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DETERMINATION OF INITIAL DATA IN TIME–FRACTIONAL WAVE EQUATION

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Abstract. In this paper, we consider a time-fractional wave equation for positive operators, including the classical Laplacian with the Dirichlet boundary condition. Determinations of initial velocity and perturbation are investigated. It is also shown that these inverse problems of determining the initial data are ill-posed. Moreover, under some conditions of well-posedness properties of the inverse problems are proved. As an appendix, we also provide some proof of the direct problems. Here, we develop the theoretical part of the inverse problems of finding the initial data for the time-fractional wave equations.

1. Introduction

In this paper, for $T > 0$ we study the time-fractional wave equation

$$\mathscr{D}_t^{\alpha} u(t) + \mathscr{L} u(t) = 0, \text{ for all } t \in (0,T), \tag{1}$$

for some general operator $\mathscr{L}$ acting on a general separable Hilbert space $\mathscr{H}$ (for simplicity, we can think about the classical Laplacian with the Dirichlet boundary condition acting on L^2). Here $\mathscr{D}_t^{\alpha}$ is the Caputo derivative (see [13]) with the order $1 < \alpha \leqslant 2$ and, in particular, $\mathscr{D}_t^{\alpha} := \partial_t^2$ for $\alpha = 2$. The Caputo fractional derivative of order $1 < \alpha < 2$ for any differentiable function g with the absolutely continuous g' on $[0,T]$ is defined by

$$\mathscr{D}_t^{\alpha}[g](t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} g''(s) ds, \quad t \in [0,T],$$

where $\Gamma(\cdot)$ is the Euler gamma function.

A physical background of the fractional in time evolutionary equations can be found in the book [10] with applications in classical mechanics, quantum mechanics, hadron spectroscopy, nuclear physics, and quantum field theory. For further discussions on the applications of the fractional wave equation we cite [16] and references therein.

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Here, we will make the following assumptions on the general operator $\mathscr{L}$. Let $\mathscr{H}$ be a separable Hilbert space and, suppose that $\mathscr{L}$ is a positive operator with the discrete spectrum $\{\lambda_\xi > 0 : \xi \in \mathscr{I}\}$ on $\mathscr{H}$, where $\mathscr{I}$ is a countable set. The main assumption in this paper is that the system of eigenfunctions $\{e_\xi \in \mathscr{H} : \xi \in \mathscr{I}\}$ of the operator $\mathscr{L}$ forms a complete orthonormal basis in the space $\mathscr{H}$.

In what follows, we will use the following property of the special function $E_{\alpha,\beta}$ and definitions related to the functional spaces.

LEMMA 1. [21, Theorem 1.6] *Suppose that $\alpha < 2$, β is an arbitrary real number and, $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then for the Mittag-Leffler function*

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)},$$

there exists a positive constant C such that

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|} \leqslant C, \tag{2}$$

for all $\mu \leqslant |arg(z)| \leqslant \pi$ and $|z| \geqslant 0$.

From [19,22,24], [23, Theorems 2.1.1 and 6.1.1] we formulate the following properties of the Mittag-Leffler function $E_{\alpha,2}(z)$.

LEMMA 2. *Suppose that $1 < \alpha < 2$.*

- *If $1 < \alpha \leqslant \frac{4}{3}$ then $E_{\alpha,2}(z)$ has no real-valued zeros;*
- *If $\frac{4}{3} < \alpha < 2$ then $E_{\alpha,2}(z)$ has a finite number of real-valued zeros;*
- *If $1 < \alpha < 2$ then $E_{\alpha,1}(z)$ has a finite number of real-valued zeros.*

For more details on zeros of the function $E_{\alpha,\beta}$, the reader is referred to the historical works [3, 4, 19, 28, 29] and to the relatively modern literature [8, 17, 22–24] and references therein.

DEFINITION 1. ([18]) Let $0 \leqslant \gamma < \infty$. We put $\mathscr{H}^\gamma := \{u \in \mathscr{H} : \mathscr{L}^\gamma u \in \mathscr{H}\}$. Then for $1 < \alpha \leqslant 2$ we denote by $\mathscr{B}^\alpha([0,T];\mathscr{H}^\gamma)$ the space of all continuous in time functions $g(t) \in \mathscr{H}^\gamma$ such that

$$\|g\|_{\mathscr{B}^\alpha([0,T];\mathscr{H}^\gamma)} := \max_{t\in[0,T]} \|g(t)\|_{\mathscr{H}^\gamma} + \max_{t\in[0,T]} \|\mathscr{D}_t^\alpha g(t)\|_{\mathscr{H}^\gamma} < \infty.$$

The space $\mathscr{B}^\alpha([0,T];\mathscr{H}^\gamma)$ equipped with the norm above is a Banach space.

1.1. Problem statements

First, we start by stating the direct problem, the properties of which we are going to use in studying the inverse problems of determining the initial data.

PROBLEM 1. (Direct Problem) Find a function $u \in \mathscr{B}^{\alpha}([0,T];\mathscr{H}) \cap C([0,T];\mathscr{H}^1)$ satisfying the equation (1), and the Cauchy conditions

$$u(0) = \varphi \in \mathscr{H}, \tag{3}$$

$$u_t(0) = \psi \in \mathscr{H}. \tag{4}$$

Here $\mathscr{H}^1 = \{u \in \mathscr{H} : \mathscr{L}u \in \mathscr{H}\}$.

The well–posed result of this direct problem is given in Appendix 3. In the main part of the paper, based on Problem 1, we will consider the following inverse problems of finding the initial conditions $u(0)$ and $u_t(0)$ from the given additional information $u(T)$ for some $T > 0$.

PROBLEM 2. (Inverse Problem 1) Find a pair of functions $(u(t), \psi)$ satisfying the equation (1) and the condition (4) from the condition (3) and additional information

$$u(T) = \phi \in \mathscr{H}. \tag{5}$$

REMARK 1. We note that instead of the condition (5) it can be considered the following intermediate condition

$$u(\tau) = \phi \in \mathscr{H}, \tag{6}$$

for any $\tau \in [0,T]$. Then Inverse Problem 1 can be studied as two joint problems: the first as an inverse problem in $t \in [0,\tau]$, and the second as a direct problem in $t \in [\tau,T]$.

PROBLEM 3. (Inverse Problem 2) Find a pair of functions $(u(t), \varphi)$ satisfying the equation (1) and the condition (3) from the condition (4) and additional data (5).

It is worth mentioning that the backward problems for the wave equation with observation in a finite time have been already investigated in the papers [2, 26], and for the time-fractional diffusion-wave equations we address to [5, 6, 11, 27, 30–32] just a few of them.

The well-posedness of the backward problem in time for the time-fractional diffusion equation firstly was established in [27]. After this work, as for $0 < \alpha < 1$, there were done a numerous number of theoretical and numerical works on the backward problems. And here especially we refer to [5] since there the well-posedness of the backward problem in time was proved for the time-fractional diffusion equation with non-symmetric elliptic operators generalising the existing results for the symmetric operators.

In [11, 30, 31] the authors studied the numerical algorithms of solving the inverse problems of determining the initial data for the time-fractional wave equations.

In [6, 32] it was considered recovering initial data for the time-fractional wave equation from the final data. The well-posedness of this inverse problem was established theoretically in [6], and in [32] it was investigated from both theoretical and numerical points of view.

In this paper, we develop the theoretical part of the inverse problems of determining the initial data for time-fractional wave equation for the abstract operator $\mathscr{L}$. The main difference of our current paper from above mentioned papers is that the operator $\mathscr{L}$ can be non-elliptic operator. For instance, the Landau Hamiltonian operator is an example of the non-elliptic operator, for more discussions and examples, see [25].

Now, we construct an example, which is showing that the stability of the solution of Inverse Problem 2 is disrupted. Let us put $\mathscr{L} := -\frac{\partial^2}{\partial x^2}$, $x \in (0,1)$, $\varphi \equiv 0$ and $\phi = \sin k\pi x$, $k \in \mathbb{N}$. By direct calculations, it is not difficult to see that the following pair of two functions

$$u(t,x) = \frac{tE_{\alpha,2}(-k\pi t^{\alpha})}{TE_{\alpha,2}(-k\pi T^{\alpha})} \sin k\pi x,$$

and

$$\psi(x) = \frac{\sin k\pi x}{TE_{\alpha,2}(-k\pi T^{\alpha})}$$

is a solution of Inverse Problem 2. Well–known that the Mittag-Leffler function $E_{\alpha,2}(z)$ has a finite number of real-valued zeros if $1 < \alpha < 2$. It is clear that $E_{\alpha,2}(-k\pi T^{\alpha})$ is equal to zero if $-k\pi T^{\alpha}$ coincides with the zeros of $E_{\alpha,2}(z)$. Therefore, the solution of Inverse Problem 2 is unstable. Similarly, one can show the instability of the solution to Inverse Problem 3. Consequently, Inverse Problems 2 and 3 are ill-posed in the sense of Hadamard.

The main purpose of this paper is to find well-posedness conditions and develop the theoretical part of the inverse problems of determining the initial data for the time-fractional wave equations, preceding the numerical algorithms.

1.2. Inverse initial velocity problem

In this subsection, we study Problem 2. First, we note that by Lemma 2 the Mittag–Leffler function $E_{\alpha,2}$ has no zeros for $1 < \alpha \leqslant \frac{4}{3}$ and a finite number of zeros for $\frac{4}{3} < \alpha < 2$. Let us denote by $\Theta := \{\theta_\eta\}_{\eta \in \mathscr{F}}$ the set of all negative zeros of the function $E_{\alpha,2}$, where $\mathscr{F}$ is some finite set. Also, denote by

$$\Lambda := \left\{ \left(-\frac{\theta_\eta}{\lambda_\xi} \right)^{\frac{1}{\alpha}} \right\}_{\xi \in \mathscr{I}, \eta \in \mathscr{F}}.$$

Then we say that the set $\mathbb{T} := \mathbb{R}_- \setminus \Lambda$ is admissible. Indeed, the set $\mathbb{T}$ is countable due to Θ and Λ are countable sets.

THEOREM 1. *Assume that* $\varphi \in \mathscr{H}, \phi \in \mathscr{H}^1$, *and* $T > 0$.

(i) Let $1 < \alpha \leqslant \frac{4}{3}$*;*

(ii) *Let* $\frac{4}{3} < \alpha < 2$*. Suppose that* T *is a sufficiently large number or from the admissible set* $\mathbb{T}$*.*

Then there exists a unique solution $(u(t), \psi)$ *of Problem 2 such that* $u \in \mathscr{B}^{\alpha}([0,T];\mathscr{H}) \cap C([0,T];\mathscr{H}^1)$ *and* $\psi \in \mathscr{H}$*. This solution can be written in the form*

$$u(t) = \sum_{\xi\in\mathscr{I}} \left[\varphi_\xi \left(E_{\alpha,1}(-\lambda_\xi t^\alpha) - \frac{E_{\alpha,1}(-\lambda_\xi T^\alpha)}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)} tE_{\alpha,2}(-\lambda_\xi t^\alpha) \right) + \phi_\xi \frac{tE_{\alpha,2}(-\lambda_\xi t^\alpha)}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)} \right] e_\xi,$$

for all $t \in [0,T]$*, and*

$$\psi = \sum_{\xi\in\mathscr{I}} \frac{\left(\phi_\xi - \varphi_\xi E_{\alpha,1}(-\lambda_\xi T^\alpha)\right) e_\xi}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)},$$

where $\varphi_\xi = (\varphi, e_\xi)_{\mathscr{H}}$ *and* $\phi_\xi = (\phi, e_\xi)_{\mathscr{H}}$*.*

REMARK 2. Below we discuss some points from Theorem 1.

- Note that, in case $1 < \alpha \leqslant \frac{4}{3}$, Problem 2 is well-posed without additional conditions on T. Here we are helped by the fact that the Mittag-Leffler function $E_{\alpha,2}$ does not have zeros for $1 < \alpha \leqslant \frac{4}{3}$, proved by Pskhu in [24].

- If $\alpha = 2$ and $\mathscr{L} := \frac{\partial^2}{\partial x^2}$, Theorem 1 coincides with the results by Sabitov et. al in [26].

1.3. Inverse initial perturbation problem

In this subsection, we study Problem 3. First, we note that for $1 < \alpha < 2$ the Mittag-Leffler function $E_{\alpha,1}$ has a finite number of real–valued zeros (see, for example, [8]).

Let us denote by $K := \{\kappa_\eta\}_{\eta\in\mathscr{F}}$ the set of all negative zeros of the function $E_{\alpha,1}$, where $\mathscr{F}$ is some finite set. Also, denote by

$$\Delta := \left\{ \left(-\frac{\kappa_\eta}{\lambda_\xi} \right)^{\frac{1}{\alpha}} \right\}_{\xi\in\mathscr{I},\eta\in\mathscr{F}}.$$

Then we say that the set

$$\mathbb{X} := \mathbb{R}_- \setminus \Delta$$

is admissible. Indeed, the set $\mathbb{X}$ is countable due to K and Δ are so.

THEOREM 2. *Suppose that $T > 0$ is a sufficiently large number or from the admissible set $\mathbb{X}$. Let us assume that $\psi \in \mathscr{H}$, $\phi \in \mathscr{H}^1$. Then there exists a unique solution $(u(t), \varphi)$ of Problem 3 such that $u \in \mathscr{B}^\alpha([0,T];\mathscr{H}) \cap C([0,T];\mathscr{H}^1)$ and $\varphi \in \mathscr{H}$. This solution can be written in the form*

$$u(t) = \sum_{\xi \in \mathscr{I}} \left[\psi_\xi \left(t E_{\alpha,2}(-\lambda_\xi t^\alpha) - \frac{T E_{\alpha,2}(-\lambda_\xi T^\alpha)}{E_{\alpha,1}(-\lambda_\xi T^\alpha)} E_{\alpha,1}(-\lambda_\xi t^\alpha) \right) + \phi_\xi \frac{E_{\alpha,1}(-\lambda_\xi t^\alpha)}{E_{\alpha,1}(-\lambda_\xi T^\alpha)} \right] e_\xi,$$

for all $t \in [0,T]$, and

$$\varphi = \sum_{\xi \in \mathscr{I}} \frac{(\phi_\xi - \psi_\xi T E_{\alpha,2}(-\lambda_\xi T^\alpha)) e_\xi}{E_{\alpha,1}(-\lambda_\xi T^\alpha)},$$

where $\psi_\xi = (\psi, e_\xi)_{\mathscr{H}}$ and $\phi_\xi = (\phi, e_\xi)_{\mathscr{H}}$.

2. Proofs of the main results

2.1. Proof of Theorem 1

Uniqueness. Similarly to the proof of Theorem 3, we introduce the function

$$u_\xi(t) = (u(t), e_\xi)_{\mathscr{H}}, \ \xi \in \mathscr{I}. \tag{7}$$

Applying $\mathscr{D}_t^\alpha$ to (7) and taking into account the equation (1), we have

$$\mathscr{D}_t^\alpha u_\xi(t) = (\mathscr{D}_t^\alpha u(t), e_\xi)_{\mathscr{H}} = -(\mathscr{L} u(t), e_\xi)_{\mathscr{H}}. \tag{8}$$

By the property $\mathscr{L} e_\xi = \lambda_\xi e_\xi$, we obtain

$$\mathscr{D}_t^\alpha u_\xi(t) + \lambda_\xi u_\xi(t) = 0, \tag{9}$$

for all $\xi \in \mathscr{I}$. According to [15], the general solution of the equation (9) can be presented by the formula

$$u_\xi(t) = A_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha) + B_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha), \tag{10}$$

where A_ξ, B_ξ are unknown coefficients. To find them we use the conditions (3) and (5):

$$\begin{aligned} u_\xi(0) &= (u(0), e_\xi)_{\mathscr{H}} = (\varphi, e_\xi)_{\mathscr{H}} = \varphi_\xi, \\ u_\xi(T) &= (u(T), e_\xi)_{\mathscr{H}} = (\phi, e_\xi)_{\mathscr{H}} = \phi_\xi. \end{aligned} \tag{11}$$

Since $\mathscr{L}\phi \in \mathscr{H}$, then we obtain

$$\frac{1}{\lambda_\xi}(\phi, \mathscr{L} e_\xi)_{\mathscr{H}} = \frac{1}{\lambda_\xi}(\mathscr{L}\phi, e_\xi)_{\mathscr{H}} = \frac{1}{\lambda_\xi}\phi_\xi^1. \tag{12}$$

From (10), by taking into account the conditions (11) and the formula (12), we find the unknown coefficients

$$A_\xi = \varphi_\xi, \; B_\xi = \frac{\frac{1}{\lambda_\xi}\phi_\xi^1 - \varphi_\xi E_{\alpha,1}(-\lambda_\xi T^\alpha)}{T E_{\alpha,2}(-\lambda_\xi T^\alpha)},$$

if for all $\xi \in \mathscr{I}$ the condition

$$E_{\alpha,2}(-\lambda_\xi T^\alpha) \neq 0 \tag{13}$$

holds true. This can be guaranteed by the assumption (i) or (ii).

Substituting the values of A_ξ and B_ξ into the formula (10), finally we get

$$\begin{aligned} u_\xi(t) =& \varphi_\xi \left(E_{\alpha,1}(-\lambda_\xi t^\alpha) - \frac{E_{\alpha,1}(-\lambda_\xi T^\alpha)}{T E_{\alpha,2}(-\lambda_\xi T^\alpha)} t E_{\alpha,2}(-\lambda_\xi t^\alpha) \right) \\ &+ \frac{1}{\lambda_\xi}\phi_\xi^1 \frac{t E_{\alpha,2}(-\lambda_\xi t^\alpha)}{T E_{\alpha,2}(-\lambda_\xi T^\alpha)}, \end{aligned} \tag{14}$$

for all $\xi \in \mathscr{I}$.

We are now in a position to show the uniqueness of the solution of Problem 2. Let $\varphi = \phi \equiv 0$ and the condition (13) holds true for all $\xi \in \mathscr{I}$. Then $\varphi_\xi = \phi_\xi^1 \equiv 0$ and from the formulas (7), (14), we have

$$(u(t), e_\xi)_{\mathscr{H}} = 0,$$

for all $\xi \in \mathscr{I}$. Further, by the basis property of the system of eigenfunctions $\{e_\xi\}_{\xi \in \mathscr{I}}$ in $\mathscr{H}$, we obtain $u(t) \equiv 0$ for all $t \in [0, T]$, including the case $u_t(0) = \psi \equiv 0$.

The uniqueness of the solution of Problem 2 is proved.

Existence. We will seek the solution $u(t)$ in the following form

$$u(t) = \sum_{\xi \in \mathscr{I}} u_\xi(t) e_\xi. \tag{15}$$

Putting (15) into (1), (3), (5), we obtain

$$\mathscr{D}_t^\alpha u_\xi(t) + \lambda_\xi u_\xi(t) = 0, \tag{16}$$

$$u_\xi(0) = \varphi_\xi, \tag{17}$$

$$u_\xi(T) = \phi_\xi, \tag{18}$$

for all $\xi \in \mathscr{I}$.

Since the formula (14) is the solution of the problem (16)–(18), by putting it into (15), we get

$$\begin{aligned} u(t) = \sum_{\xi \in \mathscr{I}} \Bigg[& \varphi_\xi \left(E_{\alpha,1}(-\lambda_\xi t^\alpha) - \frac{E_{\alpha,1}(-\lambda_\xi T^\alpha)}{T E_{\alpha,2}(-\lambda_\xi T^\alpha)} t E_{\alpha,2}(-\lambda_\xi t^\alpha) \right) \\ &+ \frac{1}{\lambda_\xi}\phi_\xi^1 \frac{t E_{\alpha,2}(-\lambda_\xi t^\alpha)}{T E_{\alpha,2}(-\lambda_\xi T^\alpha)} \Bigg] e_\xi. \end{aligned} \tag{19}$$

Now, we need to find the function ψ of the condition (4). For that we calculate the derivative of (19) by t. Let us first calculate the derivatives of the Mittag-Leffler's functions, that is, we have [13]

$$\frac{d}{dt}E_{\alpha,1}(-\lambda_\xi t^\alpha) = -\lambda_\xi t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\xi t^\alpha), \tag{20}$$

$$\frac{d}{dt}(tE_{\alpha,2}(-\lambda_\xi t^\alpha)) = E_{\alpha,1}(-\lambda_\xi t^\alpha). \tag{21}$$

Using the formulas (20)–(21), we find

$$u_t(t) = \sum_{\xi\in\mathscr{I}} \left[\frac{\lambda_\xi^{-1}\phi_\xi^1 - \varphi_\xi E_{\alpha,1}(-\lambda_\xi T^\alpha)}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}E_{\alpha,1}(-\lambda_\xi t^\alpha) - \varphi_\xi\lambda_\xi t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\xi t^\alpha)\right]e_\xi. \tag{22}$$

Finally, one can be obtained

$$\psi = u_t(t)\big|_{t=0} = \sum_{\xi\in\mathscr{I}} \frac{\left(\lambda_\xi^{-1}\phi_\xi^1 - \varphi_\xi E_{\alpha,1}(-\lambda_\xi T^\alpha)\right)e_\xi}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}. \tag{23}$$

Convergence. Now, we prove the convergence of the series $u(t)$, $\mathscr{D}_t^\alpha u(t)$, $\mathscr{L}u(t)$ and ψ. Using the asymptotic estimate of the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = O(|z|^{-1}),\ \ 0<\alpha<2,\ \ \beta>0,\ \ |\arg z|\leqslant\pi,\ \ |z|\to\infty,$$

we have

$$\left|\frac{E_{\alpha,\beta}(-\lambda_\xi t^\alpha)}{E_{\alpha,\beta}(-\lambda_\xi T^\alpha)}\right| \leqslant C,\ \ t\geqslant t_0>0,\ \ C = constant. \tag{24}$$

Then for (19) by taking into account that the Mittag-Leffler function $E_{\alpha,2}$ has no real-valued zeros for $1<\alpha\leqslant\frac{4}{3}$ (Lemma 2), one obtains

$$\begin{aligned}
\|u(t)\|_{\mathscr{H}}^2 \leqslant C&\sum_{\xi\in\mathscr{I}}|\varphi_\xi|^2\left|E_{\alpha,1}(-\lambda_\xi t^\alpha)\right|^2 \\
&+C\sum_{\xi\in\mathscr{I}}|\varphi_\xi|^2\frac{t^2|E_{\alpha,2}(-\lambda_\xi t^\alpha)|^2}{T^2|E_{\alpha,2}(-\lambda_\xi T^\alpha)|^2}|E_{\alpha,1}(-\lambda_\xi T^\alpha)|^2 \\
&+C\sum_{\xi\in\mathscr{I}}|\phi_\xi|^2\frac{t^2|E_{\alpha,2}(-\lambda_\xi t^\alpha)|^2}{T^2|E_{\alpha,2}(-\lambda_\xi T^\alpha)|^2} \\
&\overset{(2),(24)}{\leqslant} C\sum_{\xi\in\mathscr{I}}|\varphi_\xi|^2 + \frac{Ct^2}{T^2}\sum_{\xi\in\mathscr{I}}|\varphi_\xi|^2 + \frac{Ct^2}{T^2}\sum_{\xi\in\mathscr{I}}|\phi_\xi|^2 \\
&\leqslant C\|\varphi\|_{\mathscr{H}}^2 + C\|\phi\|_{\mathscr{H}}^2,
\end{aligned}$$

for any $0\leqslant t\leqslant T$. In the case $\frac{4}{3}<\alpha<2$, the last estimate is valid when T is sufficiently large or, when T is from the admissible set $\mathbb{T}$.

Finally, from the last estimate one can be shown

$$\|u\|^2_{C([0,T];\mathscr{H})} \leqslant C\|\varphi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}}.$$

We also have the convergence for ψ, that is,

$$\begin{aligned}\|\psi\|^2_{\mathscr{H}} &\leqslant C\sum_{\xi\in\mathscr{I}} \frac{|\phi^1_\xi|^2}{\lambda^2_\xi T^2|E_{\alpha,2}(-\lambda_\xi T^\alpha)|^2} + C\sum_{\xi\in\mathscr{I}} |\varphi_\xi|^2\frac{|E_{\alpha,1}(-\lambda_\xi T^\alpha)|^2}{T^2|E_{\alpha,2}(-\lambda_\xi T^\alpha)|^2}\\ &\leqslant C\|\varphi\|^2_{\mathscr{H}} + C\|\mathscr{L}\phi\|^2_{\mathscr{H}}\\ &= C\|\varphi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}^1},\end{aligned}$$

by using the estimates (2) and (24).

Let us calculate

$$\begin{aligned}\mathscr{L}u(t) &= \sum_{\xi\in\mathscr{I}}\left[\varphi_\xi\left(E_{\alpha,1}(-\lambda_\xi t^\alpha) - \frac{E_{\alpha,1}(-\lambda_\xi T^\alpha)}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}tE_{\alpha,2}(-\lambda_\xi t^\alpha)\right)\right.\\ &\quad\left.+\frac{\phi_\xi}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}tE_{\alpha,2}(-\lambda_\xi t^\alpha)\right]\mathscr{L}e_\xi\\ &= \sum_{\xi\in\mathscr{I}}\left[\lambda_\xi\varphi_\xi\left(E_{\alpha,1}(-\lambda_\xi t^\alpha) - \frac{E_{\alpha,1}(-\lambda_\xi T^\alpha)}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}tE_{\alpha,2}(-\lambda_\xi t^\alpha)\right)\right.\\ &\quad\left.+\frac{\lambda_\xi\phi_\xi}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}tE_{\alpha,2}(-\lambda_\xi t^\alpha)\right]e_\xi.\end{aligned}\tag{25}$$

In view of (1) from (25) one finds

$$\begin{aligned}\mathscr{D}^\alpha_t u(t) &= -\mathscr{L}u(t)\\ &= -\sum_{\xi\in\mathscr{I}}\left[\lambda_\xi\varphi_\xi\left(E_{\alpha,1}(-\lambda_\xi t^\alpha) - \frac{E_{\alpha,1}(-\lambda_\xi T^\alpha)}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}tE_{\alpha,2}(-\lambda_\xi t^\alpha)\right)\right.\\ &\quad\left.+\frac{\lambda_\xi\phi_\xi}{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}tE_{\alpha,2}(-\lambda_\xi t^\alpha)\right]e_\xi.\end{aligned}\tag{26}$$

Now we show the convergence of the series (25) and (26).

$$\begin{aligned}\|\mathscr{L}u(t)\|^2_{\mathscr{H}} = \|\mathscr{D}^\alpha_t u(t)\|^2_{\mathscr{H}} &\leqslant \frac{C}{t^{2\alpha}}\|\varphi\|^2_{\mathscr{H}} + \frac{Ct^2}{T^{2\alpha}}\|\varphi\|^2_{\mathscr{H}} + \frac{Ct^2}{T^2}\|\mathscr{L}\phi\|^2_{\mathscr{H}}\\ &\leqslant C\|\varphi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}^1}\end{aligned}$$

for all $0<\delta\leqslant t\leqslant T$.

From this estimate we have

$$\|\mathscr{L}u\|^2_{C([0,T];\mathscr{H})} = \|\mathscr{D}^\alpha_t u\|^2_{C([0,T];\mathscr{H})} \leqslant C\|\varphi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}^1},$$

for all $t\in[\delta,T]$, ending the proof. □

2.2. Proof of Theorem 2

Uniqueness. Now we prove the uniqueness of the solution of Problem 3 using the similarly methods as in Theorem 1. Let us introduce the function (7). So, we get the same equation (9) and the same formula for the general solution (10). In this case to find the unknown coefficients A_ξ, B_ξ in the formulas (10), we use the conditions (4)–(5):

$$u'_\xi(0) = (u_t(0), e_\xi)_{\mathscr{H}} = (\psi, e_\xi)_{\mathscr{H}} = \psi_\xi, \tag{27}$$

$$u_\xi(T) = (u(T), e_\xi)_{\mathscr{H}} = (\phi, e_\xi)_{\mathscr{H}} = \phi_\xi, \tag{28}$$

for all $\xi \in \mathscr{I}$. Repeating the arguments for $\mathscr{L}\phi \in \mathscr{H}$, we obtain

$$\frac{1}{\lambda_\xi}(\phi, \mathscr{L}e_\xi)_{\mathscr{H}} = \frac{1}{\lambda_\xi}(\mathscr{L}\phi, e_\xi)_{\mathscr{H}} = \frac{1}{\lambda_\xi}\phi_\xi^1, \tag{29}$$

for all $\xi \in \mathscr{I}$. To use the condition (27), we need to calculate the derivative of (10) by t. Namely, using the formulas (20), (21), we get

$$u'_\xi(t) = B_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha) - \lambda_\xi t^{\alpha-1} A_\xi E_{\alpha,\alpha}(-\lambda_\xi t^\alpha), \tag{30}$$

for all $\xi \in \mathscr{I}$. Now, we are able to find the coefficients A_ξ, B_ξ

$$B_\xi = \psi_\xi, \, A_\xi = \frac{\lambda_\xi^{-1}\phi_\xi^1 - \psi_\xi T E_{\alpha,2}(-\lambda_\xi T^\alpha)}{E_{\alpha,1}(-\lambda_\xi T^\alpha)},$$

when

$$E_{\alpha,1}(-\lambda_\xi T^\alpha) \neq 0, \tag{31}$$

for all $\xi \in \mathscr{I}$. By substituting the values of A_ξ and B_ξ into the formula (10), we have

$$\begin{aligned} u_\xi(t) = \psi_\xi &\left(t E_{\alpha,2}(-\lambda_\xi t^\alpha) - \frac{T E_{\alpha,2}(-\lambda_\xi T^\alpha)}{E_{\alpha,1}(-\lambda_\xi T^\alpha)} E_{\alpha,1}(-\lambda_\xi t^\alpha) \right) \\ &+ \phi_\xi^1 \frac{E_{\alpha,1}(-\lambda_\xi t^\alpha)}{\lambda_\xi E_{\alpha,1}(-\lambda_\xi T^\alpha)}. \end{aligned} \tag{32}$$

Finally, we obtain

$$\varphi_\xi = \frac{(\lambda_\xi^{-1}\phi_\xi^1 - \psi_\xi T E_{\alpha,2}(-\lambda_\xi T^\alpha))}{E_{\alpha,1}(-\lambda_\xi T^\alpha)}, \tag{33}$$

for all $\xi \in \mathscr{I}$.

Now we are in a position to prove the uniqueness of the solution of Problem 3. Let $\psi = \phi \equiv 0$ and the condition (31) holds true. Then $\psi_\xi = \phi_\xi^1 \equiv 0$ and from the formulas (7), (32) and (33) we obtain

$$(u(t), e_\xi)_{\mathscr{H}} = 0, \; (\varphi, e_\xi)_{\mathscr{H}} = 0,$$

for all $\xi \in \mathscr{I}$. Further, by the basis property of the system of functions $\{e_\xi\}_{\xi\in\mathscr{I}}$ in $\mathscr{H}$, we conclude that $u(t) \equiv 0$ and $\varphi \equiv 0$. The uniqueness is proved.

Existence. We will seek the solution of Problem 3 in the form (15). Putting (15) into the equation (1) and the conditions (4)–(5), we have

$$\mathscr{D}_t^\alpha u_\xi(t) + \lambda_\xi u_\xi(t) = 0, \tag{34}$$

$$u'_\xi(0) = \psi_\xi, \tag{35}$$

$$u_\xi(T) = \phi_\xi, \tag{36}$$

for all $\xi \in \mathscr{I}$.

It is clear that (32) is the solution of the problem (34)–(36). Then by substituting (32) into (15), we get

$$u(t) = \sum_{\xi\in\mathscr{I}} \left[\psi_\xi \left(tE_{\alpha,2}(-\lambda_\xi t^\alpha) - \frac{TE_{\alpha,2}(-\lambda_\xi T^\alpha)}{E_{\alpha,1}(-\lambda_\xi T^\alpha)} E_{\alpha,1}(-\lambda_\xi t^\alpha) \right) + \phi_\xi \frac{E_{\alpha,1}(-\lambda_\xi t^\alpha)}{E_{\alpha,1}(-\lambda_\xi T^\alpha)} \right] e_\xi,$$

and

$$\varphi = \sum_{\xi\in\mathscr{I}} \frac{(\lambda_\xi^{-1}\phi_\xi^1 - \psi_\xi TE_{\alpha,2}(-\lambda_\xi T^\alpha))e_\xi}{E_{\alpha,1}(-\lambda_\xi T^\alpha)}. \tag{37}$$

The last series make sense for sufficiently large $T > 0$ or for T from the admissible set $\mathbb{X}$ due to the fact that the Mittag-Leffler function $E_{\alpha,1}$ has a finite number of real–valued zeros for $1 < \alpha < 2$ [8]. The convergence of the series $u(t)$, $\mathscr{L}u(t)$, $\mathscr{D}_t^\alpha u(t)$ and φ can be proved by a similar way as in Theorem 1. Namely, it can be shown that

$$\|u\|_{C([0,T];\mathscr{H})} \leqslant C\|\psi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}},$$

$$\|\mathscr{L}u\|^2_{C((0,T];\mathscr{H})} = \|\mathscr{D}_t^\alpha u\|^2_{C((0,T];\mathscr{H})} \leqslant C\|\psi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}^1},$$

and

$$\|\varphi\|^2_{\mathscr{H}} \leqslant C\|\psi\|^2_{\mathscr{H}} + C\|\phi\|^2_{\mathscr{H}^1},$$

ending the proof. □

REMARK 3. We note that inverse and direct problems studied in this paper can be considered with more general Cauchy-type conditions, for example, given in [12].

3. Appendix: Direct problem

Before presenting results for Direct Problem 1, we demonstrate several examples of operator $\mathscr{L}$.

3.1. Some examples of $\mathscr{L}$

Operator $\mathscr{L}$ includes all self-adjoint operators with discrete positive spectrum, such as (for more details see [25]): Dirichlet-Laplacian, Sturm-Liouville operators, Fractional Sturm-Liouville operators, Sturm-Liouville operators with involutions, fractional Laplacian and others. Also $\mathscr{L}$ includes self-adjoint operators defined on unbounded domains and without boundary conditions. Below we give several examples of operator $\mathscr{L}$ defined on the whole $\mathbb{R}^d$.

- **Harmonic oscillator.**

  For any dimension $d \geqslant 1$, let us consider the harmonic oscillator,

$$\mathscr{L} := -\Delta + |x|^2, \ x \in \mathbb{R}^d.$$

  $\mathscr{L}$ is an essentially self-adjoint operator on $C_0^\infty(\mathbb{R}^d)$. It has a discrete spectrum, consisting of the eigenvalues

$$\lambda_k = \sum_{j=1}^{d} (2k_j + 1), \ k = (k_1, \cdots, k_d) \in \mathbb{N}^d,$$

  and with the corresponding eigenfunctions

$$\varphi_k(x) = \prod_{j=1}^{d} P_{k_j}(x_j) \mathrm{e}^{-\frac{|x|^2}{2}},$$

  which are an orthogonal basis in $L^2(\mathbb{R}^d)$. We denote by $P_l(\cdot)$ the l–th order Hermite polynomial, and

$$P_l(\xi) = a_l \mathrm{e}^{\frac{|\xi|^2}{2}} \left(x - \frac{d}{d\xi} \right)^l \mathrm{e}^{-\frac{|\xi|^2}{2}},$$

  where $\xi \in \mathbb{R}$, and

$$a_l = 2^{-l/2} (l!)^{-1/2} \pi^{-1/4}.$$

  For more information, see for example [20].

- **Landau Hamiltonian in 2D.**

  The next example is one of the simplest and most interesting models of the Quantum Mechanics, that is, the Landau Hamiltonian.

The Landau Hamiltonian in 2D is given by

$$\mathscr{L} := \frac{1}{2}\left(\left(i\frac{\partial}{\partial x} - By\right)^2 + \left(i\frac{\partial}{\partial y} + Bx\right)^2\right),$$

acting on the Hilbert space $L^2(\mathbb{R}^2)$, where $B > 0$ is some constant. The spectrum of $\mathscr{L}$ consists of infinite number of eigenvalues (see [7, 14]) with infinite multiplicity of the form

$$\lambda_n = (2n+1)B,\ n = 0,1,2,\dots,$$

and the corresponding system of eigenfunctions (see [1,9]) is

$$\begin{cases} e^1_{k,n}(x,y) = \sqrt{\dfrac{n!}{(n-k)!}} B^{\frac{k+1}{2}} \exp\Big(-\dfrac{B(x^2+y^2)}{2}\Big)(x+iy)^k L_n^{(k)}(B(x^2+y^2)),\ 0 \leqslant k, \\ e^2_{j,n}(x,y) = \sqrt{\dfrac{j!}{(j+n)!}} B^{\frac{n-1}{2}} \exp\Big(-\dfrac{B(x^2+y^2)}{2}\Big)(x-iy)^n L_j^{(n)}(B(x^2+y^2)),\ 0 \leqslant j, \end{cases}$$

where $L_n^{(\alpha)}$ are the Laguerre polynomials given by

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} (-1)^k C_{n+\alpha}^{n-k} \frac{t^k}{k!},\ \alpha > -1.$$

3.2. Well-posedness of the direct problem

We start this subsection by formulating the following theorem on the solution representation of Direct Problem 1. Indeed, this result can be partly covered by Theorem 2.3 in [27]. Sakamoto and Yamamoto studied the case when $\mathscr{L}$ is a symmetric uniformly elliptic operator. Here we deal with a more general operator not an elliptic one in particular, as given in Subsection 3.1.

THEOREM 3. *Let $\varphi, \psi \in \mathscr{H}$. Then there exists a unique solution $u \in \mathscr{B}^\alpha((0,T];\mathscr{H}) \cap C((0,T];\mathscr{H}^1)$ of Problem 1. This solution can be written in the form*

$$u(t) = \sum_{\xi\in\mathscr{I}} \varphi_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha)e_\xi + \sum_{\xi\in\mathscr{I}} \psi_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha)e_\xi,$$

for all $t \in [0,T]$, where $\varphi_\xi = (\varphi, e_\xi)_{\mathscr{H}}$ and $\psi_\xi = (\psi, e_\xi)_{\mathscr{H}}$.

At first, we prove an existence of the solution.

Proof. Existence. Since the system $\{e_\xi\}_{\xi\in\mathscr{I}}$ is an orthonormal basis in $\mathscr{H}$, we seek the solution $u(t)$ in the following form

$$u(t) = \sum_{\xi\in\mathscr{I}} u_\xi(t)e_\xi, \tag{38}$$

where $u_\xi(t)$ are unknown functions, and they can be represented by

$$u_\xi(t) = (u, e_\xi)_{\mathscr{H}},$$

for all $\xi \in \mathscr{I}$. Putting (38) into the equation (1) and conditions (3)–(4), one can has

$$\mathscr{D}_t^\alpha u_\xi(t) + \lambda_\xi u_\xi(t) = 0, \tag{39}$$

$$u_\xi(0) = \varphi_\xi, \ u'_\xi(0) = \psi_\xi, \tag{40}$$

for all $\xi \in \mathscr{I}$.

According to [15], the solution of the equation (39) satisfying the initial conditions (40) can be represented in the following form

$$u_\xi(t) = \varphi_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha) + \psi_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha), \tag{41}$$

for all $t \in [0,T]$, for all $\xi \in \mathscr{I}$. By substituting (41) into the formula (38), we obtain

$$u(t) = \sum_{\xi\in\mathscr{I}} \Big(\varphi_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha) + \psi_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha)\Big) e_\xi, \tag{42}$$

for all $t \in [0,T]$.

Now we prove the convergence of the obtained series corresponding to the functions $u(t)$, $\mathscr{D}_t^\alpha u(t)$, and $\mathscr{L}u(t)$.

$$\begin{aligned}
\|u(t)\|^2_{\mathscr{H}} &\leqslant C\left(\sum_{\xi\in\mathscr{I}} |\varphi_\xi|^2 |E_{\alpha,1}(-\lambda_\xi t^\alpha)|^2 + \sum_{\xi\in\mathscr{I}} |\psi_\xi|^2 t^2 |E_{\alpha,2}(-\lambda_\xi t^\alpha)|^2\right) \\
&\overset{(2)}{\leqslant} C\left(\sum_{\xi\in\mathscr{I}} \frac{|\varphi_\xi|^2}{(1+t^\alpha\lambda_\xi)^2} + t^2 \sum_{\xi\in\mathscr{I}} \frac{|\psi_\xi|^2}{(1+t^\alpha\lambda_\xi)^2}\right) \\
&\leqslant C\left(\sum_{\xi\in\mathscr{I}} |\varphi_\xi|^2 + \sum_{\xi\in\mathscr{I}} |\psi_\xi|^2\right) < \infty,
\end{aligned}$$

for all $t \in [0,T]$. Calculate

$$\begin{aligned}
\mathscr{L}u(t) &= \sum_{\xi\in\mathscr{I}} \varphi_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha)\mathscr{L}e_\xi + \sum_{\xi\in\mathscr{I}} \psi_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha)\mathscr{L}e_\xi \\
&= \sum_{\xi\in\mathscr{I}} \lambda_\xi \varphi_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha) e_\xi + \sum_{\xi\in\mathscr{I}} \lambda_\xi \psi_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha) e_\xi.
\end{aligned} \tag{43}$$

For $\mathscr{D}_t^\alpha u(t)$ from the equation (1), one can be shown

$$\begin{aligned}
\mathscr{D}_t^\alpha u(t) &= -\mathscr{L}u(t) \\
&= -\sum_{\xi\in\mathscr{I}} \lambda_\xi \varphi_\xi E_{\alpha,1}(-\lambda_\xi t^\alpha) e_\xi - \sum_{\xi\in\mathscr{I}} \lambda_\xi \psi_\xi t E_{\alpha,2}(-\lambda_\xi t^\alpha) e_\xi.
\end{aligned} \tag{44}$$

Using the formulas (43) and (44), we have

$$\|\mathscr{L}u(t)\|_{\mathscr{H}}^2 = \|\mathscr{D}_t^\alpha u(t)\|_{\mathscr{H}}^2 \leqslant C\left(\|\varphi\|_{\mathscr{H}}^2 + \|\psi\|_{\mathscr{H}}^2\right) < \infty,$$

for all $0 < \delta \leqslant t \leqslant T$. From the above inequalities we obtain

$$\|u\|_{C([0,T];\mathscr{H})}^2 \leqslant C \sum_{\xi\in\mathscr{I}} |\varphi_\xi|^2 + C \sum_{\xi\in\mathscr{I}} |\psi_\xi|^2 = C\left(\|\varphi\|_{\mathscr{H}}^2 + \|\psi\|_{\mathscr{H}}^2\right) < \infty,$$

$$\|u\|_{C([\delta,T];\mathscr{H}^1)}^2 = \|u\|_{\mathscr{B}^\alpha([\delta,T];\mathscr{H})}^2 \leqslant C\left(\|\varphi\|_{\mathscr{H}}^2 + \|\psi\|_{\mathscr{H}}^2\right) < \infty, \ \ \delta > 0,$$

showing the existence of the solution of Problem 1.

Uniqueness. Let us assume that there are two different solutions $u_1(t), u_2(t)$ of Problem 1. Then $u(t) = u_1(t) - u_2(t)$ is also a solution of the following homogeneous Cauchy problem

$$\mathscr{D}_t^\alpha u(t) + \mathscr{L}u(t) = 0,\ t \in [0,T], \tag{45}$$

$$u(0) = 0,\ \ u_t(0) = 0. \tag{46}$$

It can be easily shown that any solution to (45)–(46) is given by the formula (42). Indeed, by acting the operator $\mathscr{D}_t^\alpha$ to $u_\xi(t) = (u(t), e_\xi)_{\mathscr{H}}$ and taking into account the equation (45), we arrive at

$$\mathscr{D}_t^\alpha u_\xi(t) = (\mathscr{D}_t^\alpha u(t), e_\xi)_{\mathscr{H}} = -(\mathscr{L}u(t), e_\xi)_{\mathscr{H}}.$$

From the property $\mathscr{L}e_\xi = \lambda_\xi e_\xi$, we obtain the equation

$$\mathscr{D}_t^\alpha u_\xi(t) + \lambda_\xi u_\xi(t) = 0, \tag{47}$$

and in view of (46), one has

$$u_\xi(0) = 0,\ \ u'_\xi(0) = 0, \tag{48}$$

for each $\xi \in \mathscr{I}$. From the representation (41) of the solution of (47)–(48), by noting that $\varphi_\xi = \psi_\xi \equiv 0$, we obtain $(u(t), e_\xi)_{\mathscr{H}} = 0$, for all $\xi \in \mathscr{I}$. Further, by the basis property of the system of eigenfunctions $\{e_\xi\}_{\xi\in\mathscr{I}}$ in $\mathscr{H}$, we conclude that $u(t) \equiv 0$. The uniqueness of the solution of Problem 1 is proved. □

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