Fractional Differential Calculus
Volume 14, Number 1 (2024), 39–59 doi:10.7153/fdc-2024-14-03

TWO REGULARIZATION METHODS FOR A CLASS OF INVERSE FRACTIONAL PSEUDO–PARABOLIC EQUATIONS WITH INVOLUTION PERTURBATION

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(*Communicated by R. Ashurov*)

Abstract. In this study, we provide a theoretical analysis of an inverse problem governed by a time-fractional pseudo-parabolic equation with involution. The problem is characterized as ill-posed, meaning that the solution (if it exists) does not depend continuously on the measurable data. To address the inherent instability of this problem, we introduce two regularization strategies: the first employs a modified quasi-boundary value method, and the second utilizes a variant of the quasi-reversibility technique. We present convergence results under an a priori bound assumption and propose a practical a posteriori parameter selection rule.

1. Introduction

The study of inverse problems plays a pivotal role in the mathematical modeling of physical and engineering processes, where one seeks to determine unknown parameters or initial conditions based on observed data. Nevertheless, most inverse problems are inherently ill-posed, meaning that they lack unique solutions or are sensitive to small changes in data. This instability poses challenges in obtaining accurate and reliable solutions. To address this, regularization techniques are employed to introduce additional information and constraints, mitigating ill-posedness, and enhancing solution stability. Notable works such as Tikhonov's regularization, introduced by Tikhonov and Arsenin [44], and the truncated singular value decomposition (TSVD) method, as described by Hansen [21], exemplify classical regularization techniques. These techniques are indispensable for obtaining meaningful solutions in the presence of noise and limited data, making regularization a crucial component in solving ill-posed inverse problems across diverse domains. However, the landscape has seen the development of newer and more advanced methods that offer enhanced efficacy and performance in handling the intricacies of ill-posed inverse problems.

Mathematics subject classification (2020): 35R30, 35R25, 47A52, 35R11.
Keywords and phrases: Inverse problem, pseudo-parabolic problem, involution perturbation, modified quasi-boundary-value method, quasireversibility method.

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In the realm of applied mathematics, pseudo-parabolic equations are notable for their ability to model phenomena that exhibit both diffusive and wave-like characteristics. These equations find application in various modeling scenarios, including two-phase flow filtration within porous media involving dynamic capillary pressure [5], isotropic material energy dynamics [11], wave phenomena [7], and the flow dynamics of a significant class of dilute polymer solutions, commonly known as Oldroyd-B fluids [19].

Numerous researchers have focused the study of inverse problems governed by pseudo-parabolic equations (see [14, 24, 31, 32]). In the study presented in [55], the authors examined the asymptotic dynamics that characterizes solutions relevant to the inverse source problem associated with pseudo-parabolic equations. Furthermore, the investigation conducted in [1] delved into the realm of inverse problems concerning the determination of the right-hand side in pseudo-parabolic equations incorporating a p-Laplacian operator and a nonlocal integral overdetermination condition.

On the other hand, an intriguing category of partial differential equations (PDEs) involves those subject to involution perturbations. In mathematical terms, an involution is an operation that, when applied twice, yields the original outcome. In the context of PDEs, an involution perturbation refers to situations in which a PDE possesses a specific structure that is disrupted by a minor perturbation, leading to new and potentially unforeseen dynamics [51]. This concept traces its origins to the foundational contributions of Babbage [8], which were further expanded by Carleman [12] in 1932. Przeworska-Rolewicz made substantial contributions to this field, engaging in probing examinations of multifaceted inquiries related to differential equations involving involutions through a series of seminal papers [36, 37, 38]. These investigations culminated in a cohesive consolidation of her findings within a dedicated monograph [39]. For a collection of works concerning the study of inverse problems containing an involution term, refer to [2, 3, 4] and the references therein. However, it should be noted that, except for the contribution of Sassane et al. [41] and the recent work by Benabbes et al. [6], there has been a discernible lack of consideration regarding regularization and approximation techniques for these problems.

Recently, problems governed by pseudo-parabolic equations with involution perturbation have been the subject of extensive research in various mathematical models, as evidenced by the works of researchers such as [40, 42]. These equations play a significant role in engineering applications and have been explored in-depth from both theoretical and physical perspectives in previous studies and monographs. For more details on this topic, we refer to [9, 10, 13, 23, 29, 30, 35, 46, 53].

This work delves into a specific class of inverse problems generated by fractional pseudo-parabolic equations, characterized by the presence of an involution term. Even though previous studies have shed light on the theoretical development and physical motivations behind these problems, there exists a significant void in the literature regarding regularization techniques and numerical approximations for such types of problems. To fill this research gap and offer a thorough examination, this study employs two distinct methodologies to address the following fractional inverse source problem governed by a pseudo-parabolic equation with involution:

$$\begin{cases} D_t^\gamma\left[u(x,t)-u_{xx}(x,t)+\varepsilon u_{xx}(\pi-x,t)\right]-u_{xx}(x,t)+\varepsilon u_{xx}(\pi-x,t)=f(x), & (x,t)\in Q,\\ u(0,t)=u(\pi,t)=0, & t\in(0,T),\\ u(x,0)=0, & x\in(0,\pi),\\ u(x,T)=g(x), & x\in(0,\pi), \end{cases} \tag{1}$$

where $Q=(0,\pi)\times(0,T)$ and D_t^γ is the Caputo derivative for $0<\gamma<1$ defined by (see [26])

$$D_t^\gamma u(x,t)=\begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\gamma}d\tau,\ 0<\gamma<1,\\ \frac{d}{dt}u(x,t)=u_t(x,t),\ \gamma=1. \end{cases}$$

Here we specify that the problem (1) entails determining the source term $f(x)$ from the final state $g(x)$.

The first method proposed in this study uses a modified version of the quasi-boundary value method. The quasi-boundary value method, also called the non-local auxiliary boundary condition, was pioneered and advanced by Showalter [43]. This regularization technique involves substituting the final or boundary condition with a nonlocal condition, ensuring that the perturbed problem is well-posed. This approach has found application in resolving certain ill-posed problems associated with parabolic, hyperbolic, and elliptic equations. For further insights, refer to [17, 18, 27, 50] and the associated references. Instead of the traditional approach of using $u(x,T)=g(x)$ to address the unknown source problem, we adopt the approach of using $u_\mu^\delta(x,T)+\mu Lf_\mu^\delta(x)=g^\delta(x)$ (see [49]). This modified method exhibits a convergence rate of $O(\delta^{\frac{2}{3}})$ under an a priori choice of the regularization parameter, where δ represents the noise level.

The second method presented in this study is the quasi-reversibility method, which was originally introduced by Lattes and Lions [28] and has found application in tackling various categories of ill-posed problems, including inverse source problems (see [15, 47, 54]). The underlying principle of this approach involves the introduction of regularization terms that resemble reversible behaviors. These terms counteract instability and improve the accuracy of the solution (see [15, 47] and recent work [16]). In this context, we provide a convergence rate under an a priori bound assumption of the exact solution and introduce an a posteriori parameter choice rule. This rule yields a corresponding convergence rate estimate that is more practical in real-world scenarios, as discussed in [47].

The objective of this study is to assess the effectiveness and suitability of these two distinct methods in addressing the inverse source problem. By implementing these diverse strategies, we intend to strengthen the underlying theoretical framework and offer insightful perspectives relevant to real-world engineering applications.

The paper is structured as follows. Section 2 delves into problem analysis, presenting an initial result of conditional stability. In Section 3, we apply the modified quasi-boundary value method to the problem at hand, offering both a priori and a posteriori estimations. Moving forward, Section 4 introduces a second regularization approach, the quasi-reversibility method, accompanied by both a priori and a posteriori estimations through regularization parameter choice rules. The study concludes with a comprehensive summary in the final section.

2. Analysis of the problem

We define $\mathbb{N} = \{0,1,2,\ldots,\}$ and $\mathbb{N}^* = \{1,2,\ldots,\}$ to denote the sets of non-negative integers and positive integers, respectively.

Throughout the paper, we denote by $H = L^2((0,\pi);\mathbb{R})$ the Hilbert space equipped with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$, which are defined as follows:

$$\langle u,v\rangle := \int_0^\pi u(x)v(x)\,dx, \quad \|u\|^2 := \int_0^\pi |u(x)|^2\,dx.$$

Let $H^2(0,\pi) = W^{2,2}(0,\pi)$ be the usual Sobolev space equipped with the norm $\|h\|_2^2 = \|h\|^2 + \|h'\|^2 + \|h''\|^2$.

Problem (1) can be reformulated as follows:

$$\begin{cases} D_t^\gamma [u(x,t) + Lu(x,t)] + Lu(x,t) = f(x), & (x,t) \in Q, \\ u(x,0) = 0, & x \in (0,\pi), \\ u(x,T) = g(x), & x \in (0,\pi), \end{cases} \tag{2}$$

where $L : \mathscr{D}(L) \subset H \to H$, with

$$\begin{cases} \mathscr{D}(L) = H_0^1(0,\pi) \cap H^2(0,\pi) = \quad \left\{u \in H^2(0,\pi) : u(0) = u(\pi) = 0\right\}, \\ Lu(x) = -u_{xx}(x) + \varepsilon u_{xx}(\pi - x),\ x \in (0,\pi), \end{cases} \tag{3}$$

and $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leqslant \varepsilon_0 < 1$. It can be readily verified that the introduced operator is self-adjoint. For all $|\varepsilon| < 1$, the nonlocal problem (3) has the following eigenvalues, as detailed in reference [42]:

$$\lambda_{2k+1} = (1-\varepsilon)(2k+1)^2,\ k \in \mathbb{N}, \qquad \lambda_{2k} = (1+\varepsilon)(2k)^2,\ k \in \mathbb{N}^*,$$

with the following corresponding normalized eigenfunctions

$$\varphi_{2k+1}(x) = \sqrt{\frac{2}{\pi}}\sin((2k+1)x),\ k \in \mathbb{N}, \quad \varphi_{2k}(x) = \sqrt{\frac{2}{\pi}}\sin(2kx),\ k \in \mathbb{N}^*. \tag{4}$$

Under this notation, the system $\mathscr{B} = \{\varphi_{2k}\}_{k=1}^\infty \bigcup \{\varphi_{2k+1}\}_{k=0}^\infty$ constitutes an orthonormal basis in H.

For $r \geqslant 0$, let

$$\mathbb{H}_r = \mathscr{D}(L^r) = \left\{ h \in H : \sum_{k=0}^\infty \lambda_{2k+1}^{2r} |(h,\varphi_{2k+1})|^2 + \sum_{k=1}^\infty \lambda_{2k}^{2r} |(h,\varphi_{2k})|^2 < \infty \right\},$$

the Hilbert space (sub-space of H) with the norm

$$\|h\|_r = \left(\sum_{k=0}^{\infty} \lambda_{2k+1}^{2r} \left|(h, \varphi_{2k+1})\right|^2 + \sum_{k=1}^{\infty} \lambda_{2k}^{2r} \left|(h, \varphi_{2k})\right|^2 \right)^{\frac{1}{2}}.$$

The inverse problem focuses on reconstructing the function $f(x)$ from the noisy measured data $g^{\delta}(x)$. We assume that:

$$\left\| g^{\delta} - g \right\| \leqslant \delta, \tag{5}$$

where $\delta > 0$ represents the level of noise.

For the sake of simplicity, we adopt the following notations:

$$z_{2k+1}(t) = \left(1 - E_{\gamma,1}\left(-\frac{\lambda_{2k+1}}{1+\lambda_{2k+1}} t^{\gamma} \right) \right), \quad k = 0, 1, 2, \ldots, \tag{6}$$

$$z_{2k}(t) = \left(1 - E_{\gamma,1}\left(-\frac{\lambda_{2k}}{1+\lambda_{2k}} t^{\gamma} \right) \right), \quad k = 1, 2, \ldots, \tag{7}$$

For $y \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$, the Mittag-Leffler function is denoted by $E_{\alpha,\beta}(y) = \sum\limits_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k+\beta)}$. For brevity, we also use $E_{\alpha,1}(.) = E_{\alpha}(.)$. We list some properties of the Mittag-Leffler function [20, 52].

LEMMA 1. *For $0 < \alpha < 1$ and $y > 0$, $E_{\alpha}(-y)$ is decreasing. Furthermore, for all $\lambda > 0$, we have*

$$D_y^{\alpha} E_{\alpha}(-\lambda y^{\alpha}) = -\lambda E_{\alpha}(-\lambda y^{\alpha}), \tag{8}$$

$$E_{\alpha}(0) = 1, \;\; 0 \leqslant E_{\alpha}(-y) \leqslant 1, \;\; \lim_{y \longrightarrow +\infty} E_{\alpha}(-y) = 0, \tag{9}$$

and

$$\frac{d^m}{dy^m}\left(E_{\alpha}(-y)\right) = (-1)^m E_{\alpha}(-y), \quad m \in \mathbb{N}. \tag{10}$$

Now, we recall a fundamental result that will be utilized extensively throughout this paper.

LEMMA 2. [25] (Uniform estimates for the Mittag-Leffler function) *For every $0 < \alpha < 1$, the uniform estimate*

$$\frac{1}{1+c_1(\alpha) y} \leqslant E_{\alpha}(-y) \leqslant \frac{1}{1+c_2(\alpha) y}, \tag{11}$$

holds over $\mathbb{R}_+$, with the following optimal constants:

$$c_1(\alpha) = \Gamma(1-\alpha), \quad \text{and} \quad c_2(\alpha) = \Gamma(1+\alpha)^{-1}. \tag{12}$$

We start by establishing the existence and uniqueness of the solution of Problem (1). Given that the eigenfunctions (4) constitute an orthonormal basis within H, we search the functions $u(x,t)$ and $f(x)$ in the following forms:

$$u(x,t) = \sum_{k=0}^{\infty} w_k(t)\,\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} v_k(t)\,\varphi_{2k}(x), \tag{13}$$

and

$$f(x) = \sum_{k=0}^{\infty} f_{2k+1}\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} f_{2k}\varphi_{2k}(x), \tag{14}$$

where $w_k(t)$, $v_k(t)$, f_{2k} and f_{2k+1} are unknown.

According to the calculations demonstrated in [42], we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{f_{2k+1}}{\lambda_{2k+1}} z_{2k+1}(t)\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{f_{2k}}{\lambda_{2k}} z_{2k}(t)\varphi_{2k}(x). \tag{15}$$

From this representation, we find

$$f(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1}}{z_{2k+1}(T)} g_{2k+1}\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{\lambda_{2k}}{z_{2k}(T)} g_{2k}\varphi_{2k}(x). \tag{16}$$

We define the linear operator: $\mathbf{K} : L^2(0,\pi) \longrightarrow L^2(0,\pi)$ as follows:

$$\mathbf{K}f(x) = \sum_{k=0}^{\infty} \frac{f_{2k+1}}{\lambda_{2k+1}} z_{2k+1}(T)\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{f_{2k}}{\lambda_{2k}} z_{2k}(T)\varphi_{2k}(x). \tag{17}$$

We can also write this equation in the form of an integral equation of the first kind:

$$\mathbf{K}f(x) := \int_0^{\pi} k(x,\zeta)\, f(y)\, dy = g(x), \tag{18}$$

where the kernel $k(x,y) = \sum_{k=0}^{\infty} \theta_{2k+1}\varphi_{2k+1}(x)\,\varphi_{2k+1}(y) + \sum_{k=1}^{\infty} \theta_{2k}\varphi_{2k}(x)\,\varphi_{2k}(y)$.

It is clear that the kernel $k(.,.)$ is real-valued and symmetric, that is, $k(x,y) = \overline{k(y,x)}$, which allows us to conclude that $\mathbf{K}$ is a compact self-adjoint operator. Moreover, $\mathbf{K}$ is an injective operator, with eigenpairs (θ_j, φ_j):

$$\mathbf{K}\varphi_{2k} = \theta_{2k}\varphi_{2k}, \quad \mathbf{K}\varphi_{2k} = \theta_{2k}\varphi_{2k},$$

and

$$\theta_{2k} = \frac{z_{2k}(T)}{\lambda_{2k}}, \quad \theta_{2k+1} = \frac{z_{2k+1}(T)}{\lambda_{2k+1}}.$$

REMARK 1. In equation (18), the solution f is given by $f = \mathbf{K}^{-1}g$. In this situation, the algebraic inversion does not pose a problem, because the operator $\mathbf{K}$ is injective. On the other hand, this inversion is not continuous (unstable) because $\mathbf{K}$ is compact. So, we are concerned with an ill-posed problem which requires a regularization procedure.

We note that the proofs of our results are technical. To simplify, we introduce the following technical lemmas.

LEMMA 3. *For all $\lambda_j > 0,\ j = 2k$ and $j = 2k+1$, there exist positive constants a_1 and a_2 such that:*

$$z_{2k}(T) \geqslant a_1 > 0, \quad z_{2k+1}(T) \geqslant a_2 > 0. \tag{19}$$

Proof. For $T > 0$ and $\lambda_{2k} > 0$, we have

$$\frac{1}{T^\gamma} \leqslant \frac{1+\lambda_{2k}}{\lambda_{2k}T^\gamma} \leqslant \frac{1+\lambda_2}{\lambda_2 T^\gamma} = \frac{1+4(1+\varepsilon)}{4(1+\varepsilon)T^\gamma} = m_1(\varepsilon).$$

We suppose that $|\varepsilon| \leqslant \varepsilon_0 < 1$. The function $m_1(\varepsilon)$ is positive and decreasing on the interval $[-\varepsilon_0, \varepsilon_0]$. It follows that

$$m_1(\varepsilon) \leqslant m_1(-\varepsilon_0) = \frac{1+4(1-\varepsilon_0)}{4(1-\varepsilon_0)T^\gamma}.$$

From these remarks and (11), we can write

$$\begin{aligned} z_{2k}(T) \geqslant & 1 - \frac{1}{1 + \frac{\lambda_{2k}}{1+\lambda_{2k}} T^\gamma \Gamma(1+\gamma)^{-1}} = \frac{\Gamma(1+\gamma)^{-1}}{\frac{1+\lambda_{2k}}{\lambda_{2k}T^\gamma} + \Gamma(1+\gamma)^{-1}} \\ \geqslant & \frac{\Gamma(1+\gamma)^{-1}}{m_1(-\varepsilon_0) + \Gamma(1+\gamma)^{-1}} = a_1(T,\gamma,\varepsilon_0) = a_1 > 0. \end{aligned}$$

In the same way, for $\lambda_{2k+1} > 0$, we can write

$$\frac{1}{T^\gamma} \leqslant \frac{1+\lambda_{2k+1}}{\lambda_{2k+1}T^\gamma} \leqslant \frac{1+\lambda_1}{\lambda_1 T^\gamma} = \frac{1+(1-\varepsilon)}{(1-\varepsilon)T^\gamma} = m_2(\varepsilon) \leqslant m_2(\varepsilon_0) = \frac{2-\varepsilon_0}{(1-\varepsilon_0)T^\gamma},$$

which implies that

$$z_{2k+1}(T) \geqslant \frac{\Gamma(1+\gamma)^{-1}}{m_2(\varepsilon_0) + \Gamma(1+\gamma)^{-1}} = a_2(T,\gamma,\varepsilon_0) = a_2 > 0. \quad \square$$

By a direct calculation of the minimum and maximum of a real function with the help of (19), we show the following.

For $s \geqslant 0$ and $\alpha, \beta > 0$, we have

$$A(s) = \frac{s}{\alpha s^2 + \beta} \leqslant \frac{1}{2\sqrt{\beta}} \frac{1}{\sqrt{\alpha}}, \tag{20}$$

$$A_{2k+1} = \frac{\lambda_{2k+1}}{\alpha\lambda_{2k+1}^2 + z_{2k+1}(T)} \leqslant \frac{\lambda_{2k+1}}{\alpha\lambda_{2k+1}^2 + a_1} \leqslant \frac{1}{2\sqrt{a_1}} \frac{1}{\sqrt{\alpha}}, \tag{21}$$

and

$$A_{2k} = \frac{\lambda_{2k}}{\alpha\lambda_{2k}^2 + z_{2k}(T)} \leqslant \frac{\lambda_{2k}}{\alpha\lambda_{2k}^2 + a_1} \leqslant \frac{1}{2\sqrt{a_2}} \frac{1}{\sqrt{\alpha}}. \tag{22}$$

REMARK 2. By virtue of (9) and (19), we have

$$a_1 \leqslant z_{2k}(T) \leqslant 1, \;\; a_2 \leqslant z_{2k+1}(T) \leqslant 1, \;\; 1 \leqslant \frac{1}{z_{2k}(T)} \leqslant \frac{1}{a_1}, \;\; 1 \leqslant \frac{1}{z_{2k+1}(T)} \leqslant \frac{1}{a_2},$$

which implies that

$$\|g\|_1^2 = \sum_{k=1}^{\infty} \lambda_{2k}^2 |g_{2k}|^2 + \sum_{k=0}^{\infty} \lambda_{2k+1}^2 |g_{2k+1}|^2 \leqslant \|f\|^2,$$

and

$$\|f\|^2 \leqslant \kappa^2 \left(\sum_{k=1}^{\infty} \lambda_{2k}^2 |g_{2k}|^2 + \sum_{k=0}^{\infty} \lambda_{2k+1}^2 |g_{2k+1}|^2 \right) = \kappa^2 \|g\|_1^2,$$

where $\kappa = \max\left(\frac{1}{a_1}, \frac{1}{a_2}\right)$.

From this Remark, we conclude that $f \in H$ if and only if $g \in \mathbb{H}_1$.

LEMMA 4. [50] *For any $\alpha, p, \beta > 0$ and $s \geqslant \lambda_j$, $j = 1,2$, we have:*
(i) If $s \geqslant \lambda_1 = (1-\varepsilon) > 0$, then

$$F(s) = \frac{\alpha s^{2-\frac{p}{2}}}{\alpha s^2 + \beta} \leqslant \begin{cases} b_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ b_2 \alpha, & p \geqslant 4. \end{cases} \tag{23}$$

(ii) If $s \geqslant \lambda_2 = 4(1+\varepsilon) > 0$, then

$$F(s) = \frac{\alpha s^{2-\frac{p}{2}}}{\alpha s^2 + \beta} \leqslant \begin{cases} c_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ c_2 \alpha, & p \geqslant 4. \end{cases} \tag{24}$$

Here,

$$b_2 = \frac{1}{\beta \lambda_1^{\frac{p-4}{2}}}, \quad c_2 = \frac{1}{\beta \lambda_2^{\frac{p-4}{2}}} \quad \textit{and} \quad b_1 = c_1 = \frac{\left(\frac{\beta(4-p)}{p}\right)^{\frac{4-p}{p}}}{\frac{\beta(4-p)}{p} + \beta} \leqslant 1.$$

LEMMA 5. [47] *For any $\alpha, p > 0$ and $s \geqslant \lambda_j$, $j = 1,2$, we have:*
(i) If $s \geqslant \lambda_1 = (1-\varepsilon) > 0$, then

$$\hat{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s + 1} \leqslant \begin{cases} \hat{b}_1 \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \hat{b}_2 \alpha, & p \geqslant 2. \end{cases} \tag{25}$$

(ii) If $s \geqslant \lambda_2 = 4(1+\varepsilon) > 0$, then

$$\hat{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s + 1} \leqslant \begin{cases} \hat{c}_1 \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \hat{c}_2 \alpha, & p \geqslant 2. \end{cases} \tag{26}$$

Here,

$$\hat{b}_2 = \frac{1}{\lambda_1^{\frac{p-2}{2}}}, \quad \hat{c}_2 = \frac{1}{\lambda_2^{\frac{p-2}{2}}} \quad \textit{and} \quad \hat{b}_1 = \hat{c}_1 = \frac{\left(\frac{2}{p}\right)^{\frac{2-p}{2}}}{\frac{2}{p} + 1} \leqslant 1.$$

LEMMA 6. [50] *For any* $\alpha, p, \beta > 0$ *and* $s \geqslant \lambda_j,\ j = 1,2$, *we have:*
(i) If $s \geqslant \lambda_1 = (1-\varepsilon) > 0$, *then*

$$\widetilde{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s^2 + \beta} \leqslant \begin{cases} \widetilde{b}_1 \alpha^{\frac{2+p}{4}}, & 0 < p < 2, \\ \widetilde{b}_2 \alpha, & p \geqslant 2. \end{cases} \tag{27}$$

(ii) If $s \geqslant \lambda_2 = 4(1+\varepsilon) > 0$, *then*

$$\widetilde{F}(s) = \frac{\alpha s^{\frac{2-p}{2}}}{\alpha s^2 + \beta} \leqslant \begin{cases} \widetilde{c}_1 \alpha^{\frac{2+p}{4}}, & 0 < p < 2, \\ \widetilde{c}_2 \alpha, & p \geqslant 2. \end{cases} \tag{28}$$

Here,

$$\widetilde{b}_2 = \frac{1}{\beta \lambda_1^{\frac{p-2}{2}}}, \quad \widetilde{c}_2 = \frac{1}{\beta \lambda_2^{\frac{p-2}{2}}} \quad \textit{and} \quad \widetilde{b}_1 = \widetilde{c}_1 = \frac{\left(\frac{\beta(2-p)}{2+p}\right)^{\frac{2-p}{4}}}{\frac{\beta(2-p)}{2+p} + \beta} \leqslant 1.$$

Putting

$$B_{2k+1} = \frac{\alpha \lambda_{2k+1}^{2-\frac{p}{2}}}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)},$$

and

$$B_{2k} = \frac{\alpha \lambda_{2k}^{2-\frac{p}{2}}}{\alpha \lambda_{2k}^2 + z_{2k}(T)}.$$

By virtue of (19), (23) and (24), we have

$$B_{2k} \leqslant \frac{\alpha \lambda_{2k}^{2-\frac{p}{2}}}{\alpha \lambda_{2k}^2 + a_1} \leqslant \begin{cases} b_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ b_2 \alpha, & p \geqslant 4, \end{cases} \tag{29}$$

and

$$B_{2k+1} \leqslant \frac{\alpha \lambda_{2k+1}^{2-\frac{p}{2}}}{\alpha \lambda_{2k+1}^2 + a_2} \leqslant \begin{cases} c_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ c_2 \alpha, & p \geqslant 4. \end{cases} \tag{30}$$

From these inequalities, we have

$$\sup_{k \geqslant 0} B_{2k+1} \leqslant \begin{cases} b_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ b_2 \alpha, & p \geqslant 4, \end{cases} \tag{31}$$

and

$$\sup_{k \geqslant 1} B_{2k} \leqslant \begin{cases} c_1 \alpha^{\frac{p}{4}}, & 0 < p < 4, \\ c_2 \alpha, & p \geqslant 4. \end{cases} \tag{32}$$

THEOREM 1. (Conditional stability) *If* $f \in \mathbb{H}_{\frac{p}{2}}$, *i.e.,* f *satisfies the following a priori bound condition:*

$$\|f\|_{\frac{p}{2}} \leqslant E, \quad p > 0. \tag{33}$$

Then we have

$$\|f\| \leqslant CE^{\frac{2}{p+2}} \|g\|^{\frac{p}{p+2}}, \quad p > 0. \tag{34}$$

Here, the constant C *is given by* $C = \sqrt{a_1^{\frac{-2p}{p+2}} + a_2^{\frac{-2p}{p+2}}}$, *and depends on the values of* p, γ, T, *and* ε_0.

Proof. The proof is technical and is based on the use of formula (16), Hölder's inequality and estimates (19). □

REMARK 3. Drawing from the proof of Theorem 1, we can reformulate the conditional stability in the following manner:

$$\|f\| \leqslant C\left(\|f\|_{\frac{p}{2}}\right)^{\frac{2}{p+2}} \|\mathbf{K}f\|^{\frac{p}{p+2}}.$$

This indicates that we can establish an upper bound for the L^2-norm of f by estimating both $\|f\|_{\frac{p}{2}}$ and the L^2-norm of $\mathbf{K}f$.

3. Modified quasi-boundary value method and convergence rates

In this section, we adopt a variant of the modified quasi-boundary value method applied to our problem (1). We follow the same approach developed in the work [49] with an additional calculation generated by two sums of two Fourier series. In this context, we establish two convergence estimates: one based on an a priori regularization parameter selection criterion and the other on an a posteriori regularization parameter selection criterion.

Let $u_\alpha^\delta(x,t)$ denote the solution to the following regularized problem:

$$\begin{cases} D_t^\gamma\left[u_\alpha^\delta(x,t) + Lu_\alpha^\delta(x,t)\right] + Lu_\alpha^\delta(x,t) = f_\alpha^\delta(x), & (x,t) \in Q, \\ u_\alpha^\delta(0,t) = u_\alpha^\delta(\pi,t) = 0, & t \in (0,T), \\ u_\alpha^\delta(x,0) = 0, & x \in (0,\pi), \\ u_\alpha^\delta(x,T) + \alpha Lf_\alpha^\delta(x) = g^\delta(x), & x \in (0,\pi), \end{cases}$$

where $\alpha > 0$ is a regularization parameter.

Through the process of separating variables, it is evident that $u_\alpha^\delta(x,t)$ takes on the following form:

$$u_\alpha^\delta(x,t) = \sum_{k=0}^{\infty} \frac{f_{2k+1}}{\lambda_{2k+1}} z_{2k+1}(t)\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{f_{2k}}{\lambda_{2k}} z_{2k}(t)\varphi_{2k}(x). \tag{35}$$

Then we get (in the case of inexact data g^δ):

$$f_\alpha^\delta(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{2k+1}^\delta \varphi_{2k+1}(x)}{\alpha\lambda_{2k+1}^2 + z_{2k+1}(T)} + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k}^\delta \varphi_{2k}(x)}{\alpha\lambda_{2k}^2 + z_{2k}(T)}, \tag{36}$$

and (in the case of inexact data g):

$$f_\alpha(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{2k+1} \varphi_{2k+1}(x)}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k} \varphi_{2k}(x)}{\alpha \lambda_{2k}^2 + z_{2k}(T)}. \tag{37}$$

In the subsequent analysis, we present two convergence estimates for $\left\| f_\alpha^\delta - f \right\|$ while employing both an a priori and a posteriori choice rule for the regularization parameter. These convergence estimates are instrumental in evaluating the accuracy and efficacy of the regularization process in approximating the true solution $f(x)$ as we progress with the solution method.

3.1. Convergence estimate under an a priori regularization parameter choice rule

THEOREM 2. *Assuming that the a priori condition* (33) *and the noise assumption* (5) *are satisfied, then:*

1. *For* $0 < p < 4$. *If we choose* $\alpha = \left(\frac{\delta}{E}\right)^{\frac{4}{p+2}}$, *we obtain the following convergence estimate*
$$\left\| f_\alpha^\delta - f \right\| \leqslant \left(\sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} + C_1 \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

2. *For* $p \geqslant 4$. *If we choose* $\alpha = \left(\frac{\delta}{E}\right)^{\frac{2}{3}}$, *we obtain the following convergence estimate*
$$\left\| f_\alpha^\delta - f \right\| \leqslant \left(\sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} + C_2 \right) E^{\frac{1}{3}} \delta^{\frac{2}{3}},$$

where C_1, C_2 *are positive constants depending on* p, γ, T *and* ε_0.

Proof. By using the triangle inequality, we have

$$\left\| f_\alpha^\delta - f \right\| \leqslant \left\| f_\alpha^\delta - f_\alpha \right\| + \left\| f_\alpha - f \right\|. \tag{38}$$

Let us begin by providing an estimate for the first term. Using (36), (37) and (5), we have

$$\left\| f_\alpha^\delta - f_\alpha \right\|^2 = \sum_{k=0}^{\infty} \left(\frac{(g_{2k+1}^\delta - g_{2k+1}) \lambda_{2k+1}}{\alpha \lambda_{2k+1}^2 + z_{2k+1}(T)} \right)^2 + \sum_{k=1}^{\infty} \left(\frac{(g_{2k}^\delta - g_{2k}) \lambda_{2k}}{\alpha \lambda_{2k}^2 + z_{2k}(T)} \right)^2, \tag{39}$$

then

$$\left\| f_\alpha^\delta - f_\alpha \right\|^2 \leqslant \left(\left(\sup_{k \geqslant 0} A_{2k+1} \right)^2 + \left(\sup_{k \geqslant 1} A_{2k} \right)^2 \right) \delta^2, \tag{40}$$

By using (21) and (22), we derive

$$\sup_{k\geqslant 0} A_{2k+1} \leqslant \frac{1}{2\sqrt{a_1}}\frac{1}{\sqrt{\alpha}}, \tag{41}$$

and

$$\sup_{k\geqslant 1} A_{2k} \leqslant \frac{1}{2\sqrt{a_2}}\frac{1}{\sqrt{\alpha}}. \tag{42}$$

Substituting (41) and (42) in (40), we obtain

$$\left\|f_\alpha^\delta - f_\alpha\right\| \leqslant \frac{\delta}{\sqrt{\alpha}}\sqrt{\frac{1}{4a_1}+\frac{1}{4a_2}}. \tag{43}$$

Let us now estimate the second term in (38). Applying the a priori bound condition (33), we obtain

$$\|f_\alpha - f\|^2 \leqslant E^2\left(\left(\sup_{k\geqslant 0} B_{2k+1}\right)^2 + \left(\sup_{k\geqslant 1} B_{2k}\right)^2\right). \tag{44}$$

Now, using (31) and (32) combined with (43), we obtain

$$\left\|f_\alpha^\delta - f\right\| \leqslant \sqrt{\frac{1}{4a_1}+\frac{1}{4a_2}}\frac{\delta}{\sqrt{\alpha}} + \begin{cases} \sqrt{b_1^2+c_1^2}E\alpha^{\frac{p}{4}}, & 0<p<4,\\ \sqrt{b_2^2+c_2^2}E\alpha, & p\geqslant 4. \end{cases}$$

Choosing the regularization parameter α by

$$\alpha = \begin{cases} \left(\frac{\delta}{E}\right)^{\frac{4}{p+2}}, & 0<p<4,\\ \left(\frac{\delta}{E}\right)^{\frac{2}{3}}, & p\geqslant 4. \end{cases}$$

Finally, we obtain

$$\left\|f_\alpha^\delta - f\right\| \leqslant \begin{cases} \left(\sqrt{\frac{1}{4a_1}+\frac{1}{4a_2}}+C_1\right)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, & 0<p<4,\\ \left(\sqrt{\frac{1}{4a_1}+\frac{1}{4a_2}}+C_2\right)E^{\frac{1}{3}}\delta^{\frac{2}{3}}, & p\geqslant 4. \end{cases}$$

The proof is completed. $\square$

3.2. Convergence estimate under an a posteriori regularization parameter choice rule

In this subsection, we adopt an a posteriori regularization parameter choice strategy, specifically Morozov's discrepancy principle, to determine the appropriate value of the regularization parameter α. By leveraging the conditional stability estimate outlined in Theorem 1, we derive a convergence rate for the regularized solution (36). This convergence rate serves as a valuable measure of the accuracy and reliability of the regularization process in approximating the desired solution.

Morozov's discrepancy principle in our context entails identifying α under the condition:

$$\left\|Kf_\alpha^\delta - g^\delta\right\| = \tau\delta. \tag{45}$$

Here, $\tau > 1$ is a constant. As per the subsequent Lemma, it becomes evident that (45) possesses a unique solution if $\|g^\delta\| > \tau\delta > 0$.

LEMMA 7. *Set* $\rho(\alpha) = \|Kf_\alpha^\delta - g^\delta\|$, *we have the following properties:*

(a) $\rho(\alpha)$ *is a continuous function,*

(b) $\lim_{\alpha\to 0}\rho(\alpha) = 0$,

(c) $\lim_{\alpha\to\infty}\rho(\alpha) = \|g^\delta\|$,

(d) $\rho(\alpha)$ *is a monotonically increasing function on* $(0,\infty)$.

Proof. The proofs are based on Parseval's equality and technical calculations of real analysis (see [50]). $\square$

THEOREM 3. *Under the assumption that the a priori condition* (33) *and the noise assumption* (5) *are met, and if there exists a constant* $\tau > 1$ *such that* $\|g^\delta\| > \tau\delta > 0$, *and furthermore, the regularization parameter* $\alpha > 0$ *is chosen using Morozov's discrepancy principle* (45), *we can conclude the following:*

1. *For* $0 < p < 2$, *we have the following convergence estimate*

$$\left\|f_\alpha^\delta - f\right\| \leqslant \left(C(\tau+1)^{\frac{p}{p+2}} + \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}}\left(\frac{\sqrt{\widetilde{b}_1^2 + \widetilde{b}_2^2}}{\tau - 1}\right)^{\frac{2}{p+2}}\right) E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

2. *For* $p \geqslant 2$, *we have the following convergence estimate*

$$\left\|f_\alpha^\delta(x) - f(x)\right\| \leqslant \left(C(\tau+1)^{\frac{p}{p+2}} + \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}}\left(\frac{\sqrt{\widetilde{c}_1^2 + \widetilde{c}_2^2}}{\tau - 1}\right)^{\frac{1}{2}}\right) E^{\frac{1}{2}}\delta^{\frac{1}{2}}.$$

Proof. We have

$$\left\|f_\alpha^\delta - f\right\| \leqslant \left\|f_\alpha^\delta - f_\alpha\right\| + \|f_\alpha - f\|.$$

Let us start by providing an estimate for the second term. We have

$$\mathbf{K}f_\alpha^\delta(x) - g^\delta(x) = \sum_{k=0}^{\infty} \frac{-\alpha\lambda_{2k+1}^2 g_{2k+1}^\delta}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{-\alpha\lambda_{2k}^2 g_{2k}^\delta}{z_{2k}(T) + \alpha\lambda_{2k}^2} \varphi_{2k}(x),$$

$$\begin{aligned}
&\mathbf{K}\left(f_\alpha(x) - f(x)\right) \\
=&\Delta_1 + \Delta_2 \\
=&\sum_{k=0}^{\infty} \frac{-\alpha\lambda_{2k+1}^2\left(g_{2k+1} - g_{2k+1}^\delta\right)}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{-\alpha\lambda_{2k}^2\left(g_{2k} - g_{2k}^\delta\right)}{z_{2k}(T) + \alpha\lambda_{2k}^2} \varphi_{2k}(x) \\
&+ \sum_{k=0}^{\infty} \frac{-\alpha\lambda_{2k+1}^2 g_{2k+1}^\delta}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{-\alpha\lambda_{2k}^2 g_{2k}^\delta}{z_{2k}(T) + \alpha\lambda_{2k}^2} \varphi_{2k}(x).
\end{aligned}$$

By using (5) and (45), we can write

$$\begin{aligned}
\|\Delta_1\|^2 &= \sum_{k=0}^{\infty} \left[\frac{-\alpha\lambda_{2k+1}^2\left(g_{2k+1} - g_{2k+1}^\delta\right)}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2}\right]^2 + \sum_{k=1}^{\infty} \left[\frac{-\alpha\lambda_{2k}^2\left(g_{2k} - g_{2k}^\delta\right)}{z_{2k}(T) + \alpha\lambda_{2k}^2}\right]^2 \\
&\leqslant \|g - g^\delta\|^2 \leqslant \delta^2,
\end{aligned}$$

and

$$\|\Delta_2\|^2 = \|\mathbf{K}f_\alpha^\delta - g^\delta\|^2 = (\tau\delta)^2,$$

By the triangle inequality $\|\mathbf{K}(f_\alpha - f)\| = \|\Delta_1 + \Delta_2\| \leqslant \|\Delta_1\| + \|\Delta_2\|$, we obtain the desired estimate

$$\|\mathbf{K}(f_\alpha - f)\| \leqslant \delta + \tau\delta = (\tau + 1)\delta. \tag{46}$$

By employing the a priori bound condition for f, we get

$$\begin{aligned}
&\|f_\alpha - f\|_{\frac{p}{2}}^2 \\
=&\sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{2k+1}}{z_{2k+1}(T)} \frac{\alpha\lambda_{2k+1}^2}{\alpha\lambda_{2k+1}^2 + z_{2k+1}(T)}\right)^2 \lambda_{2k+1}^p + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)} \frac{\alpha\lambda_{2k}^2}{\alpha\lambda_{2k}^2 + Z_{2k}(T)}\right)^2 \lambda_{2k}^p \\
\leqslant&\sum_{k=0}^{\infty} \left(\frac{\lambda_{2k+1} g_{2k+1}}{z_{2k+1}(T)}\right)^2 \lambda_{2k+1}^p + \sum_{k=1}^{\infty} \left(\frac{\lambda_{2k} g_{2k}}{z_{2k}(T)}\right)^2 \lambda_{2k}^p = \|f\|_{\frac{p}{2}}^2 \leqslant E^2.
\end{aligned}$$

By (34) and (46), we deduce that

$$\|f_\alpha - f\| \leqslant C(\tau + 1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{47}$$

Now, let us establish the bound for the first term. Analogous to (43), we obtain

$$\left\| f_\alpha^\delta - f_\alpha \right\| \leqslant \sqrt{\frac{1}{4a_1} + \frac{1}{4a_2}} \frac{\delta}{\sqrt{\alpha}}. \tag{48}$$

From (45), there holds

$$\begin{aligned}
\tau\delta &= \left\| \sum_{k=0}^{\infty} \frac{\alpha\lambda_{2k+1}^2 g_{2k+1}^\delta \varphi_{2k+1}}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha\lambda_{2k}^2 g_{2k}^\delta \varphi_{2k}}{z_{2k}(T) + \alpha\lambda_{2k}^2} \right\| \\
&\leqslant \left\| \sum_{k=0}^{\infty} \frac{\alpha\lambda_{2k+1}^2 \left(g_{2k+1}^\delta - g_{2k+1}\right) \varphi_{2k+1}}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha\lambda_{2k}^2 \left(g_{2k}^\delta - g_{2k}\right) \varphi_{2k}}{z_{2k}(T) + \alpha\lambda_{2k}^2} \right\| \\
&\quad + \left\| \sum_{k=0}^{\infty} \frac{\alpha\lambda_{2k+1}^2 g_{2k+1} \varphi_{2k+1}(x)}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha\lambda_{2k}^2 g_{2k} \varphi_{2k}}{z_{2k+1}(T) + \alpha\lambda_{2k}^2} \right\| \\
&\leqslant \delta + J.
\end{aligned} \tag{49}$$

Once again, by employing the a priori bound condition for f, we get

$$\begin{aligned}
J^2 &= \left\| \sum_{k=0}^{\infty} \frac{\alpha\lambda_{2k+1}^2 g_{2k+1} \varphi_{2k+1}}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} + \sum_{k=1}^{\infty} \frac{\alpha\lambda_{2k}^2 g_{2k} \varphi_{2k}}{z_{2k}(T) + \alpha\lambda_{2k}^2} \right\|^2 \\
&\leqslant \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}^2 g_{2k+1}}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \right)^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}^2 g_{2k}}{z_{2k}(T) + \alpha\lambda_{2k}^2} \right)^2 \\
&\leqslant \sum_{k=0}^{\infty} \left(\frac{\alpha\lambda_{2k+1}^2 g_{2k+1}}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \frac{z_{2k+1}(T)\lambda_{2k+1}^{\frac{p}{2}}}{z_{2k+1}(T)\lambda_{2k+1}^{\frac{p}{2}}} \right)^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha\lambda_{2k}^2 g_{2k}}{z_{2k}(T) + \alpha\lambda_{2k}^2} \frac{z_{2k}(T)\lambda_{2k}^{\frac{p}{2}}}{z_{2k}(T)\lambda_{2k}^{\frac{p}{2}}} \right)^2 \\
&\leqslant E^2 \sup_{k \geqslant 0} \left(\frac{\alpha\lambda_{2k+1} z_{2k+1}(T)}{z_{2k+1}(T) + \alpha\lambda_{2k+1}^2} \right)^2 \frac{1}{\lambda_{2k+1}^p} + E^2 \sup_{k \geqslant 1} \left(\frac{\alpha\lambda_{2k} z_{2k}(T)}{z_{2k}(T) + \alpha\lambda_{2k}^2} \right)^2 \frac{1}{\lambda_{2k}^p} \\
&\leqslant E^2 \sup_{k \geqslant 0} \left(\frac{\alpha\lambda_{2k+1}^{1-\frac{p}{2}}}{a_1 + \alpha\lambda_{2k+1}^2} \right)^2 + E^2 \sup_{k \geqslant 1} \left(\frac{\alpha\lambda_{2k}^{1-\frac{p}{2}}}{a_2 + \alpha\lambda_{2k}^2} \right)^2 \\
&\leqslant \begin{cases} \left(\sqrt{\widetilde{b}_1^2 + \widetilde{c}_1^2} E \alpha^{\frac{p+2}{4}} \right)^2, & 0 < p < 2, \\ \left(\sqrt{\widetilde{b}_2^2 + \widetilde{c}_2^2} E \alpha \right)^2, & p \geqslant 2. \end{cases}
\end{aligned} \tag{50}$$

Combining (49)–(50), we obtain

$$(\tau - 1)\delta \leqslant \begin{cases} \sqrt{\widetilde{b}_1^2 + \widetilde{c}_1^2} E \alpha^{\frac{p+2}{4}}, & 0 < p < 2, \\ \sqrt{\widetilde{b}_2^2 + \widetilde{c}_2^2} E \alpha, & p \geqslant 2. \end{cases}$$

It is clear that

$$\frac{1}{\alpha} \leqslant \begin{cases} \left(\frac{\sqrt{\widetilde{b}_1^2+\widetilde{c}_1^2}}{\tau-1}\right)^{\frac{4}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{4}{p+2}}, & 0<p<2, \\ \frac{\sqrt{\widetilde{b}_2^2+\widetilde{c}_2^2}}{\tau-1}\frac{E}{\delta}, & p \geqslant 2. \end{cases} \tag{51}$$

Substituting (51) into (48), we obtain

$$\left\| f_\alpha^\delta - f_\alpha \right\| \leqslant \sqrt{\frac{1}{a_1}+\frac{1}{4a_2}} \begin{cases} \left(\frac{\sqrt{\widetilde{b}_1^2+\widetilde{c}_1^2}}{\tau-1}\right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0<p<2, \\ \left(\frac{\sqrt{\widetilde{b}_2^2+\widetilde{c}_2^2}}{\tau-1}\right)^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \geqslant 2. \end{cases} \tag{52}$$

Finally, by combining (47) and (52), we derive

$$\begin{aligned} \left\| f_\alpha^\delta - f \right\| \leqslant & C(\tau+1)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} \\ & + \sqrt{\frac{1}{4a_1}+\frac{1}{4a_2}} \begin{cases} \left(\frac{\sqrt{\widetilde{b}_1^2+\widetilde{c}_1^2}}{\tau-1}\right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0<p<2, \\ \left(\frac{\sqrt{\widetilde{b}_2^2+\widetilde{c}_2^2}}{\tau-1}\right)^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \geqslant 2. \end{cases} \end{aligned}$$

This completes the proof. □

4. Quasi-reversibility method and convergence rates

In this section, motivated by the regularizing strategy developed in [47] and the recent work [16], we present a novel quasi-reversibility method for solving problem (1) with the given perturbation (5). We provide two convergence estimates, each of which is obtained through distinct regularization parameter choice rules. The first estimate is derived using an a priori regularization parameter choice rule, whereas the second estimate is based on a posteriori regularization parameter choice rule. These convergence estimates are essential for assessing the accuracy and effectiveness of the proposed quasi-reversibility method in approximating the solutions to the given problem with a specific perturbation.

Let $u_\alpha^\delta(x,t)$ be the solution to the following regularized problem

$$\begin{cases} D_t^\gamma \left[u_\alpha^\delta(x,t) + Lu_\alpha^\delta(x,t)\right] + Lu_\alpha^\delta(x,t) = f(x) + \alpha L f(x), & \\ u_\alpha^\delta(0,t) = u_\alpha^\delta(\pi,t) = 0, & t \in (0,T), \\ u_\alpha^\delta(x,0) = 0, & x \in (0,\pi), \\ u_\alpha^\delta(x,T) = g^\delta(x), & x \in (0,\pi), \end{cases}$$

where $\alpha > 0$ is a regularization parameter.

We apply the separation of variables method, it becomes clear that $u_\alpha^\delta(x,t)$ takes on the subsequent form:

$$u_\alpha^\delta(x,t) = \sum_{k=0}^{\infty} \frac{(1+\alpha\lambda_{2k+1}) f_{2k+1}}{\lambda_{2k+1}} z_{2k+1}(t)\varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{(1+\alpha\lambda_{2k}) f_{2k}}{\lambda_{2k}} z_{2k}(t)\varphi_{2k}(x) \tag{53}$$

Then, we get (in the case of exact data)

$$f_\alpha(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{2k+1}}{(1+\alpha\lambda_{2k+1})} \frac{1}{z_{2k+1}(T)} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k}}{(1+\alpha\lambda_{2k})} \frac{1}{z_{2k}(T)} \varphi_{2k}(x). \tag{54}$$

and (in the case of inexact data)

$$f_\alpha^\delta(x) = \sum_{k=0}^{\infty} \frac{\lambda_{2k+1} g_{2k+1}^\delta}{(1+\alpha\lambda_{2k+1})} \frac{1}{z_{2k+1}(T)} \varphi_{2k+1}(x) + \sum_{k=1}^{\infty} \frac{\lambda_{2k} g_{2k}^\delta}{(1+\alpha\lambda_{2k})} \frac{1}{z_{2k}(T)} \varphi_{2k}(x). \tag{55}$$

4.1. Convergence estimate under an a priori regularization parameter choice rule

THEOREM 4. *Assuming that the a priori condition* (33) *and the noise assumption* (5) *are satisfied, then:*

1. *For* $0 < p < 2$. *If we choose* $\alpha = \left(\frac{\delta}{E}\right)^{\frac{2}{p+2}}$, *we obtain the following convergence estimate*
$$\left\| f_\alpha^\delta - f \right\| \leqslant \left(\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} + C_1 \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

2. *For* $p \geqslant 2$. *If we choose* $\alpha = \left(\frac{\delta}{E}\right)^{\frac{1}{2}}$, *we obtain*
$$\left\| f_\alpha^\delta - f \right\| \leqslant \left(\sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}} + C_2 \right) E^{\frac{1}{2}} \delta^{\frac{1}{2}},$$

   where C_1, C_2 *are positive constants depending on* p, γ, T *and* ε_0.

Proof. The proof is similar to that used in [47, Theorem 4.1], only in our case, we have two quantities, one indexed by an even index and the other by an odd index, and the constants are different. □

4.2. Convergence estimate under an a posteriori regularization parameter choice rule

In this subsection, we employ an a posteriori regularization parameter choice rule to determine the value of the regularization parameter α, for which the regularized solution (55) converges to the exact solution (54).

We adopt the discrepancy principle in the following manner:

$$\left\|\alpha L(1+\alpha L)^{-1}\left(Kf_{\alpha}^{\delta}(x)-g^{\delta}(x)\right)\right\|=\tau\delta. \tag{56}$$

Here, $\tau>1$ is a constant. As per the Lemma below, a unique solution exists for (56) when $\|g^{\delta}\|>\tau\delta>0$.

LEMMA 8. [47] *Set* $\rho(\alpha)=\left\|\alpha L(1+\alpha L)^{-1}\left(Kf_{\alpha}^{\delta}(x)-g^{\delta}(x)\right)\right\|$.
If $\|g^{\delta}\|>\tau\delta>0$, *then the following statements holds:*

(a) $\rho(\alpha)$ *is a continuous function,*

(b) $\lim_{\alpha\to 0}\rho(\alpha)=0$,

(c) $\lim_{\alpha\to\infty}\rho(\alpha)=\|g^{\delta}\|$,

(d) $\rho(\alpha)$ *is a monotonically increasing function on* $(0,\infty)$.

Proof. The proof is a straightforward result based on the fact

$$\rho(\alpha)=\left(\sum_{k=0}^{\infty}\left(\frac{\alpha\lambda_{2k+1}}{1+\alpha\lambda_{2k+1}}\right)^{4}\left(g_{2k+1}^{\delta}\right)^{2}+\sum_{k=1}^{\infty}\left(\frac{\alpha\lambda_{2k}}{1+\alpha\lambda_{2k}}\right)^{4}\left(g_{2k}^{\delta}\right)^{2}\right)^{\frac{1}{2}}. \quad \square$$

Now, using Lemma 8 with help of (19), (25), (26) along with Hölder's inequality, we derive the following result.

THEOREM 5. *Assuming that the a priori condition (33) and the noise assumption (5) are satisfied, and that there exists a constant* $\tau>1$ *such that* $\|g^{\delta}\|>\tau\delta>0$, *the regularization parameter* $\alpha>0$ *is selected using the discrepancy principle (56). Under these conditions, we have*

1. *For* $0<p<2$, *we obtain the following convergence estimate*

$$\left\|f_{\alpha}^{\delta}-f\right\|\leqslant\left(C(\tau+1)^{\frac{p}{p+2}}+\sqrt{\frac{1}{a_1^2}+\frac{1}{a_2^2}}\left(\frac{(\hat{b}_1^2+\hat{c}_1^2)}{\tau-1}\right)^{\frac{2}{p+2}}\right)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}},$$

2. *For* $p\geqslant 2$, *we obtain*

$$\left\|f_{\alpha}^{\delta}(x)-f(x)\right\|\leqslant\left(C(\tau+1)^{\frac{1}{2}}+\sqrt{\frac{1}{a_1^2}+\frac{1}{a_2^2}}\left(\frac{(\hat{b}_2^2+\hat{c}_2^2)}{\tau-1}\right)^{\frac{1}{2}}\right)E^{\frac{1}{2}}\delta^{\frac{1}{2}},$$

where

$$\hat{b}_2=\frac{1}{\lambda_1^{\frac{p-2}{2}}},\quad \hat{c}_2=\frac{1}{\lambda_2^{\frac{p-2}{2}}},\quad \hat{b}_1=\hat{c}_1=\frac{\left(\frac{4}{p+2}\right)^{\frac{2-p}{2}}}{\frac{4}{p+2}+1}\leqslant 1,\quad C=(a_1^{\frac{-p}{p+2}}+a_2^{\frac{-p}{p+2}}).$$

Proof. The proof is analogous to that used in [47, Theorem 4.2] with some additional simple calculations. $\square$

Conclusion

In this study, we extended two regularization methods previously applied to a fractional source identification problem [47, 50] to the same problem with an involutive term. We demonstrated convergence results under certain regularity conditions on the sought function using two different strategies (a priori and a posteriori). This study opens new avenues for numerical simulation to observe the influence of the involution term and may be extended to higher-dimensional problems, where $x = (x_1, x_2) \in [0,\pi] \times [0,\pi]$.

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(Received February 9, 2024)

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