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EXISTENCE AND STABILITY RESULTS FOR A PANTOGRAPH PROBLEM WITH SEQUENTIAL CAPUTO–HADAMARD DERIVATIVES

AMIRA ABDELNEBI* AND ZOUBIR DAHMANI

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Abstract. In the current paper, we look at the existence, uniqueness, and stability of solutions for a new pantograph problem with three sequential derivatives of Caputo-Hadamard type. The proposed problem admits the third-order pantograph problem as a limiting case. So, based on Banach contraction principle and Leray-Schauder fixed point theorems, two main theorems are proved. Another main result for the Ulam-Hyers stability of solutions for the problem is established. Furthermore, an illustrative example is presented to show the applicability of the existence and uniqueness result as well as the Ulam stability one.

1. Introduction

The topic of fractional integral and differential equations has proved to be valuable tools in modeling the dynamics of various systems and processes in the fields of engineering and technical sciences. For more information, we refer the reader to the research works [10, 17, 18, 20, 21, 23, 27, 28, 29] and references cited therein. Meanwhile, the variety of fractional differential operators, such as Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamards...etc, has prompted researchers to delve into this area. Much interesting and important research on differential equations with fractional calculus is devoted to model some phenomena in various applied sciences, see [1, 4, 7, 8, 12, 22, 24, 25, 32] and the references within.

Among the most famous equations, there is the pantograph equation, for which we can state that the pantograph phenomenon is an essential component of electric trains which collects electrical current from overload lines. This equation has been modeled by Ockendon and Tayler [26]. Its standard form is given by the following differential equation:

$$\begin{cases} x'(t) = Ax(t) + Bx(\omega t), \\ x(0) = x_0, \\ 0 \leqslant t \leqslant T, \quad 0 < \omega < 1. \end{cases}$$

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* Corresponding author.

In recent years, many researchers have proposed several fractional variants of the above pantograph equation. For more details, see for instance the research works [2, 3, 9, 11, 13, 14, 15, 16, 19, 30, 31, 33] and references therein.

In this sense, the authors of the paper [5] have studied the following fractional pantograph equation involving Caputo fractional derivative:

$$\begin{cases} {}^{C}D^{\alpha}[w(\tau)] = \varphi(\tau, w(\tau), w(\eta\tau)), \quad \tau \in [0,T], \\ w(0) = w_0, \\ 0 < \alpha, \eta < 1. \end{cases}$$

Very recently in [6], S. Belarbi and al. have discussed the existence and uniqueness of the following Φ-Caputo sequential pantograph fractional differential problem with integral conditions:

$$\begin{cases} {}^{c}D^{\beta,\Phi}({}^{c}D^{\alpha,\Phi}x(t) + g(t,x(t))) = f(t,x(t),x(\lambda t),{}^{c}D^{\alpha,\Phi}x(t)), \quad t \in [0,1], \\ x(0) = 0, \qquad x(1) = \int_0^1 h(s,x(s))ds, \\ 0 < \alpha, \beta < 1, \quad \lambda > 0, \end{cases}$$

where ${}^{c}D^{\beta,\Phi}$ and ${}^{c}D^{\alpha,\Phi}$ are the Φ-Caputo derivatives, the functions f, g and h are continuous.

Motivated by the above results, in this paper, we study the following three-sequential problem of pantograph type:

$$\begin{cases} D^{\alpha_1}[D^{\alpha_2}(D^{\alpha_3}x(t))] = g(t,x(t),x(\omega t),J^{\beta}x(\omega t),D^{\gamma}x(\omega t)), \\ x(1) - \mathscr{A}_1 = 0, \qquad D^{\alpha_3}x(1) - \mathscr{A}_2 = 0, \qquad D^{\alpha_2}(D^{\alpha_3}x(T)) = 0, \\ 0 < \beta < \alpha_i < 1, \quad i = 1,2,3, \quad \alpha_3 > \gamma, \quad 0 < \omega < 1, \quad \mathscr{A}_1, \mathscr{A}_2 \in \mathbb{R}, \quad t \in I, \end{cases} \tag{1}$$

where D^{α_i}, D^{γ} are the Caputo-Hadamard fractional derivatives, J^{β} is the Hadamard fractional integral, $I = [1,T]$, the function g is continuous.

The fundamental motivation of the present work has two main advantages: the first one is that the above-proposed problem is very interesting since it can admit, as a limiting case, the pantograph equation of order three, which provides a more accurate mathematical model of the motion of the pantograph arms. It takes into account the third derivative of the unknown function $x(t)$, which represents the curvature of the pantograph arms. This makes the equation more accurate than the equation of order two. The second advantage is in the use of the Caputo-Hadamard fractional derivative which combines the properties of two significant operators of Caputo and Hadamard.

This paper is systematized as follows: In Section 2, we render the rudimentary definitions and prove some lemmas that are applied throughout this paper, also we present the concepts of some fixed point theorems. In Section 3, we prove the existence of a solution and its uniqueness for the problem (1) using the Banach fixed point, we prove also a second theorem for the existence of at least one solution by applying the Leray-Schauder theorem. Then, we investigate certain types of Ulam stability. In Section 4, we give an example to illustrate two of our main results.

2. Some preliminary results

This section introduces some important definitions and preliminary lemmas, that will be needed to prove the results in the next sections. For more details, see [18, 20, 21, 23].

DEFINITION 1. (Hadamard fractional integral) The Hadamard fractional integral of order $\alpha \geqslant 0$, for a continuous function $x : [a,b] \longrightarrow \mathbb{R}$ is defined as

$$J_a^{\alpha}[x(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s}, \; \alpha > 0, \; a < t \leqslant b,$$
$$J_a^0[x(t)] = x(t),$$

where Γ is the Gamma function.

DEFINITION 2. (Caputo-Hadamard fractional derivative) Let

$$AC_\delta^m([a,b]) := \left\{ x : [a,b] \longrightarrow \mathbb{R} : \delta^{m-1} x(t) \in AC[a,b], \delta = t\frac{d}{dt} \right\}.$$

For a function $x \in AC_\delta^m([a,b])$, $m \in \mathbb{N}^*$ and $m-1 < \alpha \leqslant m$, the Caputo-Hadamard fractional derivative is given by

$$D_a^{\alpha} x(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{m-\alpha-1} \delta^m x(s) \frac{ds}{s}, & m-1 < \alpha < m \\ \delta^m x(t), & \alpha = m \end{cases}$$
$$= J_a^{m-\alpha}[\delta^m x(t)].$$

LEMMA 1. *Let $x \in AC_\delta^m([a,b])$, $m \in \mathbb{N}^*$ and $\alpha > 0$. Then, we have*

$$J_a^{\alpha}[D_a^{\alpha} x(t)] = x(t) + \sum_{i=0}^{n-1} c_i \left(\log \frac{t}{a}\right)^i,$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, $n = [\alpha] + 1$. Hence, the following fractional equation

$$D_a^{\alpha} x(t) = 0,$$

has a general solution expressed by

$$x(t) = \sum_{i=0}^{n-1} c_i \left(\log \frac{t}{a}\right)^i.$$

LEMMA 2. *For any $\alpha > 0$, $\beta > -1$,*

$$\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{a}\right)^{\beta} \frac{ds}{s} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \left(\log \frac{t}{a}\right)^{\alpha+\beta}.$$

LEMMA 3. *Let* $\beta > 0$ *and enable* $m-1 < \alpha < m$, $m \in \mathbb{N}^*$. *Then*

$$\frac{1}{\Gamma(m-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{m-\alpha-1} \delta^m \left(\log \frac{s}{a}\right)^{\beta} \frac{ds}{s} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \left(\log \frac{t}{a}\right)^{\beta-\alpha}.$$

LEMMA 4. *Let* $p, q > 0$, $f \in L^1([a,b])$. *Then,*

$$J_a^p J_a^q [x(t)] = J_a^{p+q}[x(t)], \quad t \in [a,b].$$

LEMMA 5. *Let* $q > p > 0$, $f \in L^1([a,b])$. *Then,*

$$D_a^p J_a^q [x(t)] = J_a^{q-p}[x(t)], \quad t \in [a,b].$$

THEOREM 1. (Banach contraction principle) *Let* S *be a Banach space. If* $\psi : S \longrightarrow S$ *is a contraction, then* ψ *has a unique fixed point in* S.

THEOREM 2. (Leray-Schauder Alternative) *Let* $\psi : S \longrightarrow S$ *be a completely continuous operator. If we consider the set* $\mathfrak{U}_\psi := \{x \in S : x = \sigma \psi(x) \quad \text{for some} \quad 0 < \sigma < 1\}$, *then we have:*

- *Either* ψ *has at least fixed point, or*
- *The set* $\mathfrak{U}_\psi$ *is unbounded.*

The following auxiliary result is also necessary to prove all the main results.

LEMMA 6. *Let* $G \in C([1,T])$, $t \in I$, $0 < \alpha_i \leqslant 1$. *Then the solution of the problem*

$$\begin{cases} D^{\alpha_1}[D^{\alpha_2}(D^{\alpha_3} x(t))] = G(t), \\ x(1) - \mathscr{A}_1 = 0, \qquad D^{\alpha_3} x(1) - \mathscr{A}_2 = 0, \qquad D^{\alpha_2}(D^{\alpha_3} x(T)) = 0, \end{cases} \tag{2}$$

is given by the following expression:

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} G(s) \frac{ds}{s} \\ & - \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1-1} G(s) \frac{ds}{s} \\ & + \mathscr{A}_1 \frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \mathscr{A}_2. \end{aligned} \tag{3}$$

Proof. By Lemma 1 and applying the operator I^{α_1} to Eq. (2), we can write

$$[D^{\alpha_2}(D^{\alpha_3} x(t))] = J^{\alpha_1} G(t) - c_0. \tag{4}$$

Next, by using I^{α_2} for both sides of Eq. (4), we get

$$[(D^{\alpha_3}x(t))] = J^{\alpha_1+\alpha_2}G(t) - c_0 J^{\alpha_2}(1) - c_1. \tag{5}$$

Then, using the operator I^{α_3} to Eq. (5), we conclude that

$$x(t) = J^{\alpha_1+\alpha_2+\alpha_3}G(t) - c_0 J^{\alpha_2+\alpha_3}(1) - c_1 J^{\alpha_3}(1) - c_2, \tag{6}$$

that is

$$x(t) = J^{\alpha_1+\alpha_2+\alpha_3}G(t) - c_0\frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - c_1\frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} - c_2, \tag{7}$$

where $c_i \in \mathbb{R},\ i = 0,1,2$, are arbitrary real constants.

Now, a simple calculation gives

$$c_2 = -\mathscr{A}_1.$$
$$c_1 = -\mathscr{A}_2,$$

and

$$c_0 = J^{\alpha_1}G(T).$$

Inserting the values of c_0, c_1 and c_2 in (7) provides the solution (3). □

3. Main results

This part contains the main results of the problem (1) using fixed point theory.

Let us now define the Banach space

$$S := \{x \in C(I,\mathbb{R}), D^{\gamma}x \in C(I,\mathbb{R})\},$$

and its norm

$$\| x \|_S = 2 \| x \| + \| D^{\gamma}x \|,$$

where,

$$\| x \| = \sup_{t\in I} | x(t) |, \qquad \| D^{\gamma}x \| = \sup_{t\in I} | D^{\gamma}x(t) | .$$

In view of Lemma 6, we define an operator $\psi : S \to S$ by:

$$\begin{aligned}\psi x(t) &= \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\frac{ds}{s}\\
&\quad - \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha_1-1} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\frac{ds}{s}\\
&\quad + \mathscr{A}_1\frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \mathscr{A}_2.\end{aligned}$$

Note that problem (1) has solutions if and only if the operator has ψ fixed points.

To facilitate the calculations for the reader, we need to use the following abbreviations:

$$\begin{aligned}
\Lambda_1 &= \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right),\\
\Lambda_2 &= \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right),\\
\Lambda_3 &= \Delta\Lambda_1,\\
\Lambda_4 &= M\Lambda_1 + |\mathscr{A}_1|\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mathscr{A}_2|,\\
\Lambda_5 &= \Delta\Lambda_2,\\
\Lambda_6 &= M\Lambda_2 + |\mathscr{A}_1|\frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)}.
\end{aligned}$$

Then, we need to propose the following conditions:

(P1): $g: I\times\mathbb{R}^4 \to \mathbb{R}$ is continuous function and there exists a nonnegative constant Δ_1, such that for all $t\in I$ and $u_i, v_i \in \mathbb{R}\quad (i=\overline{1:4})$, we have:

$$|g(t,u_1,u_2,u_3,u_4)-g(t,v_1,v_2,v_3,v_4)| \leqslant \Delta_1(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|).$$

(P2): There are non negative constants $\delta_0, \delta_1, \delta_2, \delta_3$ and δ_4, such that for real numbers $x_i\in\mathbb{R}, i=\overline{1:4}$, we have

$$|g(t,x_1,x_2,x_3,x_4))| \leqslant \delta_0+\delta_1|x_1|+\delta_2|x_2|+\delta_3|x_3|+\delta_4|x_4|.$$

3.1. A unique solution for the problem

THEOREM 3. *Assume that the condition $(P1)$ is satisfied and also the following inequality holds:*

$$2\Lambda_3+\Lambda_5<1. \tag{8}$$

Then, the problem (1) *has a unique solution.*

Proof. Let us put $\sup_{t\in[1,T]}|g(t,0,0,0,0)| = M$. Then, we set

$$r \geqslant \frac{2\Lambda_4+\Lambda_6}{1-(2\Lambda_3+\Lambda_5)}.$$

So, we shall prove that $\psi\mathfrak{B}_{\mathfrak{r}}\subset\mathfrak{B}_{\mathfrak{r}}$, where $\mathfrak{B}_{\mathfrak{r}} := \{x\in S: \|x\|_S\leqslant r\}$.

For $x \in \mathfrak{B}_{\mathfrak{r}}$ and for any $t \in [1,T]$, by using (Pi), $i = 1,2$, we can write

$$\begin{aligned}
&|g(t,x(t),x(\omega t),J^{\beta}x(\omega t),D^{\gamma}x(\omega t))| \\
&\leqslant |g(t,x(t),x(\omega t),J^{\beta}x(\omega t),D^{\gamma}x(\omega t)) - g(t,0,0,0,0) + g(t,0,0,0,0)| \\
&\leqslant |g(t,x(t),x(\omega t),J^{\beta}x(\omega t),D^{\gamma}x(\omega t)) - g(t,0,0,0,0)| + |g(t,0,0,0,0)| \\
&\leqslant \Delta_1(|x(t)| + |x(\omega t)| + |J^{\beta}x(\omega t)| + |D^{\gamma}x(\omega t)|) + |g(t,0,0,0,0)| \\
&\leqslant \sup_{t\in[1,T]} \left\{\Delta_1(|x(t)| + |x(\omega t)| + |J^{\beta}x(\omega t)| + |D^{\gamma}x(\omega t)|) + |g(t,0,0,0,0)|\right\} \\
&\leqslant \Delta_1 \left(2 \parallel x \parallel + \parallel J^{\beta}x \parallel + \parallel D^{\gamma}x \parallel\right) + M \\
&\leqslant \Delta_1 (2 \parallel x \parallel + \parallel D^{\gamma}x \parallel) + \Delta_1 \parallel J^{\beta}x \parallel + M \\
&\leqslant \Delta_1 \parallel x \parallel_S + \Delta_1 \frac{(\log T)^{\beta}}{\Gamma(\beta+1)} \parallel x \parallel_S + M \\
&\leqslant \Delta \parallel x \parallel_S + M \leqslant \Delta r + M,
\end{aligned}$$

where,

$$\Delta = \Delta_1 \left(1 + \frac{(\log T)^{\beta}}{\Gamma(\beta+1)}\right)$$

Using the above estimate, we obtain

$$\begin{aligned}
&|\psi x(t)| \\
&\leqslant \sup_{t\in[1,T]} \left\{\left|\frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1}\right.\right. \\
&\times g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s)) \frac{ds}{s} \\
&- \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1-1} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s)) \frac{ds}{s} \\
&\left.\left. + \mathscr{A}_1 \frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \mathscr{A}_2 \right|\right\} \\
&\leqslant \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \left|g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\right| \frac{ds}{s} \\
&+ \frac{(\log T)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right) \left|g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\right| \frac{ds}{s} \\
&+ |\mathscr{A}_1| \frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mathscr{A}_2|.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|\psi x(t)| &\leqslant (\Delta r+M)\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+(\Delta r+M)\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\\
&\quad +|\mathscr{A}_1|\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mathscr{A}_2|\\
&\leqslant \Delta\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right)r\\
&\quad +M\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right)\\
&\quad +|\mathscr{A}_1|\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mathscr{A}_2|.\\
&\leqslant \Lambda_3 r+\Lambda_4.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
D^{\gamma}\psi x(t) &= \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-\gamma-1}\\
&\quad \times g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\frac{ds}{s}\\
&\quad -\frac{(\log t)^{\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_1-1}\\
&\quad \times g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\frac{ds}{s}\\
&\quad +\mathscr{A}_1\frac{(\log t)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|D^{\gamma}\psi x(t)| &\leqslant \Delta\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}+\frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right)r\\
&\quad +M\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}+\frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right)\\
&\quad +|\mathscr{A}_1|\frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)}\\
&\leqslant \Lambda_5 r+\Lambda_6.
\end{aligned}
$$

Using the norm $\|.\|_S$, we have

$$
\begin{aligned}
\|\psi x\|_S &= 2\|\psi x\|+\|D^{\gamma}\psi x\|\\
&\leqslant 2(\Lambda_3 r+\Lambda_4)+\Lambda_5 r+\Lambda_6\\
&\leqslant (2\Lambda_3+\Lambda_5)r+2\Lambda_4+\Lambda_6\\
&\leqslant r.
\end{aligned}
$$

Consequently,

$$\psi\mathfrak{B}_{\mathfrak{r}} \subset \mathfrak{B}_{\mathfrak{r}}.$$

Now, for $x,y \in \mathfrak{B}_{\mathfrak{r}}$, and for all $t \in [1,T]$, we have:

$$\begin{aligned}
&| \psi x(t) - \psi y(t) | \\
&\leqslant \sup_{t\in[1,T]} \left\{ \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \right. \\
&\quad \times \left| \begin{array}{l} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s)) \\ -g(s,y(s),y(\omega s),J^{\beta}y(\omega s),D^{\gamma}y(\omega s)) \end{array} \right| \frac{ds}{s} - \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \\
&\quad \left. \times \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_1-1} \left| \begin{array}{l} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s)) \\ -g(s,y(s),y(\omega s),J^{\beta}y(\omega s),D^{\gamma}y(\omega s)) \end{array} \right| \frac{ds}{s} \right\} \\
&\leqslant \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \left| \begin{array}{l} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s)) \\ -g(s,y(s),y(\omega s),J^{\beta}y(\omega s),D^{\gamma}y(\omega s)) \end{array} \right| \frac{ds}{s} \\
&\quad + \frac{(\log T)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_1-1} \left| \begin{array}{l} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s)) \\ -g(s,y(s),y(\omega s),J^{\beta}y(\omega s),D^{\gamma}y(\omega s)) \end{array} \right| \frac{ds}{s} \\
&\leqslant \Delta \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)} \right) \|x-y\|_S .
\end{aligned}$$

Therefore,

$$\| \psi x - \psi y \| \leqslant \Lambda_3 \|x-y\|_S . \tag{9}$$

With the same arguments, we have

$$\begin{aligned}
&| D^{\gamma}\psi x(t) - D^{\gamma}\psi y(t) | \\
&\leqslant \Delta \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)} \right) \|x-y\|_S .
\end{aligned}$$

Thus, we obtain

$$\| D^{\gamma}\psi x - D^{\gamma}\psi y \| \leqslant \Lambda_5 \|x-y\|_S . \tag{10}$$

From (9) and (10), we get

$$\begin{aligned}
\|\psi x - \psi y\|_S &= 2\|\psi x - \psi y\| + \|D^{\gamma}\psi x - \psi y\| \\
&\leqslant (2\Lambda_3 + \Lambda_5)\|x-y\|_S .
\end{aligned}$$

Thanks to (8), we deduce that ψ is a contraction.

Hence, by Banach contraction principle, ψ has a unique fixed point which is the unique solution of the problem (1). □

3.2. At least one solution for the problem

Based on Theorem 2, we present to the reader the following second main result.

THEOREM 4. *Assume that hypothese* $(P2)$ *is satisfied. Then, problem* (1) *has at least a solution on* I, *provided that*

$$0 < \left[\delta_1 + \delta_2 + \frac{\delta_3(\log T)^{\beta}}{\Gamma(\beta+1)} + \delta_4\right][2\Lambda_1 + \Lambda_2] < 1.$$

Proof. We proceed as follows:
First of all, we shall prove that ψ is completely continuous on S:
1: The continuity of g allow us to state that ψ is continuous.
2: Let $\mathfrak{T} \in S$ be bounded. Then, $\forall x_i \in \mathfrak{T}$, $i = \overline{1,4}$, $\exists\, \Pi_1 > 0$, such that

$$|g(t,x_1,x_2,x_3,x_4)| \leqslant \Pi_1.$$

Then for all $x \in \mathfrak{T}$, we have

$$\begin{aligned}\|\ \psi x\ \| \leqslant \Pi_1 &\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right)\\ &+ |\mathscr{A}_1|\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mathscr{A}_2| < \infty.\end{aligned}$$

We have also

$$\begin{aligned}\|\ D^{\gamma}\psi x\ \| \leqslant \Pi_1 &\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right)\\ &+ |\mathscr{A}_1|\frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)}\\ &< \infty.\end{aligned}$$

Then, we deduce that ψ is uniformly bounded.
3: We show that ψ is equicontinuous.
Let $t_1, t_2 \in I$, with $t_1 < t_2$. This implies that

$$\begin{aligned}&|\ \psi x(t_1) - \psi x(t_2)\ |\\ \leqslant &\left|\frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\frac{ds}{s}\right.\\ &-\frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha_1-1} g(s,x(s),x(\omega s),J^{\beta}x(\omega s),D^{\gamma}x(\omega s))\frac{ds}{s}\\ &+\mathscr{A}_1\frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)}\end{aligned}$$

$$-\frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1}g(s,x(s),x(\omega s),J^\beta x(\omega s),D^\gamma x(\omega s))\frac{ds}{s}$$
$$+\frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_1-1}g(s,x(s),x(\omega s),J^\beta x(\omega s),D^\gamma x(\omega s))\frac{ds}{s}$$
$$\left.-\mathscr{A}_1\frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)}\right|,$$

then,

$$\begin{aligned}|\,\psi x(t_1)-\psi x(t_2)\,| \leqslant{}& \frac{\Pi_1}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}\left[(\log t_1)^{\alpha_1+\alpha_2+\alpha_3}-(\log t_2)^{\alpha_1+\alpha_2+\alpha_3}\right]\\ &+\frac{\Pi_1\,(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\left[(\log t_1)^{\alpha_2+\alpha_3}-(\log t_2)^{\alpha_2+\alpha_3}\right]\\ &+\frac{|\mathscr{A}_1|}{\Gamma(\alpha_3+1)}\left[(\log t_1)^{\alpha_3}-(\log t_2)^{\alpha_3}\right].\end{aligned}\tag{11}$$

Similarly, as before, we have

$$\begin{aligned}&|\,D^\gamma\psi x(t_1)-D^\gamma\psi x(t_2)\,|\\ \leqslant{}& \frac{\Pi_1}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}\left[(\log t_1)^{\alpha_1+\alpha_2+\alpha_3-\gamma}-(\log t_2)^{\alpha_1+\alpha_2+\alpha_3-\gamma}\right]\\ &+\frac{\Pi_1\,(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\left[(\log t_1)^{\alpha_2+\alpha_3-\gamma}-(\log t_2)^{\alpha_2+\alpha_3-\gamma}\right]\\ &+\frac{|\mathscr{A}_1|}{\Gamma(\alpha_3-\gamma+1)}\left[(\log t_1)^{\alpha_3-\gamma}-(\log t_2)^{\alpha_3-\gamma}\right].\end{aligned}\tag{12}$$

The right hand sides of (11) and (12) is independent of x, and tend to zero as $t_1\to t_2$. Therefore, the operator ψ is thus equicontinuous.

As a consequence of the previous steps and thanks to Arzela-Ascoli theorem, we conclude that ψ is completely continuous.

Now, we show that $\mathfrak{U}_\psi := \{x\in S : x=\sigma\psi(x),\ 0<\sigma<1\}$, is bounded.

Let $x\in\mathfrak{U}_\psi$. Then, we have $x=\sigma\psi(x)$ for some $0<\sigma<1$, and then for each $t\in I$, we can write

$$x(t)=\frac{\sigma}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1}g(s,x(s),x(\omega s),J^\beta x(\omega s),D^\gamma x(\omega s))\frac{ds}{s}$$
$$-\frac{\sigma(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_1-1}g(s,x(s),x(\omega s),J^\beta x(\omega s),D^\gamma x(\omega s))\frac{ds}{s}$$
$$+\mathscr{A}_1\frac{\sigma(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)}+\sigma\mathscr{A}_2.$$

Then, we get

$$\begin{aligned}\|x\| \leqslant & \left[\delta_1+\delta_2+\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}\delta_3+\delta_4\right]\\ & \times\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right]\|x\|_S\\ & +\delta_0\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right]\\ & +|\mathscr{A}_1|\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mathscr{A}_2|.\end{aligned}$$

In the same way, we obtain:

$$\begin{aligned}\|D^{\gamma}x\| \leqslant & \left[\delta_1+\delta_2+\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}\delta_3+\delta_4\right]\\ & \times\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}+\frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right]\|x\|_S\\ & +\delta_0\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}+\frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right]\\ & +|\mathscr{A}_1|\frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x\|_S & \leqslant 2\|x\|+\|D^{\gamma}x\|\\ & \leqslant 2\left[\delta_1+\delta_2+\frac{\delta_3(\log T)^{\beta}}{\Gamma(\beta+1)}+\delta_4\right]\\ & \quad\times\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right]\|x\|_S\\ & \quad+2\delta_0\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}+\frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)}\right]\\ & \quad+2|\mathscr{A}_1|\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)}+2|\mathscr{A}_2|\\ & \quad+\left[\delta_1+\delta_2+\frac{\delta_3(\log T)^{\beta}}{\Gamma(\beta+1)}+\delta_4\right]\\ & \quad\times\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}+\frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right]\|x\|_S\\ & \quad+\delta_0\left[\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}+\frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)}\right]\\ & \quad+|\mathscr{A}_1|\frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)}.\end{aligned}$$

Hence,

$$\|x\|_S \leqslant \left\{ \delta_0 \left[2\left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)} \right) + \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)} \right) \right] + |\mathscr{A}_1| \left[2\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)} \right] + 2|\mathscr{A}_2| \right\} \Big/ \left\{ 1 - 2\left[\delta_1+\delta_2 + \frac{\delta_3(\log T)^{\beta}}{\Gamma(\beta+1)} + \delta_4 \right] \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{(\log T)^{\alpha_2+\alpha_3}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1+1)} \right) + \left[\delta_1+\delta_2 + \frac{\delta_3(\log T)^{\beta}}{\Gamma(\beta+1)} + \delta_4 \right] \left(\frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \frac{(\log T)^{\alpha_2+\alpha_3-\gamma}(\log T)^{\alpha_1}}{\Gamma(\alpha_2+\alpha_3-\gamma+1)\Gamma(\alpha_1+1)} \right) \right\}.$$

Consequently,

$$\|x\|_S \leqslant \frac{\delta_0\,[2\Lambda_1+\Lambda_2] + |\mathscr{A}_1| \left[2\frac{(\log T)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \frac{(\log T)^{\alpha_3-\gamma}}{\Gamma(\alpha_3-\gamma+1)} \right] + 2|\mathscr{A}_2|}{1 - \left[\delta_1+\delta_2+\frac{\delta_3(\log T)^{\beta}}{\Gamma(\beta+1)} + \delta_4 \right] [2\Lambda_1+\Lambda_2]}.$$

So, $\mathfrak{U}_\psi$ is bounded.

We conclude by Theorem 2 that ψ admits at least one fixed point which is a solution of (1). □

3.3. Ulam type stabilities

In this section, we discuss some Ulam type stabilities for the solutions of (1).

Let $\varepsilon > 0$ and the function φ in S. We consider the following inequalities:

$$\left| D^{\alpha_1}[D^{\alpha_2}(D^{\alpha_3}y(t))] - g(t,y(t),y(\omega t),J^{\beta}y(\omega t),D^{\gamma}y(\omega t)) \right| \leqslant \varepsilon, \quad t \in I. \tag{13}$$

$$\left| D^{\alpha_1}[D^{\alpha_2}(D^{\alpha_3}y(t))] - g(t,y(t),y(\omega t),J^{\beta}y(\omega t),D^{\gamma}y(\omega t)) \right| \leqslant \varepsilon\varphi(t), \quad t \in I. \tag{14}$$

We introduce the following definitions.

DEFINITION 3. The problem (1) is Ulam-Hyers stable if there exists a real number $\mathfrak{C}_{\mathfrak{f}} > 0$, such that for each $\varepsilon > 0$ and for each solution $y \in S$ of (13), there exists a solution $x \in S$ of (1), with

$$\|y-x\|_S \leqslant \mathfrak{C}_{\mathfrak{f}}\varepsilon.$$

DEFINITION 4. We say that problem (1) is generalized Ulam-Hyers stable if there exists $\mathfrak{F} \in C(\mathbb{R}^+,\mathbb{R}^+)$, with $\mathfrak{F}(0) = 0$, such that for any $\varepsilon > 0$, and for each solution $y \in S$ to the inequality (13), there exists a solution $x \in S$ of (1), with

$$\|y-x\|_S \leqslant \mathfrak{F}(\varepsilon).$$

DEFINITION 5. We say that problem (1) is Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $\mathfrak{C}_{\mathfrak{f},\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in S$ of (14), there is a solution $x \in S$ of (1), with

$$\|y-x\|_S \leqslant \varepsilon \mathfrak{C}_{\mathfrak{f},\varphi}\varphi(t).$$

Now, we are ready to study the stability of solutions for (1).

THEOREM 5. *The solution of* (1) *is Ulam-Hyers stable if the hypotheses of Theorem* 3 *hold.*

Proof. Let $\varepsilon > 0$ and let $y \in X$ be a function that satisfies the inequality (13). Then, by integrating (13) with the same conditions as in (1), we obtain

$$\left| \begin{array}{c} y(t) - \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} G_y(s) \frac{ds}{s} \\ + \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1-1} G_y(s) \frac{ds}{s} \\ -\mathscr{A}_1 \frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} - \mathscr{A}_2. \end{array} \right| \leqslant \varepsilon \times \frac{(\log t)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)},$$

and let us assume that x is a solution of problem (1) given by

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} G(s) \frac{ds}{s} \\ & - \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1-1} G(s) \frac{ds}{s} \\ & + \mathscr{A}_1 \frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \mathscr{A}_2, \end{aligned}$$

where $G(s) = g(x, x(s), x(\omega s), J^\beta x(\omega s), D^\gamma x(\omega s))$ and $G_y(s) = g(y, y(s), y(\omega s), J^\beta y(\omega s), D^\gamma y(\omega s))$.

For each $t \in I$, we have

$$|y(t)-x(t)| \leqslant \left| \begin{array}{c} \varepsilon \times \frac{(\log t)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} [G_y(s)-G(s)] \frac{ds}{s} \\ + \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_1-1} [G_y(s)-G(s)] \frac{ds}{s} \end{array} \right|.$$

From hypothese $(P1)$, we get

$$\|x-y\| \leqslant \frac{\varepsilon(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \Lambda_3 \|x-y\|_S.$$

Similarly,

$$\|D^{\gamma}x - D^{\gamma}y\| \leqslant \frac{\varepsilon(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \Lambda_5\|x-y\|_S,$$

which implies that

$$\begin{aligned}\|x-y\|_S &\leqslant 2\frac{\varepsilon(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + 2\Lambda_3\|x-y\|_S \\ &\quad + \frac{\varepsilon(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)} + \Lambda_5\|x-y\|_S, \\ &\leqslant \left[\frac{2\varepsilon(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{\varepsilon(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}\right] + (2\Lambda_3+\Lambda_5)\|x-y\|_S.\end{aligned}$$

Consequently, we get

$$\|x-y\|_S \leqslant \left[\frac{\frac{2(\log T)^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} + \frac{(\log T)^{\alpha_1+\alpha_2+\alpha_3-\gamma}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\gamma+1)}}{1-(2\Lambda_3+\Lambda_5)}\right]\varepsilon = \mathfrak{C}_{\mathfrak{f}}\varepsilon.$$

Hence, the solution of problem (1) is stable in the Ulam-Hyers sense. □

REMARK 1. By taking $\mathfrak{F}(\varepsilon) = \mathfrak{C}_{\mathfrak{f}}\varepsilon$, we can state that the considered problem (1) is generalized Ulam-Hyers stable.

THEOREM 6. *Assume that the hypotheses of Theorem 3 and*

(P3): The function $\varphi \in C(I, \mathbb{R}^+)$ *is increasing and there exist* $\theta_{\varphi,\alpha} > 0$ *such that, for each* $t \in I$ *we have*

$$J^{\alpha}\varphi(t) \leqslant \theta_{\varphi,\alpha}\varphi(t), \tag{15}$$

are valid.

Then the solution of (1) is Ulam-Hyers-Rassias stable.

Proof. By integrating the inequality (14) and using (15), we get

$$\begin{aligned}&\left| \begin{array}{c} y(t) - \frac{1}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1+\alpha_2+\alpha_3-1} G_y(s)\frac{ds}{s} \\ + \frac{(\log t)^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_1)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha_1-1} G_y(s)\frac{ds}{s} \\ -\mathscr{A}_1\frac{(\log t)^{\alpha_3}}{\Gamma(\alpha_3+1)} - \mathscr{A}_2. \end{array} \right| \\ &\leqslant \varepsilon J^{\alpha_1+\alpha_2+\alpha_3}\varphi(t) \leqslant \varepsilon\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3}\varphi(t),\end{aligned}$$

where (14) has $y \in S$ as a proposed solution.

Let $x \in S$ be the unique solution of (1). Then, for each $t \in I$, we have

$$\|x-y\| \leqslant \varepsilon\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3}\varphi(t) + \Lambda_3\|x-y\|_S.$$

With the same arguments as before, we obtain

$$\|D^{\gamma}x - D^{\gamma}y\| \leqslant \varepsilon\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3-\gamma}\varphi(t) + \Lambda_5\|x-y\|_S.$$

Therefore

$$\begin{aligned}\|x-y\|_S &\leqslant \left(2\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3} + \theta_{\varphi,\alpha_1+\alpha_2+\alpha_3-\gamma}\right)\varepsilon\varphi(t) + (2\Lambda_3+\Lambda_5)\|x-y\|_S \\ &\leqslant \varepsilon \times \frac{\left(2\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3} + \theta_{\varphi,\alpha_1+\alpha_2+\alpha_3-\gamma}\right)}{1-(2\Lambda_3+\Lambda_5)}\varphi(t) = \varepsilon\mathfrak{C}_{\mathfrak{f},\varphi}\varphi(t).\end{aligned}$$

Then, (1) is Ulam-Hyers-Rassias stable. □

4. An example

As an application of our results, we consider the following problem:

$$\begin{cases} D^{0.9}[D^{0.93}(D^{0.86}x(t))] = g(t,x(t),x(\frac{1}{6}t),J^{0.45}x(\frac{1}{6}t),D^{0.80}x(\frac{1}{6}t)), \quad t\in[1,e], \\ x(1)-\frac{9}{10}=0, \qquad D^{0.86}x(1)+\frac{26}{3}=0, \qquad D^{0.93}(D^{0.86}x(e))=0, \end{cases} \tag{16}$$

where,

$$g(t,u,v,w,z) = \frac{\sin(\pi t)}{100\ln(t+1)} + \frac{2}{70}u + \frac{11}{1997}v + \frac{1}{20^3}\sin(t)w + \frac{1}{30^3}z.$$

So, for any $u_i, v_i \in \mathbb{R}$ $(i=1,2,3,4)$ and $t\in[1,e]$, we have

$$\begin{aligned}&|g(t,u_1,u_2,u_3,u_4) - g(t,v_1,v_2,v_3,v_4)| \\ &\leqslant \frac{2}{70}(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|).\end{aligned}$$

The condition (P1) holds with $\Delta_1 = \frac{2}{70}$.

Using the given data, we find that

$$\Delta = \Delta_1\left(1+\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}\right) = 0.0608,$$

and also, we have

$$\Lambda_3 = 0.0861, \quad \Lambda_5 = 0.0971.$$

Moreover,

$$2\Lambda_3+\Lambda_5 = 0.2694 < 1.$$

Consequently, Theorem 3 implies that (1) has a unique solution on I.

Also, thanks to Theorem 5, the unique solution of (16) is Ulam-Hyers stable.

5. Conclusion & perspectives

In this study, we have proposed a new pantograph problem with three sequential derivatives of Caputo-Hadamard type. The existence, uniqueness and different types of Ulam-stability of solutions for this problem has been addressed. The problem was solved using the Leray-Schauder and Banach fixed point theorems. The solutions are shown to be unique and stable in the sense of Ulam-Hyers and Ulam-Hyers-Rassias. For validation, a numerical example has been given to illustrate our main results.

Regarding future perspectives, this work opens some new possibilities for us to develop numerical methods for solving the new pantograph problem. Also, The stability of solutions to this problem could be investigated using various of methods, such as Lyapunov's stability theorem.

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Amira Abdelnebi
Department of Mathematics
Laboratory of Pure and Applied Mathematics
UMAB, University of Mostaganem
Algeria
e-mail: amira.math27@gmail.com

Zoubir Dahmani
Faculty of Sciences, Saad DAHLAB Blida 1 University
Laboratory of Pure and Applied Maths, UMAB University
Algeria
e-mail: zzdahmani@yahoo.fr