{\alpha-1}}{\Gamma(\alpha)}\left[1+\frac{t_0^{1-\rho}}{(T^{1-\rho}+t_0^{1-\rho})}\right] = L_3 \ (let). \end{aligned}$$

Hence, for any $t_1,t_2 \in I$, we have,

$$D_H\Big((\mathfrak{T}\mathscr{X})(t_2) \ominus_g (\mathfrak{T}\mathscr{X})(t_1,\mathbf{0})\Big) \leqslant \int_{t_1}^{t_2} D_H((\mathfrak{T}\mathscr{X})'(\tau),\mathbf{0})d\tau \leqslant L_3(t_2-t_1). \tag{28}$$

This means that $\mathfrak{T}$ is equicontinuous on $[t_0, T]$ hence by Arzela-Ascoli theorem, $\mathfrak{T}$ is completely continuous. Hence $\mathfrak{T}$ has a fixed point and hence the BVP (1) has at least one solution. $\square$

5. Applications

5.1. Example 1

Consider the following fuzzy OBC-fractional anti periodic differential equation

$$\begin{aligned}
&{}_{1}^{OBC}D_t^{1.5,0.5}\mathscr{X}(t) = \frac{1}{6}\arctan\left(\frac{\tan\mathscr{X}}{2}\right) + \frac{\ln t}{1+t^2},\quad t\in[1,2],\\
&\mathscr{X}(1) = -1\odot\mathscr{X}(2),\ \mathscr{X}'(1) = -1\odot\mathscr{X}'(2),\ \ \mathscr{X}(1),\mathscr{X}(2),\mathscr{X}'(1),\mathscr{X}'(2)\in\mathbb{F}_{\mathbb{R}},
\end{aligned} \tag{29}$$

Computing the following,

$$\begin{aligned}
D_H(\mathsf{f}(t,\mathscr{X}_1(t)),\mathsf{f}(t,\mathscr{X}_2(t))) =&D_H\Bigg(\frac{1}{6}\arctan\left(\frac{\tan\mathscr{X}_1}{2}\right) + \frac{\ln t}{1+t^2},\\
&\frac{1}{2}\arctan\left(\frac{\tan\mathscr{X}_2}{2}\right) + \frac{\ln t}{1+t^2}\Bigg)\\
=&\frac{1}{6}D_H\left(\arctan\left(\frac{\tan\mathscr{X}_1}{2}\right)\ominus_g\arctan\left(\frac{\tan\mathscr{X}_2}{2}\right),\mathbf{0}\right)
\end{aligned}$$

Using the mean value theorem for fuzzy functions [13], we get $L = 1/3$. Which allows us to conclude that,

$$\frac{L(T^\rho - t_0^\rho)^\alpha}{2\rho^\alpha\Gamma(\alpha+1)}\left[3 + \frac{t_0^{1-\rho}\alpha}{(T^{1-\rho}+t_0^{1-\rho})}\right] < 0.25927 < 1.$$

Hence, the problem under consideration has a unique solution as it satisfies the requirement of the Theorem 3.

5.2. Example 2

Consider the following fuzzy OBC-fractional anti periodic differential equation

$$\begin{aligned}
&{}_{1}^{OBC}D_t^{1.5,0.5}\mathscr{X}(t) = \frac{\sin\mathscr{X}}{(t^2+3)(D_H(\mathscr{X},0)+1)} + t^2e^{-t},\quad t\in[1,3],\\
&\mathscr{X}(1) = -1\odot\mathscr{X}(3),\ \mathscr{X}'(1) = -1\odot\mathscr{X}'(3),\ \ \mathscr{X}(1),\mathscr{X}(3),\mathscr{X}'(1),\mathscr{X}'(3)\in\mathbb{F}_{\mathbb{R}},
\end{aligned} \tag{30}$$

Clearly f, here, is continuous as it constitutes only continuous functions. Now, we have,

$$
\begin{aligned}
D_H(\mathsf{f}(t,\mathscr{X}(t)),0) = & D_H\left(\frac{\sin\mathscr{X}}{(t^2+3)(D_H(\mathscr{X},0)+1)}+t^2e^{-t},0\right)\\
& \leqslant \frac{D_H(\sin\mathscr{X},0)}{(|t|^2+3)(D_H(\mathscr{X},0)+1)}+|t^2e^{-t}| \leqslant \frac{1}{4}+\frac{9}{e}=3.5609=L_1
\end{aligned}
$$

That is, our problem fulfills the conditions stated in Theorem 4 and hence has at least one solution.

5.3. Example 3

With an intent to provide an example which is analyzed numerically, consider the following,

$$
\begin{aligned}
&{}_{0}^{OBC}D_t^{\alpha,\rho}\,\mathscr{X}(t)=\cos\left(\frac{t}{5}\right),\ \ t\in[0,1],\\
&\mathscr{X}(0)=-1\odot\mathscr{X}(1),\quad \mathscr{X}(0),\mathscr{X}(1)\in\mathbb{F}_{\mathbb{R}},
\end{aligned}
\tag{31}
$$

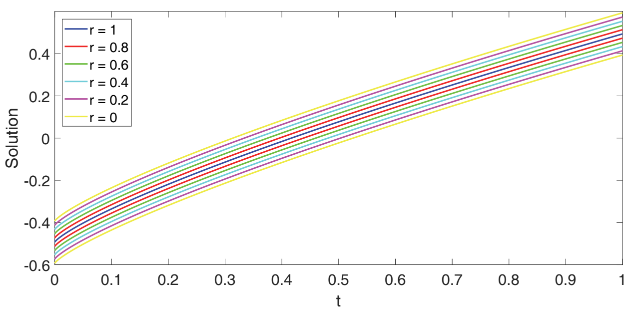

Figure 1: $\alpha=0.8,\ \rho=1.1$

t	$\mathscr{X}$ $(r=1)$	$\mathscr{X}_1$ $(r=0.8)$	$\mathscr{X}_2$ $(r=0.8)$	$\mathscr{X}_3$ $(r=0.6)$	$\mathscr{X}_4$ $(r=0.6)$	$\mathscr{X}_5$ $(r=0.4)$	$\mathscr{X}_6$ $(r=0.4)$	$\mathscr{X}_7$ $(r=0.2)$	$\mathscr{X}_8$ $(r=0.2)$	$\mathscr{X}_9$ $(r=0)$	$\mathscr{X}_{10}$ $(r=0)$
0.10	−0.10	−0.12	−0.08	−0.14	−0.06	−0.16	−0.04	−0.18	−0.02	−0.20	0.00
0.20	0.03	0.02	0.06	0.00	0.08	−0.02	0.10	−0.04	0.12	−0.06	0.14
0.30	0.13	0.12	0.16	0.10	0.18	0.08	0.20	0.06	0.22	0.04	0.24
0.40	0.21	0.20	0.24	0.18	0.26	0.16	0.28	0.14	0.30	0.12	0.32
0.50	0.28	0.27	0.31	0.25	0.33	0.23	0.35	0.21	0.37	0.19	0.39
0.60	0.34	0.33	0.37	0.31	0.39	0.29	0.41	0.27	0.43	0.25	0.45
0.70	0.40	0.38	0.42	0.36	0.44	0.34	0.46	0.32	0.48	0.30	0.50
0.80	0.45	0.43	0.47	0.41	0.49	0.39	0.51	0.37	0.53	0.35	0.55
0.90	0.49	0.48	0.52	0.46	0.54	0.44	0.56	0.42	0.58	0.40	0.60
1.00	0.54	0.52	0.56	0.50	0.58	0.48	0.60	0.46	0.62	0.44	0.64

Table 1: $\alpha=0.4,\ \rho=1.1$

The pots for the solution of the above problem for different values of α and ρ and tabular data for some of the plots are given.

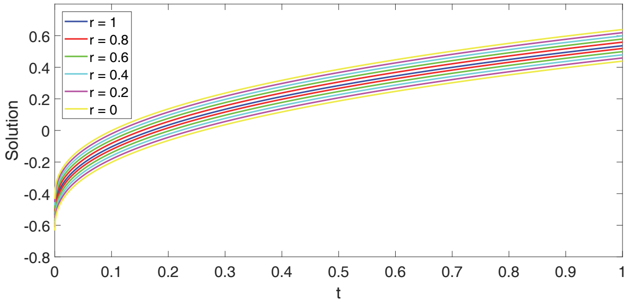

Figure 2: $\alpha = 0.4,\ \rho = 1.1$

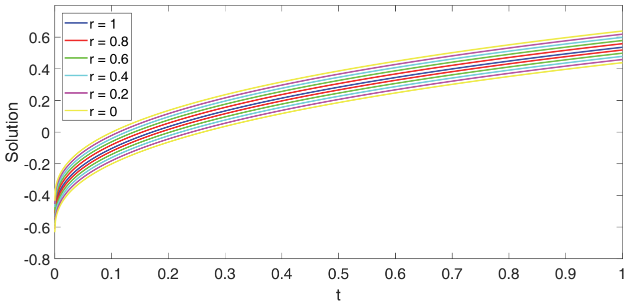

Figure 3: $\alpha = 0.8,\ \rho = 0.9$

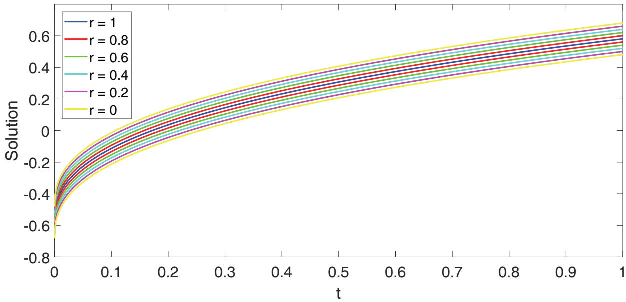

Figure 4: $\alpha = 0.4,\ \rho = 0.9$

These tables and plots exhibit that the problem taken is consistent with the fuzzy OBC-fractional derivative.

t	$\mathscr{X}$ $(r=1)$	$\mathscr{X}_1$ $(r=0.8)$	$\mathscr{X}_2$ $(r=0.8)$	$\mathscr{X}_3$ $(r=0.6)$	$\mathscr{X}_4$ $(r=0.6)$	$\mathscr{X}_5$ $(r=0.4)$	$\mathscr{X}_6$ $(r=0.4)$	$\mathscr{X}_7$ $(r=0.2)$	$\mathscr{X}_8$ $(r=0.2)$	$\mathscr{X}_9$ $(r=0)$	$\mathscr{X}_{10}$ $(r=0)$
0.10	−0.11	−0.13	−0.09	−0.15	−0.07	−0.17	−0.05	−0.19	−0.03	−0.21	−0.01
0.20	0.04	0.02	0.06	0.00	0.08	−0.02	0.10	−0.04	0.12	−0.06	0.14
0.30	0.14	0.12	0.16	0.10	0.18	0.08	0.20	0.06	0.22	0.04	0.24
0.40	0.23	0.21	0.25	0.19	0.27	0.17	0.29	0.15	0.31	0.13	0.33
0.50	0.31	0.29	0.33	0.27	0.35	0.25	0.37	0.23	0.39	0.21	0.41
0.60	0.37	0.35	0.39	0.33	0.41	0.31	0.43	0.29	0.45	0.27	0.47
0.70	0.43	0.41	0.45	0.39	0.47	0.37	0.49	0.35	0.51	0.33	0.53
0.80	0.49	0.47	0.51	0.45	0.53	0.43	0.55	0.41	0.57	0.39	0.59
0.90	0.54	0.52	0.56	0.50	0.58	0.48	0.60	0.46	0.62	0.44	0.64
1.00	0.58	0.56	0.60	0.54	0.62	0.52	0.64	0.50	0.66	0.48	0.68

Table 2: $\alpha = 0.4,\ \rho = 0.9$

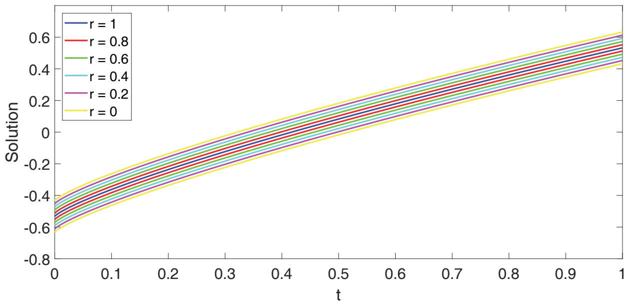

Figure 5: $\alpha = 0.8,\ \rho = 1$

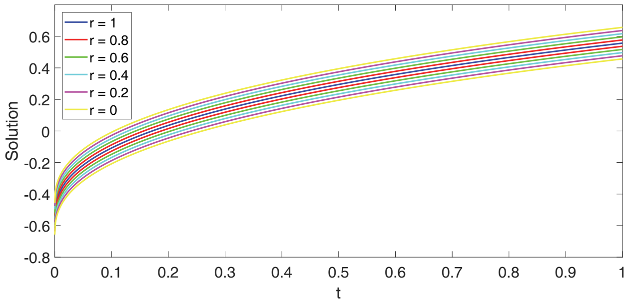

Figure 6: $\alpha = 0.4,\ \rho = 1$

t	$\mathscr{X}$ $(r=1)$	$\mathscr{X}_1$ $(r=0.8)$	$\mathscr{X}_2$ $(r=0.8)$	$\mathscr{X}_3$ $(r=0.6)$	$\mathscr{X}_4$ $(r=0.6)$	$\mathscr{X}_5$ $(r=0.4)$	$\mathscr{X}_6$ $(r=0.4)$	$\mathscr{X}_7$ $(r=0.2)$	$\mathscr{X}_8$ $(r=0.2)$	$\mathscr{X}_9$ $(r=0)$	$\mathscr{X}_{10}$ $(r=0)$
0.10	−0.11	−0.13	−0.09	−0.15	−0.07	−0.17	−0.05	−0.19	−0.03	−0.21	−0.01
0.20	0.03	0.01	0.05	−0.01	0.07	−0.03	0.09	−0.05	0.11	−0.07	0.13
0.30	0.14	0.12	0.16	0.10	0.18	0.08	0.20	0.06	0.22	0.04	0.24
0.40	0.22	0.20	0.24	0.18	0.26	0.16	0.28	0.14	0.30	0.12	0.32
0.50	0.29	0.27	0.31	0.25	0.33	0.23	0.35	0.21	0.37	0.19	0.39
0.60	0.36	0.34	0.38	0.32	0.40	0.30	0.42	0.28	0.44	0.26	0.46
0.70	0.41	0.39	0.43	0.37	0.45	0.35	0.47	0.33	0.49	0.31	0.51
0.80	0.47	0.45	0.49	0.43	0.51	0.41	0.53	0.39	0.55	0.37	0.57
0.90	0.51	0.49	0.53	0.47	0.55	0.45	0.57	0.43	0.59	0.41	0.61
1.00	0.56	0.54	0.58	0.52	0.60	0.50	0.62	0.48	0.64	0.46	0.66

Table 3: $\alpha = 0.4,\ \rho = 1$

6. Conclusion

Present work is divided mainly into two parts to fulfill the objectives of this paper. Firstly, we have modified and defined Odibat-Baleanu-Caputo (OBC) fractional derivative, which is a generalized Caputo-type fractional derivative, in fuzzy environment. Secondly, we studied a class of fuzzy anti periodic boundary value problems of order $1 < \alpha < 2$ under OBC fractional derivative. Some results on the existence and uniqueness of the solutions of the class have been established under suitable assumptions on the concerned nonlinear functions via contraction principal and the Schauder fixed point theorem. At the end, we have presented three examples as application, first one illustrates the results established in Theorem 3 and the second one does the same for Theorem 4. In the last example, we employed a numerical technique which is a generalization of the Euler's method for fuzzy OBC-fractional derivative.

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