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INITIAL–BOUNDARY VALUE PROBLEMS TO THE TIME–SPACE NONLOCAL DIFFUSION EQUATION

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Abstract. This article investigates a time-fractional space-nonlocal diffusion equation in a bounded domain. The fractional operators are defined rigorously, using the Caputo fractional derivative of order β and the Riemann-Liouville fractional integral of order α, where $0<\alpha<\beta\leqslant 1$. The solution is expressed as a series involving the two-parameter Mittag-Leffler function and orthonormal eigenfunctions of the Sturm-Liouville operator. The convergence of the series is investigated, and conditions for the solution to belong to a specific function space are established. The uniqueness of the solution is demonstrated and the continuity of the solution in the specified domain is confirmed through the uniform convergence of the series.

1. Introduction

Over the course of millennia, fractional partial differential equations (FPDEs) have evolved into essential tools for representing complex systems and anomalous phenomena [8]–[9]. A comprehensive exploration of the applications of these equations across disciplines such as chemistry, technology, and physics is presented in the book [20].

Furthermore, differential equations where the unknown function and its derivatives are evaluated with changes to time or space variables are known as differential equations with modified arguments, or general functional differential equations. Equations containing involutions can be distinguished among them [7].

DEFINITION 1. ([21]) A function $\alpha(x)\not\equiv x$ that maps a set of real numbers, Γ, onto itself and satisfies on Γ the condition

$$\alpha(\alpha(x))=x, \quad \text{or} \quad \alpha^{-1}(x)=\alpha(x)$$

is called an involution on Γ.

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Equations containing involution are equations with an alternating deviation (at $x^* < x$ being equations with advanced, and at $x^* > x$ being equations with delay, where x^* is a fixed point of the mapping $\alpha(x)$).

We recommend the reader to the books of Skubachevskii [17], Wu [22] and Cabada and Tojo [7] for general information regarding partial functional differential equations and specifically for properties of equations with involutions.

Consider the mapping $S:\mathbb{R}\to\mathbb{R}$, of the type $Sx := 1-x$. Obviously, the mapping is involution, i.e. $S^2 = I$, where I is the identity mapping.

Using these mapping, we introduce the operator

$$L_x u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + \varepsilon\frac{\partial^2}{\partial x^2}u(1-x,t)$$

where $\varepsilon \geqslant 0$.

It should be noted that the direct and inverse problems for the diffusion and fractional diffusion equations with involutions were covered in [1, 2, 3, 4, 5, 10, 6, 16, 18, 19].

Motivated the above papers, we consider the the following equation

$$D^{\beta}_{0+}u(x,t) - I^{\alpha}_{0+}[L_x u](x,t) = 0, \text{ in } \Omega = \{(x,t) : 0<x<1,\ 0<t<T\}, \tag{1}$$

with initial and boundary conditions

$$u(x,0) = \varphi(x) \text{ on } x\in[0,1], \tag{2}$$

$$u(0,t) = u(1,t) = 0,\ 0\leqslant t\leqslant T, \tag{3}$$

where $0<\alpha<\beta\leqslant 1$ and the function $\varphi(x)$ is continuous.

In addition, the operator I^{α}_{0+} is the Riemann-Liouville fractional integral of order $\alpha>0$, defined as

$$I^{\alpha}_{0+}u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(x,s)\,ds,\ t\in(0,T],$$

and the operator D^{β}_{0+} stands for the Caputo fractional derivative of order $\beta\in(0,1)$ is defined by

$$D^{\beta}_{0+}u(x,t) = I^{1-\beta}_{0+}\left[\frac{\partial}{\partial t}u(x,t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\frac{\partial}{\partial s}u(x,s)\,ds.$$

LEMMA 1. [9, p. 95] *If* $0<\beta<1$ *for* $f\in AC[0,T]$, $f\in C'(0,T)$ *then*

$$I^{\beta}_{0+}\left[D^{\beta}_{0+}f(t)\right] = f(t) - f(0),$$

holds true.

LEMMA 2. [9, p. 101] *Let* $f \in C[0,T]$. *If* $\alpha+\beta \leqslant 1$, *then*

$$I_{0+}^{\beta}\left[I_{0+}^{\alpha} f(t)\right] = I_{0+}^{\alpha+\beta} f(t).$$

DEFINITION 2. The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, is defined by (see [9, p. 42])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \ (z,\beta \in \mathbb{C});\ \Re(\alpha) > 0).$$

LEMMA 3. [15, p. 9] *For every* $\lambda \geqslant 0$ *one has the optimal bounds*

$$\left|E_{\alpha,\beta}(-\lambda t^{\alpha})\right| \leqslant \frac{C}{1+|\lambda t^{\alpha}|} \leqslant C,\ t \geqslant 0,\ \beta \geqslant 0,$$
$$\lambda t^{\alpha}\left|E_{\alpha,\beta}(-\lambda t^{\alpha})\right| \leqslant C,\ 0<\alpha<2,\ \beta \in \mathbb{C}.$$

2. Main results

This section summarizes the key findings of this article.

THEOREM 1. *Let* $\varphi(x) \in C[0,1]$, $\varphi'(x) \in L_2(0,1)$, *then the unique solution of problem* (1)–(3) *is the function* $u(x,t) \in C(\overline{\Omega})$, *which has the form*

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k X_k(x)\left[E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})\right], \tag{4}$$

where

$$\varphi_k = \sqrt{2}\int_0^1 \varphi(x)\sin(k\pi x)dx$$

and $E_{\alpha,\beta}(-z)$ *is the two-parameter Mittag-Leffler function.*

Proof. As operator L_x with Dirichlet conditions has the eigenvalues $\{\lambda_k \geqslant 0, k \in N\}$ and corresponding orthonormal eigenfunctions $\{X_k(x), k \in N\}$ (see [13, 11]), we can represent any solution of (1)–(3) as

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x)T_k(t),\ (x,t) \in (0,1)\times(0,T). \tag{5}$$

The function φ can be represented in the form

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k X_k(x),\ x \in (0,1),$$

where

$$\varphi_k = \sqrt{2}\int_0^1 \varphi(x)X_k(x)dx.$$

By substituting (5) into the equations (1)–(3), we have a separate problem for the variable t

$$D^{\beta}_{0+}T_k(t)+\lambda_k I^{\alpha}_{0+}T_k(t)=0,\ t>0, \tag{6}$$

and respect to x

$$X_k''(x)+\varepsilon X_k''(1-x)+\lambda_k X_k(x)=0, \tag{7}$$

$$X_k(0)=X_k(1)=0. \tag{8}$$

The orthonormal eigenfunctions and related eigenvalues of the Dirichlet problem (7)–(8) are $X_k(x)=\sin(k\pi x)$ and $\lambda_k=(1+(-1)^{k+1}\varepsilon)(k\pi)^2$, respectively.

Then, applying the operator I^{β}_{0+} to equation (6), we get

$$I^{\beta}_{0+}[D^{\beta}_{0+}T_k(t)+\lambda_k I^{\alpha}_{0+}T_k(t)]=0.$$

In view of Lemma 1 and Lemma 2, it holds

$$\lambda_k(I^{\alpha+\beta}_{0+}T_k)(t)+T_k(t)=T_k(0),\ t>0,$$

The integral equation has a unique solution (see [9], p. 231)

$$T_k(t)=T_k(0)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta}). \tag{9}$$

Therefore, we deduce that

$$u(x,t)=\sum_{k=1}^{\infty}\varphi_k X_k(x)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta}), \tag{10}$$

where $0<\alpha<\beta\leqslant 1$, $\varphi_k=(\varphi,X_k)$, $X_k(x)=\sqrt{2}\sin(k\pi x)$, $\lambda_k=(1+(-1)^{k+1}\varepsilon)(k\pi)^2$.

Next, we have to show that $u(x,t)\in C(\overline{\Omega})$ for $\overline{\Omega}=\{(x,t):0\leqslant x\leqslant 1, 0\leqslant t\leqslant T\}$. For this, we have to show the uniform convergence of series (10) in a closed domain $\overline{\Omega}$.

Now, let us estimate the coefficients φ_k, which given by

$$\varphi_k=(\varphi,X_k)=\sqrt{2}\int_0^1\varphi(x)\sin(k\pi x)dx.$$

Integrating by parts, we have

$$\begin{aligned}\varphi_k &= \sqrt{2}\int_0^1 \varphi(x)d\left[-\frac{\cos(k\pi x)}{k\pi}\right]\\ &= -\varphi(x)\frac{\sqrt{2}\cos(k\pi x)}{k\pi}\Big|_{x=0}^{x=1}+\frac{\sqrt{2}}{k\pi}\int_0^1\varphi'(x)\cos(k\pi x)dx.\end{aligned}$$

If the conditions $\varphi(0)=\varphi(1)=0$ are hold true, then it yields that

$$\varphi_k=\frac{1}{k\pi}\varphi_k^{(1)}, \tag{11}$$

where

$$\varphi_k^{(1)}=\int_0^1\sqrt{2}\varphi'(x)\cos(k\pi x)dx. \tag{12}$$

Using Lemma 3, and $|X_k(x)|\leqslant C$, we obtain

$$|u(x,t)|\leqslant C\sum_{k=1}^{\infty}|\varphi_k|=C\sum_{k=1}^{\infty}\frac{1}{k}|\varphi_k^{(1)}|.$$

Consequently, we study the convergence of the series $\sum\limits_{k=1}^{\infty}\frac{1}{k}|\varphi_k^{(1)}|$. Due to the Cauchy-Schwarz inequality, it follows that

$$\sum_{k=1}^{\infty}\frac{1}{k}|\varphi_k^{(1)}|\leqslant\sqrt{\sum_{k=1}^{\infty}\frac{1}{k^2}}\sqrt{\sum_{k=1}^{\infty}\left|\varphi_k^{(1)}\right|^2}.$$

Moreover, it is well-known that, the system $Y_k(x)=\{\sqrt{2}\cos(k\pi x)\}_{k=1}^{\infty}$ is orthonormal in $L_2(0,1)$ and for any function $g(x)\in L_2(0,1)$ the Bessel inequality holds

$$\sum_{k=1}^{\infty}|g_k|^2\leqslant\parallel g(x)\parallel_{L_2}^2=\int_0^1 g^2(x)dx,$$

where

$$g_k=(g,Y_k)=\sqrt{2}\int_0^1 g(x)\cos(k\pi x)dx.$$

So, if

$$g(x)\in L_2(0,1)\Leftrightarrow\int_0^1 g^2(x)dx<\infty,$$

then

$$\sum_{k=1}^{\infty}|g_k|^2<\infty,$$

i.e. the series converges.

Further, if $\varphi'(x)\in L_2(0,1)$, then for coefficients $\varphi_k^{(1)}$ of equality (12) using Bessel's inequality, we conclude $\sum\limits_{k=1}^{\infty}|\varphi_k^{(1)}|<\infty$.

Thus, if $\varphi'(x)\in L_2(0,1)$, then the number series $\sum\limits_{k=1}^{\infty}\frac{1}{k}|\varphi_k^{(1)}|$ converges, when the conditions are holds

$$\varphi(x)\in C[0,1],\ \varphi'(x)\in L_2[0,1],\ \varphi(0)=\varphi(1)=0. \tag{13}$$

Consequently, the series (10) converges uniformly in the closed region $\overline{\Omega}$, which give us $u \in C(\overline{\Omega})$.

Now let us show that the solution is unique. Assume that $u_1(x,t)$ and $u_2(x,t)$ are solutions to the problem (1)–(3). We choose $u(x,t) = u_1(x,t) - u_2(x,t)$ so that $u(x,t)$ satisfies the equation and the initial and boundary conditions (2)–(3).

Consider the following identity

$$T_k(t) = \int_0^1 u(x,t)\sin(k\pi x)dx, \ k \in N, \ t \geqslant 0. \tag{14}$$

Noting (6), we apply the operator D^{β}_{0+} to the left side of the equation (14)

$$\begin{aligned}(D^{\beta}_{0+}T_k)(t) &= \int_0^1 (D^{\beta}_{0+}u)(x,t)\sin(k\pi x)dx \\ &= -(k\pi)^2 I^{\alpha}_{0+}\int_0^1 u(x,t)\sin(k\pi x)dx \\ &= -(k\pi)^2 (I^{\alpha}_{0+}T_k)(t), \ k \in N, \ t \geqslant 0.\end{aligned}$$

As a result of (2) and (3) we have

$$T_k(0) = \int_0^1 u(x,0)\sin(k\pi x)dx = \int_0^1 \varphi(x)\sin(k\pi x)dx = 0.$$

In view of (9) we deduce that

$$T_k(t) = T_k(0)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta}) = 0.$$

Since $T_k(0) = 0$, which means $u(x,t) = 0$. Hence $u_1(x,t) = u_2(x,t)$, and the problem (1)–(3) has a unique solution.

By applying the operators D^{β}_{0+} and I^{α}_{0+} to the identity (10), we get

$$\begin{aligned}D^{\beta}_{0+}u(x,t) &= D^{\beta}_{0+}\left[\sum_{k=1}^{\infty}\varphi_k X_k(x)E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})\right] \\ &= \sum_{k=1}^{\infty}\varphi_k X_k(x) D^{\beta}_{0+}\left[E_{\alpha+\beta,1}(-\lambda_k t^{\alpha+\beta})\right] \\ &= -\sum_{k=1}^{\infty}\varphi_k X_k(x)\lambda_k t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\end{aligned} \tag{15}$$

and

$$\begin{aligned}I^{\alpha}_{0+}[L_x u](x,t) &= D^{\beta}_{0+}u(x,t) \\ &= -\sum_{k=1}^{\infty}\varphi_k X_k(x)\lambda_k t^{\alpha}E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta}).\end{aligned} \tag{16}$$

Next, we show that $D^{\beta}_{0+}u \in C(\Omega)$ and $I^{\alpha}_{0+}u \in C(\Omega)$.

Let δ be an arbitrary, sufficiently small positive number. Then for all $0 < \delta \leqslant t$, taking into account (15), (16) and Lemma 3, we arrive at

$$\begin{aligned}
|D_{0+}^{\beta}u(x,t)| &= |I_{0+}^{\alpha}[L_x u](x,t)| \\
&= \left|\sum_{k=1}^{\infty}\varphi_k X_k(x)\lambda_k t^{\alpha} E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\right| \\
&= \left|\sum_{k=1}^{\infty}\varphi_k X_k(x) t^{-\beta}\left[\lambda_k t^{\alpha+\beta} E_{\alpha+\beta,\alpha+1}(-\lambda_k t^{\alpha+\beta})\right]\right| \\
&\leqslant C\sum_{k=1}^{\infty}|\varphi_k|.
\end{aligned}$$

If the conditions (13) holds, the series $\sum_{k=1}^{\infty}|\varphi_k|$ converges, and then the series (15) and (16) representing the function $D_{0+}^{\beta}u(x,t)$ and $I_{0+}^{\alpha}[L_x u](x,t)$ converges uniformly in any closed subdomain $\overline{\Omega}_{\delta}$ of the domain Ω.

Consequently, since the value of δ is arbitrary, we have $D_{0+}^{\beta}u \in C(\Omega)$ and $I_{0+}^{\alpha}[L_x u] \in C(\Omega)$. This completes the proof. $\square$

3. Conclusion

In this paper, the main results include the presentation of well-known properties associated with fractional operators and the establishment of a classical solution to this problem. The key conclusions are summarized using a theorem that provides an explicit form of the solution. The solution is expressed as a series including the two-parameter Mittag-Leffler function and orthonormal eigenfunctions of the Sturm-Liouville operator. The uniqueness of the solution is proved, which guarantees that the problem has a unique solution.

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