Fractional Differential Calculus
Volume 14, Number 1 (2024), 109–169 doi:10.7153/fdc-2024-14-07

RECENT RESULTS ON FRACTIONAL LYAPUNOV–TYPE INEQUALITIES: A SURVEY

SOTIRIS K. NTOUYAS*, BASHIR AHMAD AND JESSADA TARIBOON

(*Communicated by M. Andrić*)

Abstract. This survey paper complements to our previous review papers on Lyapunov-type inequalities and contains some of the most recent results on these inequalities for fractional boundary value problems involving a variety of fractional derivative operators and boundary conditions. In precise terms, we have included the results related to Riemann–Liouville, Caputo, mixed Riemann–Liouville and Caputo, Riesz-Caputo, ψ-Caputo, Hadamard, Katugampola, Hilfer, ψ-Hilfer, proportional, variable order Hadamard, partial, systems of Riemann-Liouville, bi-ordinal Hilfer-Katugampola, and ψ-Hilfer fractional derivative operators. The Lyapunov-type inequalities for discrete fractional boundary value problems are also presented.

1. Introduction

Inequalities are found to be a valuable tool in developing many branches of mathematics and handling a variety of problems of applied nature. Among the well-known inequalities, the Lyapunov's inequality is of fundamental importance as it has been successfully applied to establish the theoretical aspects of ordinary and partial differential equations, difference equations, dynamic equations on time scales, oscillation theory, disconjugacy, Hamiltonian systems, etc.; for details, see the text [2] and the references cited therein. Concerning the application of Lyapunov-type inequalities to eigenvalue problems, we refer the reader to the monograph [37]. For application of Lyapunov's inequality in control systems and stability of switched systems, for example, see [14, 6] and [18], respectively. The role of generalized Lyapunov inequalities in studying stochastic linear systems can be found in [5].

Let us recall that Lyapunov's inequality was firstly proposed and proved by Lyapunov [30], in which the necessary condition for the problem

$$\begin{cases} z''(t)+r(t)z(t)=0, \quad y_1<t<y_2, \\ z(y_1)=z(y_2)=0, \end{cases}$$

to have a nontrivial classical solution was

$$\int_{y_1}^{y_2} |r(s)|ds > \frac{4}{y_2-y_1}.$$

Mathematics subject classification (2020): 26A33, 34A08, 26D10, 34B27.
Keywords and phrases: Lyapunov-type inequality, fractional derivative, fractional integral, boundary value problems.

* Corresponding author.

During the last few decades, an overwhelming interest has been observed in developing the subject of fractional calculus. Numerous definitions of fractional derivatives and integrals were introduced. Accordingly, initial and boundary value problems involving different kinds of fractional differential operators were studied. This development motivated many researchers to work on the Lyapunov-type inequalities for fractional order boundary value problems. A fruitful technique for finding Lyapunov-type inequality for the fractional differential equations is to convert the fractional boundary value problem (FBVP for short) into an equivalent integral equation and then find the maximum of the Green's function involved in it.

Recently, in the survey [31], the Lyapunov-type inequalities for FBVP were discussed in detail. Here, it is imperative to mention that the survey in [31] is continuation of the survey papers [32] and [33]. In the present survey, we continue our efforts to collect the most recent results on Lyapunov-type inequalities for FBVP appearing in the literature after the publication of the surveys [31, 32, 33]. Precisely, a comprehensive and up-to-date review of Lyapunov-type inequalities for boundary value problems involving different kinds of fractional derivative operators and boundary conditions will be described.

The rest of the paper is organized as follows. We present Lyapunov-type inequalities for the problems containing fractional derivative due to Riemann–Liouville in Section 2, Caputo in Section 3, Riemann–Liouville and Caputo (mixed) in Section 4, Riesz-Caputo in Section 5, ψ-Caputo in Section 6, Hadamard in Section 7, Katugampola in Section 8, and Hilfer in Section 9. The Lyapunov-type inequalities for proportional, variable order Hadamard type, and partial fractional derivative operators are respectively discussed in Sections 10, 11 and 12. Section 13 deals with Lyapunov-type inequalities for systems of Riemann-Liouville fractional differential equations, while Lyapunov-type inequalities for bi-ordinal Hilfer-Katugampola and ψ-Hilfer fractional derivative operators are described in Section 14. In Section 15, we present Lyapunov-type inequalities for a discrete FBVP. We present all the results on Lyapunov-type inequalities without proofs. However, a complete reference for the details of each result elaborated in this survey is mentioned for the convenience of the reader.

2. Lyapunov-type inequalities for FBVP involving Riemann–Liouville fractional derivative

Let us first recall some basic definitions [22, 38].

DEFINITION 1. The left and right Riemann-Liouville fractional integrals $I^{\zeta}_{y_1+}$ and $I^{\zeta}_{y_2-}$ of order $\zeta > 0$ for a function $g \in L_1[y_1, y_2]$ are respectively defined by

$$(I^{\zeta}_{y_1+}g)(x) = \frac{1}{\Gamma(\zeta)} \int_{y_1}^{x} (x-t)^{\zeta-1} g(t)dt,$$

$$(I^{\zeta}_{y_2-}g)(x) = \frac{1}{\Gamma(\zeta)} \int_{x}^{y_2} (t-x)^{\zeta-1} g(t)dt,$$

where $\Gamma(\zeta)$ is the Euler Gamma function given by $\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt$ and

$$(I^0_{y_1-}g)(x)=(I^0_{y_1+}g)(x)=g(x).$$

DEFINITION 2. The left and right Riemann-Liouville fractional derivatives $D^\zeta_{y_1+}g$ and $D^\zeta_{y_2-}g$ of order $\zeta\in(n-1,n]$, where $g,g^{(n)}\in L_1[y_1,y_2]$, are respectively defined by

$$(D^\zeta_{y_1+}g)(x)=\Big(\frac{d}{dx}\Big)^n(I^{n-\zeta}_{y_1+}g)(x)\quad\text{and}\quad(D^\zeta_{y_2-}g)(x)=\Big(-\frac{d}{dx}\Big)^n(I^{n-\zeta}_{y_2-}g)(x).$$

DEFINITION 3. The left and right Caputo fractional derivatives ${}^CD^\zeta_{y_1+}g$ and ${}^CD^\zeta_{y_2-}g$ of order $\zeta\in(n-1,n]$ for a function $g\in AC^n[y_1,y_2]$ are respectively given by

$$({}^CD^\zeta_{y_1+}g)(x)=(I^{n-\zeta}_{y_1+}D^ng)(x)\quad\text{and}\quad({}^CD^\zeta_{y_2-}g)(x)=(-1)^n(I^{n-\zeta}_{y_2-}D^ng)(x).$$

In 2019, Y. Wang and Q. Wang [50] considered the following multi-point FBVP:

$$\begin{cases}D^\zeta_{y_1+}z(t)+r(t)z(t)=0,\ \ y_1<t<y_2,\ \ 2<\zeta\leqslant 3,\\ z(y_1)=z'(y_1)=0,\quad D^{\beta+1}_{y_1+}z(y_2)=\sum\limits_{i=1}^{m-2}\theta_i\,D^\beta_{y_1+}z(\xi_i),\end{cases}\tag{1}$$

where $D^\zeta_{y_1+}$ denotes the standard Riemann-Liouville fractional derivative operator of order ζ, $\zeta>\beta+2$, $0\leqslant\beta<1$, $y_1<\xi_1<\xi_2<\ldots<\xi_{m-2}<y_2$, $\theta_i\geqslant 0$ $(i=1,2,\ldots,m-2)$, $0\leqslant\sum_{i=1}^{m-2}\theta_i(\xi_i-y_1)^{\zeta-\beta-1}<(\zeta-\beta-1)(y_2-y_1)^{\zeta-\beta-2}$ and $r:[y_1,y_2]\to\mathbb{R}$ is a continuous function.

LEMMA 1. *A function $z\in C([y_1,y_2],\mathbb{R})$ is a solution of the FBVP (1) if and only if z satisfies the integral equation*

$$z(t)=\int_{y_1}^{y_2}G(t,s)r(s)z(s)ds+T(t)\int_{y_1}^{y_2}\Big(\sum_{i=1}^{m-2}\theta_i\,H(\tau,s)r(s)z(s)\Big)ds,$$

where

$$G(t,s)=\frac{1}{\Gamma(\zeta)}\begin{cases}\dfrac{(t-y_1)^{\zeta-1}(y_2-s)^{\zeta-\beta-2}}{(y_2-y_1)^{\zeta-\beta-2}}-(t-s)^{\zeta-1}, & y_1\leqslant s\leqslant t\leqslant y_2,\\ \dfrac{(t-y_1)^{\zeta-1}(y_2-s)^{\zeta-\beta-2}}{(y_2-y_1)^{\zeta-\beta-2}}, & y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

$$H(t,s)=\frac{1}{\Gamma(\zeta)}\begin{cases}\dfrac{(t-y_1)^{\zeta-\beta-1}(y_2-s)^{\zeta-\beta-2}}{(y_2-y_1)^{\zeta-\beta-2}}-(t-s)^{\zeta-\beta-1}, & y_1\leqslant s\leqslant t\leqslant y_2,\\ \dfrac{(t-y_1)^{\zeta-\beta-1}(y_2-s)^{\zeta-\beta-2}}{(y_2-y_1)^{\zeta-\beta-2}}, & y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

$$T(t) = \frac{(t-y_1)^{\zeta-1}}{(\zeta-\beta-1)(y_2-y_1)^{\zeta-\beta-2} - \sum_{i=1}^{m-2}\theta_i(\xi_i-y_1)^{\zeta-\beta-1}}, \ t \geqslant a.$$

LEMMA 2. *The function $G(t,s)$ defined in Lemma 1 satisfies the properties:*

(i) $G(t,s) \geqslant 0$, *for all* $(t,s) \in [y_1,y_2]\times[y_1,y_2]$;

(ii) $G(t,s)$ *is non-decreasing with respect to the first variable;*

(iii) $0 \leqslant G(y_1,s) \leqslant G(t,s) \leqslant G(y_2,s) = \frac{1}{\Gamma(\zeta)}(y_2-s)^{\zeta-\beta-2}\Big[(y_2-y_1)^{\beta+1} - (y_2-s)^{\beta+1}\Big]$, $(t,s)\in[y_1,y_2]\times[y_1,y_2]$;

(iv) *For any* $s \in [y_1,y_2]$,

$$\max_{s\in[y_1,y_2]} G(y_2,s) = \frac{\beta+1}{\zeta-1}\left(\frac{\zeta-\beta-2}{\zeta-1}\right)^{\frac{\zeta-\beta-2}{\beta+1}}\frac{(y_2-y_1)^{\zeta-1}}{\Gamma(\zeta)}.$$

LEMMA 3. *The function $H(t,s)$ defined in Lemma 1 satisfies the properties:*

(i) $H(t,s) \geqslant 0$, *for all* $(t,s) \in [y_1,y_2]\times[y_1,y_2]$;

(ii) $H(t,s)$ *is non-decreasing with respect to the first variable;*

(iii) $0 \leqslant H(y_1,s) \leqslant H(t,s) \leqslant H(y_2,s) = \frac{1}{\Gamma(\zeta)}(y_2-s)^{\zeta-\beta-2}(s-y_1)$, $(t,s)\in[y_1,y_2]\times[y_1,y_2]$;

(iv) $\max_{s\in[y_1,y_2]} H(y_2,s) = H(y_2,s^*) = \frac{(\zeta-\beta-2)^{\zeta-\beta-2}}{\Gamma(\zeta)}\left(\frac{y_2-y_1}{\zeta-\beta-1}\right)^{\zeta-\beta-1}$,
where $s^* = \frac{\zeta-\beta-2}{\zeta-\beta-1}y_1 + \frac{1}{\zeta-\beta-1}y_2$.

Now, we are ready to give the Lyapunov-type inequality for the FBVP (1).

THEOREM 1. *If a continuous nontrivial solution to the FBVP (1) exists, then*

$$\int_{y_1}^{y_2}(y_2-s)^{\zeta-\beta-2}\Big[(y_2-y_1)^{\beta+1}-(y_2-s)^{\beta+1}+\sum_{i=1}^{m-2}\theta_i T(y_2)(s-y_1)\Big]|r(s)|ds \geqslant \Gamma(\zeta),$$

and

$$\int_{y_1}^{y_2}|r(s)|ds \geqslant \frac{\Gamma(\zeta)}{Q},$$

where

$$Q = \frac{\beta+1}{\zeta-1}\left(\frac{\zeta-\beta-2}{\zeta-1}\right)^{\frac{\zeta-\beta-2}{\beta+1}}(y_2-y_1)^{\zeta-1} + \sum_{i=1}^{m-2}\theta_i T(y_2)(\zeta-\beta-2)^{\zeta-\beta-2}\left(\frac{y_2-y_1}{\zeta-\beta-1}\right)^{\zeta-\beta-1}.$$

In 2022, Dhar and Neugebauer [8] studied the FBVP:

$$\begin{cases} D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0, & y_1<t<y_2, \\ z(y_1)=0, \quad D^{\beta}_{y_1+}z(y_2)=0, \end{cases} \tag{2}$$

where $\zeta\in(1,2]$, $\beta\in[0,\zeta-1]$, $D^{\zeta}_{y_1+}$, $D^{\beta}_{y_1+}$ are Riemann-Liouville derivative operators of orders ζ and β, respectively, and $r\in C([y_1,y_2],\mathbb{R})$.

LEMMA 4. *A function $z\in C([y_1,y_2],\mathbb{R})$ is a solution to the FBVP (2) if and only if it satisfies the integral equation:*

$$z(t)=\int_{y_1}^{y_2}G(t,s)r(s)z(s)ds,$$

where $G(t,s)$ is the Green's functions given by

$$G(t,s)=\frac{1}{\Gamma(\zeta)}\begin{cases} \dfrac{(t-y_1)^{\zeta-1}(y_2-s)^{\zeta-1-\beta}}{(y_2-y_1)^{\zeta-1-\beta}}-(t-s)^{\zeta-1}, & y_1\leqslant s\leqslant t\leqslant y_2, \\ \dfrac{(t-y_1)^{\zeta-1}(y_2-s)^{\zeta-1-\beta}}{(y_2-y_1)^{\zeta-1-\beta}}, & y_1\leqslant t\leqslant s\leqslant y_2. \end{cases}$$

LEMMA 5. *The Green's function $G(t,s)$, defined in Lemma 4, satisfies the properties:*

(1) $G(t,s)\geqslant 0$ *for all* $(t,s)\in[y_1,y_2]\times[y_1,y_2]$;

(2) $\max_{t\in[y_1,y_2]}G(t,s)\leqslant G(s,s)$ *for* $s\in[y_1,y_2]$;

(3) $G(s,s)$ *has a unique maximum at* $\bar{s}=\dfrac{(\zeta-1)y_2+(\zeta-1-\beta)y_1}{2\zeta-2-\beta}$ *given by*

$$G(\bar{s},\bar{s})=\frac{1}{\Gamma(\zeta)}\Big(\frac{(y_2-y_1)(\zeta-1)}{2\zeta-2-\beta}\Big)^{\zeta-1}\Big(\frac{\zeta-1-\beta}{2\zeta-2-\beta}\Big)^{\zeta-1-\beta};$$

(4) $\displaystyle\int_{y_1}^{y_2}G(t,s)ds\leqslant\frac{(\zeta-1)^{\zeta-1}}{(\zeta-\beta)^{\zeta}\Gamma(\beta+1)}(y_2-y_1)^{\zeta}.$

Now, we present a Lyapunov-type inequality for the FBVP (2).

THEOREM 2. *If the FBVP (2) has a nontrivial continuous solution $z\in C([y_1,y_2],\mathbb{R})$ and $z(t)\neq 0$ on (y_1,y_2), where q is a real and continuous function defined on $[y_1,y_2]$, then*

$$\int_{y_1}^{y_2}r_+(s)ds\geqslant\Gamma(\zeta)\Big(\frac{2\zeta-2-\beta}{(y_2-y_1)(\zeta-1)}\Big)^{\zeta-1}\Big(\frac{2\zeta-2-\beta}{\zeta-1-\beta}\Big)^{\zeta-1-\beta},$$

where $r_+(s)=\max\{r(s),0\}$.

REMARK 1. The FBVP (2) has no nontrivial solution if

$$\int_{y_1}^{y_2} r_+(s)ds \leqslant \Gamma(\zeta)\Big(\frac{2\zeta-2-\beta}{(y_2-y_1)(\zeta-1)}\Big)^{\zeta-1}\Big(\frac{2\zeta-2-\beta}{\zeta-1-\beta}\Big)^{\zeta-1-\beta}.$$

In 2023, Eloe et al. considered in [10] the (n,p)-type nonlinear FBVP:

$$\begin{cases} D^{\zeta}_{y_1+}z(t)+\ell(t)Fz(t)=0, & y_1<t<y_2, \\ y^{(i)}(y_1)=0, \ \ i=0,1,\dots,n-2, & D^{p}_{y_1+}z(y_2)=0, \end{cases} \tag{3}$$

where $y_2>y_1$, $\zeta\in(n-1,n]$, $n\geqslant 2$ is an integer, $p\in[0,\zeta-1]$ and $D^{(\cdot)}_{y_1+}$ denotes the Riemann-Liouville derivative operator, $\ell\in C([y_1,y_2],\mathbb{R})$ and $F:[y_1,y_2]\to[y_1,y_2]$.

LEMMA 6. *Let $x\in C([y_1,y_2],\mathbb{R})$, $\zeta\in(n-1,n]$, where $n\geqslant 2$ denotes an integer, and $p\in[0,\zeta-1]$. Then, $z\in C([y_1,y_2],\mathbb{R})$ is a solution to the FBVP:*

$$\begin{cases} D^{\zeta}_{y_1+}z(t)+x(t)=0, & y_1<t<y_2, \\ y^{(i)}(y_1)=0, \ \ i=0,1,\dots,n-2, & D^{p}_{y_1+}z(y_2)=0, \end{cases} \tag{4}$$

if and only if

$$z(t)=\int_{y_1}^{y_2} G_\zeta(t,s)x(s)ds,$$

where

$$G_\zeta(t,s)=\frac{1}{\Gamma(\zeta)}\begin{cases} \dfrac{(y_2-s)^{\zeta-p-1}(t-y_1)^{\zeta-1}}{(y_2-y_1)^{\zeta-p-1}}-(t-s)^{\zeta-1}, & y_1\leqslant s\leqslant t\leqslant y_2, \\ \dfrac{(y_2-s)^{\zeta-p-1}(t-y_1)^{\zeta-1}}{(y_2-y_1)^{\zeta-p-1}}, & y_1\leqslant s\leqslant t\leqslant y_2, \end{cases}$$

if $p<\zeta-1$, and

$$G_\zeta(t,s)=\frac{1}{\Gamma(\zeta)}\begin{cases} (t-y_1)^{\zeta-1}-(t-s)^{\zeta-1}, & y_1\leqslant s\leqslant t\leqslant y_2, \\ (t-y_1)^{\zeta-1}, & y_1\leqslant s\leqslant t\leqslant y_2, \end{cases}$$

if $p=\zeta-1$.

Now, we find the bounds for the Green's function $G_\zeta(t,s)$ for three cases:

(I) $1<\zeta\leqslant 2$, and $0\leqslant p\leqslant\zeta-1$;

(II) $2<\zeta$, and $0\leqslant p<1$;

(III) $2<\zeta$, and $1\leqslant p\leqslant\zeta-1$.

LEMMA 7. *Assume that $1<\zeta\leqslant 2$ and $0\leqslant p\leqslant\zeta-1$. Then, the Green's function defined in Lemma 6 has the properties:*

(i) $G_\zeta(t,s) \geqslant 0$, *for all* $t,s \in [y_1,y_2]$;

(ii) $\max_{t\in[y_1,y_2]} G_\zeta(t,s) = G_\zeta(s,s)$

$$= \frac{1}{\Gamma(\zeta)} \begin{cases} \dfrac{(y_2-s)^{\zeta-p-1}(s-y_1)^{\zeta-1}}{(y_2-y_1)^{\zeta-p-1}}, & \forall s \in [y_1,y_2], \ \textit{if} \ p < \zeta-1, \\ (s-y_1)^{\zeta-1}, & \forall s \in [y_1,y_2], \ \textit{if} \ p = \zeta-1; \end{cases}$$

(iii) $\max_{s\in[y_1,y_2]} G_\zeta(s,s)$

$$= \begin{cases} \dfrac{1}{\Gamma(\zeta)} \left[\dfrac{\zeta-p-1}{2(\zeta-1)-p}\right]^{\zeta-p-1} \left[\dfrac{\zeta-1}{2(\zeta-1)-p}\right]^{\zeta-1}, & \textit{if} \ p < \zeta-1, \\ \dfrac{(y_2-y_1)^{\zeta-1}}{\Gamma(\zeta)}, & \textit{if} \ p = \zeta-1. \end{cases}$$

LEMMA 8. *Assume that* $2 < \zeta$ *and* $0 \leqslant p \leqslant \zeta-1$. *Then, the Green's function defined in Lemma 6 has the properties:*

(i) $G_\zeta(t,s) \geqslant 0$, *for all* $t,s \in [y_1,y_2]$;

(ii) $\max_{t\in[y_1,y_2]} G_\zeta(t,s) = G_\zeta(s^*,s)$, *where* $s^* = \dfrac{s-y_2D}{1-D}$ *and* $D = \left(\dfrac{y_2-s}{y_2-y_1}\right)^{\frac{\zeta-p-1}{\zeta-2}}$;

(iii) $\max_{s\in[y_1,y_2]} G_\zeta(s^*,s) = \dfrac{(y_2-y_1)^{\zeta-p-1}\varpi_{\zeta,p}^{\zeta-p-1}(1-\varpi_{\zeta,p})^{\zeta-1}}{\Gamma(\zeta)\left(1-\varpi_{\zeta,p}^{\zeta-p-1}\right)^{\zeta-2}}$, *where* $\varpi_{\zeta,p}$ *is the unique solution of the nonlinear equation:*

$$\varpi_{\zeta,p}^{\frac{2\zeta-p-3}{\zeta-2}} - \left(2-\frac{p}{\zeta-1}\right)\varpi + \frac{\zeta-p-1}{\zeta-1} = 0,$$

in the interval $\left(0, \left[\left(2-\dfrac{p}{\zeta-1}\right)\left(\dfrac{\zeta-2}{2\zeta-p-3}\right)\right]^{\frac{\zeta-2}{\zeta-p-1}}\right) \subset (0,1)$.

LEMMA 9. *Assume that* $2 < \zeta$ *and* $1 \leqslant p \leqslant \zeta-1$. *Then, the Green's function defined in Lemma 6 has the properties:*

(i) $G_\zeta(t,s) \geqslant 0$, *for all* $t,s \in [y_1,y_2]$;

(ii) $G_\zeta(t,s) \leqslant G_\zeta(y_2,s)$ *for* $t,s \in [y_1,y_2]$;

(iii) $\max_{s\in[y_1,y_2]} G_\zeta(y_2,s) = \dfrac{(y_2-y_1)^{\zeta-1}}{\Gamma(\zeta)}$.

We need the following assumptions to obtain Lyapunov-type inequalities for the FBVP (3):

(A_1) $F : C[y_1,y_2] \to C[y_1,y_2]$ and there exists $\eta > 0$, independent of z such that, if $y \in C[y_1,y_2]$, then $\|Fy\|_\infty \leqslant \eta \|y\|_\infty$;

(A_2) $\ell \in C([y_1,y_2],\mathbb{R})$.

THEOREM 3. *Assume that* (A_1) *and* (A_2) *hold. Then, for* $1 < \zeta < 2$ *and* $0 \leqslant p \leqslant \zeta - 1$, *the estimate*

$$\int_{y_1}^{y_2} |\ell(s)|ds < \frac{\Gamma(\zeta)}{\eta}\left[\frac{2(\zeta-1)-p}{\zeta-p-1}\right]^{\zeta-p-1}\left[\frac{2(\zeta-1)-p}{\zeta-1}\right]^{\zeta-1}\left(\frac{1}{y_2-y_1}\right)^{\zeta-1}$$

implies that the FBVP (3) *has only the trivial solution* $z(t) \equiv 0$.

THEOREM 4. *Let* $2 < \zeta$ *and* $0 \leqslant p < 1$. *If* (A_1) *and* (A_2) *hold, then the estimate*

$$\int_{y_1}^{y_2} |\ell(s)|ds < \frac{\Gamma(\zeta)\left(1-\varpi_{\zeta,p}^{\frac{\zeta-p-1}{\zeta-2}}\right)^{\zeta-2}}{\eta(y_2-y_1)^{\zeta-p-1}\varpi_{\zeta,p}^{\zeta-p-1}(1-\varpi_{\zeta,p})^{\zeta-1}}$$

implies that the FBVP (3) *has only the trivial solution* $z(t) \equiv 0$, *where* $\varpi_{\zeta,p}$ *is the unique solution of the nonlinear equation:*

$$\varpi^{\frac{2\zeta-p-3}{\zeta-2}} + \left(\frac{p}{\zeta-1} - 1\right)\varpi + \frac{\zeta-p-1}{\zeta-1} = 0,$$

in the interval $\left(0, \left[\left(2-\frac{p}{\zeta-1}\right)\left(\frac{\zeta-2}{2\zeta-p-3}\right)\right]^{\frac{\zeta-2}{\zeta}-p-1}\right)$.

THEOREM 5. *Suppose that* (A_1) *and* (A_2) *hold and* $1 \leqslant p \leqslant \zeta - 1$ *with* $2 < \zeta$. *Then, the estimate*

$$\int_{y_1}^{y_2} |\ell(s)|ds < \frac{\Gamma(\zeta)}{(y_2-y_1)^{\zeta-1}\eta}$$

implies that the FBVP (3) *has only the trivial solution* $z(t) \equiv 0$.

In [42], Silva studied the following FBVP with (left) Riemann-Liouville fractional derivative operator:

$$\begin{cases} (D^{\zeta}_{y_1+}z)(t) + (D^{\beta}_{y_1+}rz)(t) = 0, \ \ t \in [y_1,y_2], \\ z(y_1) = z(y_2) = 0, \end{cases} \tag{5}$$

with $0 \leqslant y_1 < y_2$, $\zeta \in (1,2)$, $\beta \in (0,\zeta-1)$ and $r : [y_1,y_2] \to \mathbb{R}$ is a continuous function.

LEMMA 10. *Let* $1 < \zeta < 2$, $0 < \beta < \zeta - 1$ *and* $r \in C([y_1,y_2],\mathbb{R})$. *Then, the integral solution* z *of FBVP* (5) *is*

$$z(t) = \int_{y_1}^{y_2} G(t,s)r(s)z(s)ds,$$

where the Green's function $G(t,s)$ is defined by

$$G(t,s)=\frac{1}{\Gamma(\zeta-\beta)}\begin{cases}\dfrac{(t-y_1)^{\zeta-1}(y_2-s)^{\zeta-\beta-1}}{(y_2-y_1)^{\zeta-1}}, & y_1\leqslant t\leqslant s\leqslant y_2,\\ \dfrac{(t-y_1)^{\zeta-1}(y_2-s)^{\zeta-\beta-1}}{(y_2-y_1)^{\zeta-1}}-(t-s)^{\zeta-\beta-1}, & y_1\leqslant s\leqslant t\leqslant y_2.\end{cases}$$

LEMMA 11. *For any $(t,s)\in[y_1,y_2]\times[y_1,y_2]$, the maximum value of the Green's function defined in Lemma 10 is given by*

$$\max_{t,s\in[y_1,y_2]}|G(t,s)|=\frac{1}{\Gamma(\zeta-\beta)}\Big(\frac{\zeta-1}{2\zeta-\beta-2}\Big)^{\zeta-1}\Big(\frac{\zeta-\beta-1}{2\zeta-\beta-2}\Big)^{\zeta-\beta-1}(y_2-y_1)^{\zeta-\beta-1},$$

with $\zeta\in(1,2)$ and $\beta\in(0,\zeta-1)$.

In the following theorem, we give a Lyapunov-type inequality for the FBVP (5).

THEOREM 6. *Let z be a nontrivial continuous solution to the FBVP (5), then*

$$\int_{y_1}^{y_2}|r(s)|ds\geqslant\frac{\Gamma(\zeta-\beta)(y_2-y_1)^{1+\beta-\zeta}}{\Big(\frac{\zeta-1}{2\zeta-\beta-2}\Big)^{\zeta-1}\Big(\frac{\zeta-\beta-1}{2\zeta-\beta-2}\Big)^{\zeta-\beta-1}}.$$

3. Lyapunov-type inequalities for FBVP involving Caputo fractional derivative operator

In [46], Toprakseven studied the Liouville-Caputo type FBVP:

$$\begin{cases}{}^{C}D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0, & 2<\zeta\leqslant 3,\ y_1\leqslant t\leqslant y_2,\\ z(y_1)=z(y_3)=z(y_2)=0, & y_1<y_3<y_2,\end{cases}\tag{6}$$

where ${}^{C}D^{\zeta}_{y_1+}$ is the Liouville-Caputo derivative and $r\in C([y_1,y_2],\mathbb{R})$.

LEMMA 12. *Let $\zeta=\beta+1\in(2,3]$ with $\beta\in(1,2]$. Then, $z\in C^3[y_1,y_2]$ is a solution to the FBVP (6) if and only if z is a solution of the integral equation*

$$z(t)=\int_{x_1}^{x_2}\Big(\int_b^t G(\eta,s)d\eta\Big)p(s)z(s)ds,\quad x_1\in[y_1,y_3],\ x_2\in[y_3,y_2],$$

where the Green's function $G(t,s)$ is given by

$$G(t,s)=\frac{1}{\Gamma(\zeta-1)}\begin{cases}\dfrac{t-x_1}{x_2-x_1}(x_2-s)^{\zeta-2}-(t-s)^{\zeta-2}, & x_1\leqslant s\leqslant t\leqslant x_2,\\ \dfrac{t-x_1}{x_2-x_1}(x_2-s)^{\zeta-2}, & x_1\leqslant t\leqslant s\leqslant x_2.\end{cases}$$

We now present Hartman-Wintner-type inequalities for the FBVP (6).

THEOREM 7. *If the FBVP* (6) *has a solution* $z(t) \not\equiv 0$ *for* $t \in (y_1, y_3) \cup (y_3, y_2)$*, then one of the following Hartman-Wintner-type inequalities holds:*

$$(A) \quad \int_{y_1}^{y_2} (s-y_1)(y_2-s)^{\zeta-2} r_-(s)ds > \Gamma(\zeta-1)(\zeta-1),$$

$$(B) \quad \int_{y_1}^{y_2} (s-y_1)(y_2-s)^{\zeta-2} r_+(s)ds > \Gamma(\zeta-1)(\zeta-1),$$

$$(C) \quad \int_{y_1}^{y_3} (s-y_1)(y_2-s)^{\zeta-2} r_-(s)ds + \int_{y_3}^{y_2} (s-y_1)(y_2-s)^{\zeta-2} r_+(s)ds > \Gamma(\zeta-1)(\zeta-1),$$

where $r_-(s) = \max\{-r(s), 0\}$ *and* $r_+(s) = \max\{r(s), 0\}$.

As the first consequence of Theorem 7, we have the following Lyapunov-type inequalities.

COROLLARY 1. *If the problem* (6) *has a nontrivial solution, then one of the following Lyapunov-type inequalities holds:*

$$(1) \quad \int_{y_1}^{y_2} r_-(s)ds > \frac{\Gamma(\zeta-1)(\zeta-1)(\zeta-1)^{\zeta-1}}{(y_2c-y_2)^{\zeta-1}(\zeta-2)^{\zeta-2}},$$

$$(2) \quad \int_{y_1}^{y_2} r_+(s)ds > \frac{\Gamma(\zeta-1)(\zeta-1)(\zeta-1)^{\zeta-1}}{(y_2-y_1)^{\zeta-1}(\zeta-2)^{\zeta-2}},$$

$$(3) \quad \int_{y_1}^{y_3} r_-(s)ds + \int_{y_3}^{y_2} r_+(s)ds > \frac{\Gamma(\zeta-1)(\zeta-1)(\zeta-1)^{\zeta-1}}{(y_2-y_1)^{\zeta-1}(\zeta-2)^{\zeta-2}}.$$

Consequently, we get

$$\int_{y_1}^{y_2} |r(s)|ds > \frac{\Gamma(\zeta-1)(\zeta-1)(\zeta-1)^{\zeta-1}}{(y_2-y_1)^{\zeta-1}(\zeta-2)^{\zeta-2}}.$$

As a second consequence of Theorem 7, we find a lower bound for the eigenvalues of the FBVP considered in the next corollary.

COROLLARY 2. *If* λ *is an eigenvalue of the FBVP:*

$$\begin{cases} {}^C D_{y_1}^{\zeta} z(t) + \lambda z(t) = 0, & t \in [y_1, y_2], \quad \zeta \in (2,3], \\ z(y_1) = z(y_3) = z(y_2) = 0, & y_1 < y_3 < y_2, \end{cases}$$

then

$$|\lambda| > \frac{\zeta(\zeta-1)^2\Gamma(\zeta-1)}{(y_2-y_1)^{\zeta}}.$$

In [44], Srivastava et al. studied the FBVP:

$$\begin{cases} {}^{C}D^{\zeta}z(t)+r(t)z(t))=0, \ \ y_1<t<y_2, \ \ 1<\zeta\leqslant 2, \\ z(y_1)-d_1z'(y_1)=e_1\Psi[z], \\ z(y_2)-d_2z'(y_2)=e_2\Phi[z], \end{cases} \tag{7}$$

where

$$\Psi[z]=\int_{y_1}^{y_2} z(t)dA(t), \quad \Phi[z]=\int_{y_1}^{y_2} z(t)dB(t),$$

are the Riemann-Stieltjes integrals, A,B are functions of bounded variations, and d_1, d_2, e_1, and e_2 are nonnegative real constants.

In the subsequent results, we use the notation: $\lambda=\lambda_1\lambda_4+\lambda_2\lambda_3$, where $\lambda_1=1-e_1\Psi[1]$, $\lambda_2=d_1+e_1(\Psi[\hat{t}]-y_2\Psi[1])$, $\lambda_3=1-e_2\Phi[1]$, $\lambda_4=y_2-y_1-d_2-e_2(\Phi[\hat{t}]-y_2\Phi[1])$, and

$$\Psi[1]=\int_{y_1}^{y_2} dA(t), \quad \Psi[\hat{t}]=\int_{y_1}^{y_2} tdA(t), \quad \Phi[1]=\int_{y_1}^{y_2} dB(t), \quad \Phi[\hat{t}]=\int_{y_1}^{y_2} tdB(t),$$

$$J_A(s)=\int_{s}^{y_2}(t-s)^{\zeta-1}dA(t), \quad J_B(s)=\int_{s}^{y_2}(t-s)^{\zeta-1}dB(t).$$

LEMMA 13. *The unique solution $z\in C(y_1,y_2)\cap L(y_1,y_2)$ of the FBVP (7) is given by the integral equation*

$$z(t)=\int_{y_1}^{y_2} G(t,s)r(s)z(s)ds,$$

where $G(t,s)$ is the Green's function represented as

$$G(t,s)=G_1(t,s)+G_2(t,s),$$

with

$$\begin{aligned} G_1(t,s) &= \frac{1}{\lambda\Gamma(\zeta)}\Big[e_1(-\lambda_4+\lambda_3(t-y_1))J_A(s)-e_2(\lambda_2+\lambda_1(t-y_1))J_B(s)\Big], \\ G_2(t,s) &= \frac{1}{\lambda\Gamma(\zeta)}(y_2-s)^{\zeta-2}H(t,s), \end{aligned}$$

and

$$H(t,s)=\begin{cases} (\lambda_2+\lambda_1(t-y_1)(y_2-s-d_2(\zeta-1)) \\ -\lambda(t-s)^{\zeta-1}(y_2-s)^{2-\zeta}, & y_1\leqslant s\leqslant t\leqslant y_2, \\ (\lambda_2+\lambda_1(t-y_1)(y_2-s-d_2(\zeta-1)), & y_1\leqslant t\leqslant s\leqslant y_2. \end{cases}$$

Next, we obtain the upper estimates on the Green's function to obtain a Lyapunov-type inequality for the FBVP (7).

LEMMA 14. *(i) Let* $e_1 \in \left(0, \frac{1}{\Psi[1]}\right)$, $e_2 \in \left(0, \frac{1}{\Phi[1]}\right)$ *and* $\lambda > 0$. *Then*

$$|G_1(t,s)| \leqslant \frac{(y_2-s)^{\zeta-2}}{\lambda\Gamma(\zeta)} \max\{e_1(y_2-y_1)\Psi[1](d_1+e_2(y_2-y_1)\Phi[1]), \\ e_2(y_2-y_1)\Phi[1](d_1+y_2-y_1)\}, \quad y_1 \leqslant s,t \leqslant y_2;$$

(ii) Let $1-e_1\Psi[1]>0$, $1-e_2\Phi[1]>0$, $\lambda>0$, $\lambda(y_2-y_1-d_2(\zeta-1)) \geqslant \lambda_2$. *Then, for* $y_1 \leqslant s,t \leqslant y_2$, *we have*

$$G_2(t,s) \leqslant \frac{(y_2-s)^{\zeta-2}}{\lambda\Gamma(\zeta)} \max\left\{\frac{[\lambda_2+\lambda_1(y_2-y_1-d_2(\zeta-1))]^2}{4\lambda_1}, \frac{(\lambda+\lambda_1 d_2(\zeta-1))^2}{4\lambda_1}\right\}.$$

THEOREM 8. *Let* $1-e_1\Psi[1]>0$, $1-e_2\Phi[1]>0$, $\lambda>0$, $\lambda(y_2-y_1-d_2(\zeta-1)) \geqslant \lambda_2$. *If a continuous nontrivial solution of the FBVP* (7) *exists, then*

$$\int_{y_1}^{y_2} (y_2-s)^{\zeta-2}|r(s)|ds \geqslant \frac{\lambda\Gamma(\zeta)}{\Delta_1+\Delta_2},$$

where

$$\Delta_1 = \max\{(y_2-y_1)e_1\Psi[1](d_2+e_2(y_2-y_1)\Phi[1]), (y_2-y_1)e_2\Phi[1](d_1+y_2-y_1)\},$$

$$\Delta_2 = \max\left\{\frac{[\lambda_2+\lambda_1(y_2-y_1-d_2(\zeta-1))]^2}{4\lambda_1}, \frac{(\lambda+\lambda_1 d_2(\zeta-1))^2}{4\lambda_1}\right\}.$$

In [24], Laadjal and Ma discussed Lyapunov-type inequalities for the Langevin type boundary value problems:

$$\begin{cases} {}^C D^{\beta}_{y_1+}\left({}^C D^{\zeta}_{y_1+}+\lambda\right)z(t)+r(t)z(t)=0, \quad y_1<t<y_2, \\ z(y_1)=z(y_2)=0, \end{cases} \tag{8}$$

and

$$\begin{cases} {}^C D^{\beta}_{y_1+}\left({}^C D^{\zeta}_{y_1+}+\lambda\right)z(t)+r(t)z(t)=0, \quad y_1<t<y_2, \\ z(y_1)=0={}^C D^{\gamma}_{y_1+}z(y_2), \end{cases} \tag{9}$$

where (either $0<\beta\leqslant 1$ or $1<\beta\leqslant 2$), $0<\zeta$, $\gamma\leqslant 1$, $\lambda\in\mathbb{R}$, such that $1<\beta+\zeta\leqslant 2$ and $\gamma<\beta$, ${}^C D^{\beta}_{y_1+}$, ${}^C D^{\zeta}_{y_1+}$ and ${}^C D^{\gamma}_{y_1+}$ respectively denote the Liouville-Caputo fractional derivative operators of order β, ζ and γ, and $r:[y_1,y_2]\to\mathbb{R}$ is a continuous function.

We consider two cases according to the values of β: (i) $0<\beta\leqslant 1$ and (ii) $1<\beta\leqslant 2$. In each case, we discuss the problems (8) and (9). By using the Green's function and its properties for each problem, we obtain the Lyapunov-type inequalities.

3.1. Discussion of FBVP (8) for the case $0<\beta\leqslant 1$

LEMMA 15. *Assume that* $0<\beta\leqslant 1$. *A function* $z\in C([y_1,y_2],\mathbb{R})$ *is a solution to the FBVP* (8) *if and only if it satisfies the integral equation*

$$z(t)=\int_{y_1}^{y_2}\lambda G_\beta(t,s)z(s)ds+\int_{y_1}^{y_2}G_{\zeta+\beta}(t,s)r(s)z(s)ds,$$

where

$$G_\delta(t,s)=\frac{1}{\Gamma(\delta)}\begin{cases}\left(\frac{t-y_1}{y_2-y_1}\right)^\beta(y_2-s)^{\delta-1}-(t-s)^{\delta-1}, & y_1\leqslant s<t\leqslant y_2,\\ \left(\frac{t-y_1}{y_2-y_1}\right)^\beta(y_2-s)^{\delta-1}, & y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

with $\delta\in\{\beta,\zeta+\beta\}$.

LEMMA 16. *For* $0<\beta\leqslant 1$, *the functions* G_δ, $\delta\in\{\beta,\zeta+\beta\}$, *defined in Lemma 15, satisfy the properties:*

(i) $\displaystyle\max_{t\in[y_1,y_2]}\int_{y_1}^{y_2}|G_\beta(t,s)|ds=\frac{(y_2-y_1)^\beta}{2^{2\beta-1}\Gamma(\beta+1)}$;

(ii) $\displaystyle\max_{t,s\in[y_1,y_2]}|G_{\zeta+\beta}(t,s)|=\frac{[(\zeta+\beta-1)(y_2-y_1)]^{\zeta+\beta-1}\beta^\beta}{(\beta+2\beta-1)^{\beta+2\beta-1}\Gamma(\zeta+\beta)}$.

In consequence, we express the Lyapunov-type inequality for the FBVP (8) as follows.

THEOREM 9. *Assume that* $0<\beta\leqslant 1$. *If a continuous nontrivial solution to the FBVP* (8) *exists, then*

$$\int_{y_1}^{y_2}|r(s)|ds\geqslant\frac{(\beta+2\beta-1)^{\beta+2\beta-1}\Gamma(\zeta+\beta+1)}{[(\zeta+\beta-1)(y_2-y_1)]^{\zeta+\beta-1}\beta^\beta}\left(1-\frac{|\lambda|(y_2-y_1)^\beta}{2^{2\beta-1}\Gamma(\beta+1)}\right).$$

3.1.1. Discussion of FBVP (9) for the case $0<\beta\leqslant 1$

LEMMA 17. *Assume that* $0<\beta\leqslant 1$. *A function* $z\in C([y_1,y_2],\mathbb{R})$ *is a solution to the FBVP* (9) *if and only if it satisfies the integral equation*

$$z(t)=\int_{y_1}^{y_2}\lambda\bar{G}(t,s)z(s)ds+\int_{y_1}^{y_2}(y_2-s)^{\zeta+\beta-\gamma-1}G(t,s)r(s)z(s)ds,$$

where

$$\bar{G}(t,s)=\frac{1}{\Gamma(\beta+1}\begin{cases}\frac{(\beta-\gamma)(t-y_1)^\beta}{(y_2-y_1)^{\beta-\gamma}}(y_2-s)^{\beta-\gamma-1}-\beta(t-s)^{\beta-1}, & y_1\leqslant s<t\leqslant y_2,\\ \frac{(\beta-\gamma)(t-y_1)^\beta}{(y_2-y_1)^{\beta-\gamma}}(y_2-s)^{\beta-\gamma-1}, & y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

and

$$G(t,s) = \begin{cases} \dfrac{\Gamma(\beta+1-\gamma)(t-y_1)^{\beta}}{\Gamma(\beta+1)\Gamma(\zeta+\beta-\gamma)(y_2-y_1)^{\beta-\gamma}} \\ -\dfrac{(t-s)^{\zeta+\beta-1}}{\Gamma(\zeta+\beta)(y_2-s)^{\zeta+\beta-\gamma-1}}, & y_1 \leqslant s \leqslant t \leqslant y_2, \\ \dfrac{\Gamma(\beta+1-\gamma)(t-y_1)^{\beta}}{\Gamma(\beta+1)\Gamma(\zeta+\beta-\gamma)(y_2-y_1)^{\beta-\gamma}}, & y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

LEMMA 18. *For $0<\beta\leqslant 1$, the functions $\bar{G}$ and G defined in Lemma 17 satisfy the properties:*

(i) For any $t\in[y_1,y_2]$, $\displaystyle\int_{y_1}^{y_2}|\bar{G}(t,s)|ds \leqslant \frac{2(y_2-y_1)^{\beta}}{\Gamma(\beta+1)}$;

(ii) $\displaystyle\max_{t,s\in[y_1,y_2]}|G(t,s)| = C$, *where*

$$C = (y_2-y_1)^{\gamma}\max\Big\{\frac{\Gamma(\beta+1-\gamma)}{\Gamma(\zeta+\beta-\gamma)\Gamma(\beta+1)}, \\ \frac{(1-\zeta)}{\beta\Gamma(\zeta+\beta)}\Big(\frac{\Gamma(\beta+1-\gamma)\Gamma(\zeta+\beta-\gamma)}{\Gamma(\zeta+\beta-\gamma)\Gamma(\beta)}\Big)^{\frac{\zeta+\beta-1}{\zeta-1}}, \text{ with } \zeta<1\Big\}. \tag{10}$$

In relation to the FBVP (9), we have the following Lyapunov-type inequality.

THEOREM 10. *Assume that $0<\beta\leqslant 1$. If a continuous nontrivial solution to the FBVP (9) exists, then*

$$\int_{y_1}^{y_2}(y_2-s)^{\zeta+\beta-\gamma-1}|r(s)|ds \geqslant \frac{1}{C}\Big(1-\frac{2|\lambda|(y_2-y_1)^{\beta}}{\Gamma(\beta+1)}\Big),$$

where C is given in (10).

3.2. Discussion of FBVP (8) for the case $1<\beta\leqslant 2$

LEMMA 19. *Assume that $1<\beta\leqslant 2$. A function $z\in C^2([y_1,y_2],\mathbb{R})$ is a solution to the FBVP (8) if and only if it satisfies the integral equation*

$$z(t) = \int_{y_1}^{y_2}\lambda\widehat{G}_{\beta}(t,s)z(s)ds + \int_{y_1}^{y_2}\widehat{G}_{\zeta+\beta}(t,s)r(s)z(s)ds,$$

where

$$\widehat{G}_{\delta}(t,s) = \frac{1}{\Gamma(\delta)}\begin{cases} \dfrac{t-y_1}{y_2-y_1}(y_2-s)^{\delta-1}-(t-s)^{\delta-1}, & y_1\leqslant s<t\leqslant y_2, \\ \dfrac{t-y_1}{y_2-y_1}(y_2-s)^{\delta-1}, & y_1\leqslant t\leqslant s\leqslant y_2, \end{cases}$$

with $\delta\in\{\beta,\zeta+\beta\}$.

LEMMA 20. *For* $1 < \beta \leqslant 2$, *the functions* $\widehat{G}_\delta, \delta \in \{\beta, \zeta+\beta\}$, *defined in Lemma* 19, *satisfy the properties:*

(i) *For any* $t \in [y_1, y_2]$, *we have* $\int_{y_1}^{y_2} |\widehat{G}_\beta(t,s)|ds = \frac{2(y_2-y_1)^\beta}{\Gamma(\beta+1)}$;

(ii) $\max_{t,s\in[y_1,y_2]} |\widehat{G}_{\zeta+\beta}(t,s)| = \frac{[(\zeta+\beta-1)(y_2-y_1)]^{\zeta+\beta-1}}{(\zeta+\beta)^{\zeta+\beta}\Gamma(\zeta+\beta)}$.

The Lyapunov-type inequality for the FBVP (8) is given in the following result.

THEOREM 11. *Assume that* $1 < \beta \leqslant 2$. *If a continuous nontrivial solution to the FBVP* (8) *exists, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant \frac{(\zeta+\beta)^{\zeta+\beta}\Gamma(\zeta+\beta)}{[(\zeta+\beta-1)(y_2-y_1)]^{\zeta+\beta-1}}\Big(1 - \frac{2|\lambda|(y_2-y_1)^\beta}{\Gamma(\beta+1)}\Big).$$

3.2.1. Discussion of FBVP (9) for the case $1 < \beta \leqslant 2$

LEMMA 21. *Assume that* $1 < \beta \leqslant 2$. *A function* $z \in C^2([y_1,y_2],\mathbb{R})$ *is a solution to the FBVP* (9) *if and only if*

$$z(t) = \int_{y_1}^{y_2} \lambda \tilde{G}_1(t,s)z(s)ds + \int_{y_1}^{y_2} (y_2-s)^{\zeta+\beta-\gamma-1}\tilde{G}_2(t,s)r(s)z(s)ds,$$

where

$$\tilde{G}_1(t,s) = \begin{cases} \dfrac{\Gamma(2-\gamma)(t-y_1)}{\Gamma(\beta-\gamma)(y_2-y_1)^{1-\gamma}}(y_2-s)^{\beta-\gamma-1} - \dfrac{1}{\Gamma(\beta)}(t-s)^{\beta-1}, & y_1 \leqslant s < t \leqslant y_2, \\ \dfrac{\Gamma(2-\gamma)(t-y_1)}{\Gamma(\beta-\gamma)(y_2-y_1)^{1-\gamma}}(y_2-s)^{\beta-\gamma-1}, & y_1 \leqslant t \leqslant s \leqslant y_2, \end{cases}$$

and

$$\tilde{G}_2(t,s) = \begin{cases} \dfrac{\Gamma(2-\gamma)(t-y_1)}{\Gamma(\zeta+\beta-\gamma)(y_2-y_1)^{1-\gamma}} - \dfrac{(t-s)^{\zeta+\beta-1}}{\Gamma(\zeta+\beta)(y_2-s)^{\zeta+\beta-\gamma-1}}, & y_1 \leqslant s \leqslant t \leqslant y_2, \\ \dfrac{\Gamma(2-\gamma)(t-y_1)}{\Gamma(\zeta+\beta-\gamma)(y_2-y_1)^{1-\gamma}}, & y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

LEMMA 22. *Let* $1 < \beta \leqslant 2$. *The functions* $\tilde{G}_1$ *and* $\tilde{G}_2$ *defined in the previous lemma, satisfy the properties:*

(i) *For any* $t \in [y_1, y_2]$, *we have*

$$\int_{y_1}^{y_2} |\tilde{G}_1(t,s)|ds \leqslant \Big(\frac{\Gamma(2-\gamma)}{\Gamma(\beta-\gamma-1)} + \frac{1}{\Gamma(\beta+1)}\Big)(y_2-y_1)^\beta;$$

(ii) $\max_{t,s\in[y_1,y_2]} |\tilde{G}_2(t,s)| = \tilde{C}$, *where* $\tilde{C} = C$ (C *is defined by* (10) *with* $1 < \beta \leqslant 2$).

Now, we state the Lyapunov-type inequality for the FBVP (9) as follows.

THEOREM 12. *Assume that* $1 < \beta \leqslant 2$. *If a nontrivial continuous solution to the FBVP* (9) *on* $[y_1, y_2]$ *exists, then*

$$\int_{y_1}^{y_2} (y_2 - s)^{\zeta+\beta-\gamma-1}|r(s)|ds \geqslant \frac{1}{\tilde{C}}\Big(1 - |\lambda|(y_2 - y_1)^{\beta}\Big(\frac{\Gamma(2-\gamma)}{\Gamma(\beta-\gamma-1)} + \frac{1}{\Gamma(\beta+1)}\Big)\Big).$$

4. Lyapunov-type inequalities for FBVP involving mixed Riemann-Liouville and Caputo fractional derivatives

In 2022, Liu and Li [26] studied the FBVP:

$$\begin{cases} {}^{C}D^{\beta}_{y_1+}D^{\zeta}_{y_1+}z(t) + r(t)z(t) = 0, \ \ y_1 < t < y_2, \\ z(y_2) = 0, \quad pz(y_1) = \gamma D^{\zeta}_{y_2-}z(y_1), \end{cases} \tag{11}$$

where $0 < \zeta \leqslant \beta \leqslant 1$, $1 < \zeta + \beta \leqslant 2$, $p\gamma \leqslant 0$ and $p \neq 0$, ${}^{C}D^{\beta}_{y_1+}$ denotes the left Caputo derivative operator of order β, $D^{\zeta}_{y_2-}$ denotes the right Riemann-Liouville derivative operator of order ζ and $r \in C([y_1, y_2], \mathbb{R})$.

LEMMA 23. *A function* $z \in C([y_1, y_2], \mathbb{R})$ *is a solution to the FBVP* (11) *if and only if*

$$z(t) = \int_{y_1}^{y_2} G(t,s)r(s)z(s)ds,$$

where $G(t,s)$ *is the Green's functions given by*

$$G(t,s) = \frac{1}{\Gamma(\zeta)\Gamma(\beta)} \begin{cases} \dfrac{(y_2-t)^{\zeta}}{\eta\Gamma(1+\zeta)} \displaystyle\int_s^{y_2} (\tau - y_1)^{\zeta-1}(\tau - s)^{\beta-1}d\tau \\ \quad - \displaystyle\int_t^{y_2} (\tau - t)^{\zeta-1}(\tau - s)^{\beta-1}d\tau, & y_1 \leqslant s \leqslant \tau \leqslant t \leqslant y_2, \\ \dfrac{(y_2-t)^{\zeta}}{\eta\Gamma(1+\zeta)} \displaystyle\int_s^{y_2} (\tau - y_1)^{\zeta-1}(\tau - s)^{\beta-1}d\tau \\ \quad - \displaystyle\int_s^{y_2} (\tau - t)^{\zeta-1}(\tau - s)^{\beta-1}d\tau, & y_1 \leqslant s \leqslant t \leqslant \tau \leqslant y_2, \end{cases}$$

with $\eta = \dfrac{(y_2-y_1)^{\zeta}}{\Gamma(1+\zeta)} - \dfrac{\gamma}{p}$.

LEMMA 24. *The Green's function* $G(t,s)$ *given in Lemma* 23 *satisfies the property:*

$$|G(t,s)|$$

$$\leqslant \begin{cases} \dfrac{(\beta\eta\Gamma(1+\zeta))^{\zeta+\beta-1}}{\Gamma(\zeta)\Gamma(\beta)(\zeta+\beta-1)(\zeta+\beta)^{\zeta+\beta}(y_2-y_1)^{(\zeta-1)(\zeta+\beta-1)}}, & (y_2-y_1)^{\zeta} > -\frac{\gamma}{p}\beta\Gamma(\zeta), \\ \dfrac{(y_2-y_1)^{\zeta+\beta-1}(\beta\eta\Gamma(1+\zeta)-(\zeta+\beta-1)(y_2-y_1)^{\zeta})}{\Gamma(\zeta)\Gamma(1+\zeta)\Gamma(1+\beta)(\zeta+\beta-1)\eta}, & (y_2-y_1)^{\zeta} \leqslant -\frac{\gamma}{p}\beta\Gamma(\zeta), \end{cases}$$

for $t,s \in [y_1,y_2]\times[y_1,y_2]$.

Now, we are ready to present the Lyapunov-type inequality for problem (11).

THEOREM 13. *If the FBVP* (11) *has a nontrivial solution* $z \in C([y_1,y_2],\mathbb{R})$, *where* r *is a real and continuous function in* $[y_1,y_2]$, *then*

$$\int_{y_1}^{y_2} |r(s)|ds$$

$$\geqslant \begin{cases} \dfrac{\Gamma(\zeta)\Gamma(\beta)(\zeta+\beta-1)(\zeta+\beta)^{\zeta+\beta}(y_2-y_1)^{(\zeta-1)(\zeta+\beta-1)}}{(\beta\eta\Gamma(1+\zeta))^{\zeta+\beta-1}}, & (y_2-y_1)^{\zeta} > -\frac{\gamma}{p}\beta\Gamma(\zeta), \\ \dfrac{\Gamma(\zeta)\Gamma(1+\zeta)\Gamma(1+\beta)(\zeta+\beta-1)\eta}{(y_2-y_1)^{\zeta+\beta-1}(\beta\eta\Gamma(1+\zeta)-(\zeta+\beta-1)(y_2-y_1)^{\zeta})}, & (y_2-y_1)^{\zeta} > -\frac{\gamma}{p}\beta\Gamma(\zeta). \end{cases}$$

5. Lyapunov-type inequalities for Riesz-Caputo fractional derivative

In this section, we present Lyapunov-type inequalities for the Riesz-Caputo type FBVP. Unlike the other fractional operators, the salient feature of the Riesz-Caputo fractional derivative operator is that it uses both left and right fractional derivatives possessing nonlocal memory effects. This property of the Riesz-Caputo derivative is important in the mathematical modeling of physical processes on a finite domain when the present states depend both on the past and future memory effects.

DEFINITION 4. [22] The Riesz fractional integral of a function $g \in C([y_1,y_2],\mathbb{R})$ of order $\zeta > 0$ is defined as

$$I_{y_1}^{\zeta} g(t) = \frac{1}{2\Gamma(\zeta)} \int_{y_1}^{y_2} |t-s|^{\zeta-1} g(s)ds = \frac{1}{2}\Big(I_{y_1}^{\zeta} g(t) + {}_{y_2}I^{\zeta} g(t)\Big), \quad t \in [y_1,y_2],$$

where $I_{y_1}^{\zeta}$ and ${}_{y_2}I^{\zeta}$ are the left and right Riemann-Liouville fractional integral operators of order $\zeta > 0$.

DEFINITION 5. [22] The Riesz-Caputo fractional derivative of a function $g \in C^{n+1}([y_1,y_2],\mathbb{R})$ of order $\zeta \in (n,n+1]$, $n \in \mathbb{N}$, is defined as

$$\begin{aligned} {}^{RC}D_{y_1}^{\zeta} g(t) &= \frac{1}{\Gamma(n+1-\zeta)} \int_{y_1}^{y_2} |t-s|^{n-\zeta} g^{(n+1)} ds \\ &= \frac{1}{2}\Big({}^{C}D_{y_1}^{\zeta} g(t) + (-1)^{n+1}\, {}^{C}_{y_2}D^{\zeta} g(t)\Big), \end{aligned}$$

where ${}^{C}D^{\zeta}_{y_1}g(t)$ and ${}^{C}_{y_2}D^{\zeta}$ are respectively the left and right Riemann-Liouville fractional derivative operators of order $\zeta > 0$.

In [47], Toprakseven et al. studied the anti-periodic FBVP:

$$\begin{cases} {}^{RC}D^{\zeta}_{y_1+}z(t) + r(t)z(t) = 0, \ \ \zeta \in (1,2], \ \ y_1 \leqslant t \leqslant y_2, \\ z(y_1) + z(y_2) = 0 = z'(y_1) + z'(y_2), \end{cases} \tag{12}$$

where ${}^{RC}D^{\zeta}_{y_1+}$ is the Riesz-Caputo fractional derivative operator of order $\zeta > 0$ and $r \in C([y_1, y_2], \mathbb{R})$.

LEMMA 25. *Let $1 < \zeta \leqslant 2$ and $r : [y_1, y_2] \to \mathbb{R}$ be a continuous function. Then, z is a solution to the boundary value problem (12) if and only if z is the solution to the integral equation*

$$z(t) = \int_{y_1}^{y_2} (y_2 - s)^{\zeta-2} G(t,s) r(s) z(s) ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \dfrac{1}{2\Gamma(\zeta)}(t-s)^{\zeta-1}(y_2-s)^{2-\zeta} - \dfrac{y_2-y_1}{4\Gamma(\zeta-1)}, & y_1 \leqslant s \leqslant t \leqslant y_2, \\ \dfrac{1}{2\Gamma(\zeta)}(s-t)^{\zeta-1}(y_2-s)^{2-\zeta} - \dfrac{y_2-y_1}{4\Gamma(\zeta-1)}, & y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

LEMMA 26. *The Green's function $G(t,s)$ given in Lemma 25 is such that*

$$|G(t,s)| \leqslant \frac{(3-\zeta)(y_2-y_1)}{4\Gamma(\zeta)}, \ \ y_1 \leqslant t, s \leqslant y_2.$$

Now, we are ready to state the Lyapunov inequality for the FBVP (12).

THEOREM 14. *Let $1 < \zeta \leqslant 2$ and $r : [y_1, y_2] \to \mathbb{R}$ be a continuous function. If there exists a nonzero solution to the FBVP (12), then*

$$\int_{y_1}^{y_2} (y_2 - s)^{\zeta-2} |r(s)| ds > \frac{4\Gamma(\zeta)}{(3-\zeta)(y_2-y_1)}.$$

As an application of the above Lyapunov-type inequality, we find a bound on the eigenvalues of a FBVP.

COROLLARY 3. *Let $\zeta \in (1,2]$. If a nonzero solution exists for the fractional boundary value problem:*

$$\begin{cases} {}^{RC}D^{\zeta}_{y_1}z(t) + \lambda z(t) = 0, \ \ y_1 \leqslant t \leqslant y_2, \\ z(y_1) + z(y_2) = 0 = z'(y_1) + z'(y_2), \end{cases}$$

then the eigenvalue $\lambda \in \mathbb{R}$ is such that $|\lambda| > \dfrac{4(\zeta-1)\Gamma(\zeta)}{(3-\zeta)(y_2-y_1)^{\zeta}}$.

6. Lyapunov-type inequalities for FBVP involving ψ-Caputo fractional derivative

Let $[y_1,y_2]$, $0<y_1<y_2<\infty$, be an interval and $\psi:[y_1,y_2]\to\mathbb{R}$ be a function such that $\psi'(t)>0$ for every $t\in[y_1,y_2]$.

DEFINITION 6. [22] The ψ-Riemann-Liouville fractional integral of order $\zeta>0$ for an integrable function $g:[y_1,y_2]\to\mathbb{R}$ with respect to the function $\psi:[y_1,y_2]\to\mathbb{R}$ is defined by

$$I_{y_1+}^{\zeta,\psi}g(t)=\frac{1}{\Gamma(\zeta)}\int_{y_1}^{t}\psi'(s)(\psi(t)-\psi(s))^{\zeta-1}g(s)ds.$$

DEFINITION 7. [4] Let $n\in\mathbb{N}$ and $\psi,g\in C^n([y_1,y_2],\mathbb{R})$. The ψ-Caputo fractional derivative of order ζ for the function z is defined by

$${}^{C}D_{y_1+}^{\zeta,\psi}g(t)=I_{y_1+}^{n-\zeta,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}g(s),$$

with $n=[\zeta]+1$ for $\zeta\notin\mathbb{N}$ and $n=\zeta$ for $\zeta\in\mathbb{N}$, where $[\zeta]$ denotes the largest integer less than or equal to ζ.

A Lyapunov-type inequality was obtained in 2022 by Rezapour el al. [40] for a thermostat control model involving generalized ψ-operators given by

$$\begin{cases} {}^{C}D^{\zeta,\psi}z(t)+K(t,z(t))=0, & y_1<t<y_2,\\ {}^{C}D^{1,\psi}z(y_1)=0, & z(\xi)+\mu\,{}^{C}D^{\zeta-1,\psi}z(y_2)=0, \end{cases} \tag{13}$$

where $\zeta\in(1,2]$, $\xi\in(y_1,y_2)$, $\mu>0$ and ${}^{C}D^{1,\psi}=\frac{1}{\psi'(t)}\frac{d}{dt}$, which is the generalized ψ-Caputo derivative of order one, $K:[y_1,y_2]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and ${}^{C}D^{\gamma,\psi}$ denotes the generalized ψ-Caputo fractional derivative of order $\gamma\in\{1,\zeta,\zeta-1\}$.

LEMMA 27. *Let $\zeta\in(1,2]$, $\xi\in(y_1,y_2)$, $\mu>0$ and $\omega\in C([y_1,y_2],\mathbb{R})$. A function $z\in C([y_1,y_2],\mathbb{R})$ is a solution to the linear thermostat ψ-model:*

$$\begin{cases} {}^{C}D^{\zeta,\psi}z(t)+\omega(t)=0, & t\in[y_1,y_2],\\ {}^{C}D^{1,\psi}z(y_1)=0, & z(\xi)+\mu\,{}^{C}D^{\zeta-1,\psi}z(y_2)=0, \end{cases} \tag{14}$$

if it satisfies the integral equation

$$z(t)=\int_{y_1}^{y_2}G_{\psi}(t,s)\psi'(s)\omega(s)ds,$$

where

$$G_\psi(t,s)=\begin{cases}-\dfrac{(\psi(t)-\psi(s))^{\zeta-1}}{\Gamma(\zeta)}+\dfrac{(\psi(\xi)-\psi(s))^{\zeta-1}}{\Gamma(\zeta)}+\mu, & y_1\leqslant s\leqslant \min\{\xi,t\},\\ \dfrac{(\psi(\xi)-\psi(s))^{\zeta-1}}{\Gamma(\zeta)}+\mu, & y_1\leqslant t\leqslant s\leqslant \xi,\\ -\dfrac{(\psi(t)-\psi(s))^{\zeta-1}}{\Gamma(\zeta)}+\mu, & \xi\leqslant s\leqslant t\leqslant y_2,\\ \mu, & \max\{\xi,t\}\leqslant s\leqslant y_2.\end{cases}$$

LEMMA 28. *The Green's function defined in the above lemma has the properties:*

(i) $\min\limits_{y_1\leqslant t,s\leqslant y_2} G_\psi(t,s)=-\dfrac{(\psi(y_2)-\psi(\xi))^{\zeta-1}}{\Gamma(\zeta)}+\mu;$

(ii) $\max\limits_{y_1\leqslant t,s\leqslant y_2} G_\psi(t,s)=\mu+\dfrac{(\psi(\xi)-\psi(y_1))^{\zeta-1}}{\Gamma(\zeta)};$

(iii) $\displaystyle\int_{y_1}^{y_2}|G_\psi(t,s)|\psi'(s)ds\leqslant \max\Big\{\frac{(\psi(\xi)-\psi(y_1))^{\zeta}}{\Gamma(\beta+1)}+\mu(\psi(y_2)-\psi(y_1)),\ \frac{(\psi(y_2)-\psi(\xi))^{\zeta}}{\Gamma(\beta+1)}-\mu(\psi(y_2)-\psi(y_1))\Big\}$, *for each* $t\in[y_1,y_2]$.

A Lyapunov-type inequality for the thermostat control ψ-model (13) is now presented.

THEOREM 15. *Assume that:*

(A) There exist $\phi:[y_1,y_2]\to\mathbb{R}$ *and a positive, concave and non-decreasing function* $\Omega:\mathbb{R}\to\mathbb{R}$ *such that* $|K(t,z)|\leqslant|\phi(t)||\Omega(z)|$ *for each* $t\in[y_1,y_2]$ *and* $z\in\mathbb{R}$.

If $\psi'(\cdot)\phi(\cdot)\in L^1[y_1,y_2]$ *and the fractional thermostat control* ψ*-model* (13) *has a non-trivial solution* $z\in C([y_1,y_2],\mathbb{R})$*, then*

$$\int_{y_1}^{y_2}\psi'(s)|\phi(s)|ds$$
$$\geqslant \min\Big\{\frac{\Gamma(\zeta)}{\mu\Gamma(\zeta)+(\psi(\xi)-\psi(y_1))^{\zeta-1}},\frac{\Gamma(\zeta)}{|(\psi(y_2)-\psi(\xi))^{\zeta-1}-\mu\Gamma(\zeta)|}\Big\}\frac{\|z\|}{\Omega(\|z\|)}.$$

If $\mu\geqslant\dfrac{(\psi(y_2)-\psi(\xi))^{\zeta-1}}{\Gamma(\zeta)}$*, then*

$$\int_{y_1}^{y_2}\psi'(s)|\phi(s)|ds\geqslant\frac{\Gamma(\zeta)\|z\|}{(\mu\Gamma(\zeta)+(\psi(\xi)-\psi(y_1))^{\zeta-1})\Omega(\|z\|)}.$$

As an example we can state the existence and uniqueness result for thermostat control ψ-model (13).

THEOREM 16. *Assume that:*

(B) There exist $\phi : [y_1, y_2] \to [0, \infty)$ *such that* $|K(t,z) - K(t,x)| \leqslant \phi(t)|z - x|$, *for each* $(t,y), (t,x) \in [y_1, y_2] \times \mathbb{R}$.

If $\psi \in C^1[y_1, y_2]$, $\phi \in L^1[y_1, y_2]$ *and*

$$\|\phi\|_{L^1[y_1,y_2]} < \frac{1}{\|\psi'\|} \min\left\{ \frac{\Gamma(\zeta)}{\mu\Gamma(\zeta) + (\psi(\xi) - \psi(y_1))^{\zeta-1}}, \frac{\Gamma(\zeta)}{|(\psi(y_2) - \psi(\xi))^{\zeta-1} - \mu\Gamma(\zeta)|} \right\},$$

then the fractional thermostat control ψ*-model* (13) *has a unique solution.*

In 2022, Long [27] studied the boundary value problem:

$$\begin{cases} {}^{C}D^{\zeta,\psi}_{y_1+} z(t) + r(t)z(t) = 0, & t \in [y_1, y_2], \\ z(y_1) = 0, \quad z(y_2) = 0, \end{cases} \tag{15}$$

where $1 < \zeta \leqslant 2$, $\psi \in C^1([y_1, y_2], \mathbb{R})$, ${}^{C}D^{\zeta,\psi}_{y_1+}$ is the ψ-Caputo fractional derivative operator of order ζ and $r : [y_1, y_2] \to \mathbb{R}$ is a continuous function.

LEMMA 29. *A function* $z \in C([y_1, y_2], \mathbb{R})$ *is a solution to the FBVP* (15) *if and only if*

$$z(t) = \frac{1}{\Gamma(\zeta)} \int_{y_1}^{y_2} G(t,s)[\psi(y_2) - \psi(s)]^{\zeta-1} \psi'(s) r(s) z(s) ds,$$

where $G(t,s)$ *is the Green's function given by*

$$G(t,s) = \begin{cases} \dfrac{\psi(t) - \psi(y_1)}{\psi(y_2) - \psi(y_1))} - \left(\dfrac{\psi(t) - \psi(s)}{\psi(y_2) - \psi(s)} \right)^{\zeta-1}, & y_1 \leqslant s < t \leqslant y_2, \\ \dfrac{\psi(t) - \psi(y_1)}{\psi(y_2) - \psi(y_1))}, & y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

Now, we present a Lyapunov-type inequality for the FBVP (15).

THEOREM 17. *If* z *is a nontrivial solution to the FBVP* (15)*, then*

$$\int_{y_1}^{y_2} [\psi(y_2) - \psi(y_1)] \psi'(s) |r(s)| ds \geqslant \Gamma(\zeta).$$

7. Lyapunov-type inequalities for Hadamard-type FBVP

Let $z : (y_1, y_2) \to \mathbb{R}$, where $0 < y_1 < y_2 < \infty$. Define the space $AC^n_\delta[y_1, y_2]$ as

$$AC^n_\delta[y_1, y_2] = \left\{ z : [y_1, y_2] \to \mathbb{R} | \delta^{n-1} z(t) \in AC[y_1, y_2],\ \delta = t\frac{d}{dt} \right\},$$

where $AC[y_1, y_2]$ denotes the space of all absolutely continuous real valued functions on $[y_1, y_2]$.

DEFINITION 8. ([22, 3]) The left-sided Hadamard fractional integral of order $\zeta > 0$ for a function $g : [y_1, y_2] \to \mathbb{R}$ is defined by

$$^{H}I^{\zeta}_{y_1+}g(t) = \frac{1}{\Gamma(\zeta)} \int_{y_1}^{t} \left(\ln\frac{t}{s}\right)^{\zeta-1} g(s)\frac{ds}{s},$$

provided that the integral exists.

DEFINITION 9. ([22, 3]) Let $\zeta > 0$, $n = [\zeta] + 1$. The left-side Hadamard fractional derivative of order ζ for a function $g : [y_1, y_2] \to \mathbb{R}$ is defined by

$$^{H}D^{\zeta}_{y_1+}g(t) = \frac{1}{\Gamma(n-\zeta)} \left(t\frac{d}{dt}\right)^{n} \int_{y_1}^{t} \left(\ln\frac{t}{s}\right)^{n-\zeta-1} g(s)\frac{ds}{s},$$

provided that the integral exists.

In 2022, Zhang et al. [54] established Lyapunov-type inequalities for a fractional Langevin-type equation involving Caputo-Hadamard fractional derivative subject to mixed boundary conditions:

$$\begin{cases} {}^{C}_{H}D^{\beta}_{y_1+}({}^{C}_{H}D^{\zeta}_{y_1+} + p(t))z(t) + r(t)z(t) = 0, \ \ 0 < y_1 < t < y_2, \\ z(y_1) = {}^{C}_{H}D^{\zeta}_{y_1+}z(y_1) = 0, \quad z(y_2) = 0, \end{cases} \tag{16}$$

where ${}^{C}_{H}D^{\kappa}_{y_1+}$ denotes the Caputo-Hadamard fractional derivative of order $\kappa \in \{\zeta, \beta\}$, $0 < \beta < 1 < \zeta < 2$, and $p, r \in C([y_1, y_2], \mathbb{R})$.

LEMMA 30. *A function $z \in C([y_1, y_2], \mathbb{R})$ is a solution to the FBVP (16) if and only if it satisfies the integral equation:*

$$z(t) = \int_{y_1}^{y_2} G_1(t,s)r(s)z(s)ds + \int_{y_1}^{y_2} G_2(t,s)p(s)z(s)ds,$$

where $G_1(t,s)$ and $G_2(t,s)$ are Green's functions given by

$$G_1(t,s) = \frac{1}{s\Gamma(\beta+\zeta)} \begin{cases} \frac{\ln(t/y_1)}{\ln(y_2/y_1)}\left(\ln\frac{y_2}{s}\right)^{\beta+\zeta-1} - \left(\ln\frac{t}{s}\right)^{\beta+\zeta-1}, & 0 < y_1 \leqslant s \leqslant t \leqslant y_2, \\ \frac{\ln(t/y_1)}{\ln(y_2/y_1)}\left(\ln\frac{y_2}{s}\right)^{\beta+\zeta-1}, & 0 < y_1 \leqslant t \leqslant s \leqslant y_2, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{s\Gamma(\zeta)} \begin{cases} \frac{\ln(t/y_1)}{\ln(y_2/y_1)}\left(\ln\frac{y_2}{s}\right)^{\zeta-1} - \left(\ln\frac{t}{s}\right)^{\zeta-1}, & 0 < y_1 \leqslant s \leqslant t \leqslant y_2, \\ \frac{\ln(t/y_1)}{\ln(y_2/y_1)}\left(\ln\frac{y_2}{s}\right)^{\zeta-1}, & 0 < y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

LEMMA 31. *The Green's function $G_1(t,s)$ defined in Lemma 30 satisfies the properties:*

(i) $G_1(t,s)$ *is nonnegative continuous function in* $[y_1,y_2]\times[y_1,y_2]$;

(ii) $G_1(t,s)\leqslant \dfrac{(\ln(y_2/y_1))^{\beta+\zeta-1}}{y_1\Gamma(\beta+\zeta)}$ *for any* $(t,s)\in[y_1,y_2]\times[y_1,y_2]$.

In the following theorem, we present the Lyapunov-type inequality for the FBVP (16).

THEOREM 18. *If the FBVP* (16) *has a nontrivial continuous solution* $z\in C([y_1,y_2],\mathbb{R})$, *where* r *is a real and continuous function in* $[y_1,y_2]$, *then*

$$\int_{y_1}^{y_2}(|r(s)|+|p(s)|)ds \geqslant \frac{y_1(\ln(y_2/y_1))^{1-\zeta}}{\max\left\{\dfrac{(\ln(y_2/y_1))^{\beta}}{\Gamma(\beta+\zeta)},\dfrac{(\zeta-1)^{\zeta-1}}{\zeta^{\zeta}\Gamma(\zeta)}\right\}}.$$

Zhang et al. [54] also established Lyapunov-type inequalities for the fractional p-Laplacian Langevin-type equation involving Caputo-Hadamard fractional derivatives subject to mixed boundary conditions:

$$\begin{cases} {}^{C}_{H}D^{\eta}_{y_1+}\phi_p[({}^{C}_{H}D^{\gamma}_{y_1+}+u(t))z(t)]+r(t)\phi_p(z(t))=0, & 0<y_1<t<y_2,\\ z(y_1)={}^{C}_{H}D^{\gamma}_{y_1+}z(y_1)=0, \quad z(y_2)={}^{C}_{H}D^{\gamma}_{y_1+}z(y_2)=0, \end{cases} \tag{17}$$

where ${}^{C}_{H}D^{\chi}_{y_1+}$ denotes the Caputo-Hadamard fractional derivative of order $\chi\in\{\gamma,\eta\}$, $1<\gamma<1$, $\eta<2$, and $u,r\in C([y_1,y_2],\mathbb{R})$.

LEMMA 32. *Let* $\dfrac{1}{p}+\dfrac{1}{q}=1$, *then a function* $z\in C([y_1,y_2],\mathbb{R})$ *is a solution to the FBVP* (17) *if and only if*

$$z(t)=\int_{y_1}^{y_2}G(t,s)u(s)z(s)ds-\int_{y_1}^{y_2}G(t,s)\phi_q\Big(\int_{y_1}^{y_2}H(s,\tau)r(\tau)\phi_p(z(\tau))d\tau\Big)ds,$$

where the kernel functions $G(t,s)$ *and* $H(t,s)$ *are given by*

$$G(t,s)=\frac{1}{s\Gamma(\gamma)}\begin{cases}\dfrac{\ln(t/y_1)}{\ln(y_2/y_1)}\left(\ln\dfrac{y_2}{s}\right)^{\gamma-1}-\left(\ln\dfrac{t}{s}\right)^{\gamma-1}, & 0<y_1\leqslant s\leqslant t\leqslant y_2,\\ \dfrac{\ln(t/y_1)}{\ln(y_2/y_1)}\left(\ln\dfrac{y_2}{s}\right)^{\gamma-1}, & 0<y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

and

$$H(t,s)=\frac{1}{\tau\Gamma(\eta)}\begin{cases}\dfrac{\ln(s/y_1)}{\ln(y_2/y_1)}\left(\ln\dfrac{y_2}{\tau}\right)^{\eta-1}-\left(\ln\dfrac{s}{\tau}\right)^{\eta-1}, & 0<y_1\leqslant s\leqslant t\leqslant y_2,\\ \dfrac{\ln(s/y_1)}{\ln(y_2/y_1)}\left(\ln\dfrac{y_2}{s}\right)^{\eta-1}, & 0<y_1\leqslant t\leqslant s\leqslant y_2.\end{cases}$$

THEOREM 19. *If the FBVP (17) has a nontrivial continuous solution $z \in C[y_1, y_2]$, where r is a real and continuous function in $[y_1, y_2]$, then either*

$$\int_{y_1}^{y_2} |u(s)|ds \geqslant \frac{y_1 \gamma^\gamma \Gamma(\gamma)}{[(\gamma-1)\ln(y_2/y_1)]^{\gamma-1}},$$

or

$$\int_{y_1}^{y_2} |r(s)|ds$$
$$\geqslant \phi_p\left\{\frac{y_1\gamma^\gamma\Gamma(\gamma) - [(\gamma-1)\ln(y_2/y_1)]^{\gamma-1}\int_{y_1}^{y_2}|u(s)|ds}{(y_2-y_1)[(\zeta-1)\ln(y_2/y_1)]^{\gamma-1}}\right\}\frac{y_1\eta^\eta\Gamma(\eta)}{[(\eta-1)\ln(y_2/y_1)]^{\eta-1}}.$$

8. Lyapunov-type inequalities for FBVP involving Katugampola fractional derivative

Let us first define the Katugampola fractional integral and derivative operators.

DEFINITION 10. [21] Let $\zeta > 0$, $\rho > 0$ and $-\infty < y_1 < y_2 < \infty$. The left-sided and right-sided Katugampola integrals of fractional order ζ are respectively defined for $g \in L^p(y_1, y_2)$, $p \geqslant 1$, as

$$I_{y_1+}^{\zeta,\rho} g(t) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_{y_1}^{t} \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\zeta}} g(\tau)d\tau,$$

$$I_{y_2-}^{\zeta,\rho} g(t) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} \int_{t}^{y_2} \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\zeta}} g(\tau)d\tau.$$

DEFINITION 11. [21] Let $\zeta > 0$, $\rho > 0$, $n = [\zeta] + 1$, $0 < y_1 < t < y_2 \leqslant \infty$. The left-sided and right-sided Katugampola derivatives of fractional order ζ are respectively defined for $g \in L^p(y_1, y_2)$, $p \geqslant 1$, as

$$D_{y_1+}^{\zeta,\rho} g(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^n I_{y_1+}^{n-\zeta,\rho} g(t),$$

$$D_{y_2-}^{\zeta,\rho} g(t) = \left(-t^{1-\rho}\frac{d}{dt}\right)^n I_{y_2-}^{n-\zeta,\rho} g(t).$$

In [29], Lupinska considered the boundary value problem:

$$\begin{cases} D_{y_1+}^{\zeta,\rho} z(t) + r(t)z(t) = 0, \ \ y_1 < t < y_2, \ \ 1 < \zeta \leqslant 2, \\ z(y_1) = z'(y_2) = 0, \end{cases} \tag{18}$$

where $D_{y_1+}^{\zeta,\rho}$ denotes the Katugampola fractional derivative of order ζ and $r : [y_1, y_2] \to \mathbb{R}$ is a continuous function.

LEMMA 33. *The function $z \in C([y_1, y_2], \mathbb{R})$ is a solution to the FBVP (18) if and only if*

$$z(t) = \int_{y_1}^{y_2} G(t,s) r(s) z(s) ds,$$

where the Green's function $G(t,s)$ is given by

$$G(t,s) = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)} s^{\rho-1} \begin{cases} \left(\frac{y_2^\rho - y_1^\rho}{y_2^\rho - s^\rho}\right)^{2-\zeta} (t^\rho - y_1^\rho)^{\zeta-1}, & y_1 \leqslant t \leqslant s \leqslant y_2, \\ \left(\frac{y_2^\rho - y_1^\rho}{y_2^\rho - s^\rho}\right)^{2-\zeta} (t^\rho - y_1^\rho)^{\zeta-1} - (t^\rho - s^\rho), & y_1 \leqslant s \leqslant t \leqslant y_2. \end{cases}$$

LEMMA 34. *The function $G(t,s)$ defined in Lemma 33 satisfies the properties:*

(i) $G(t,s) \geqslant 0$ *for all* $t \in [y_1, y_2]$, $s \in [y_1, y_2]$;

(ii) $\max_{t \in [y_1, y_2]} G(t,s) = G(s,s) \leqslant \frac{\rho^{1-\zeta}(y_2^\rho - y_1^\rho)}{\Gamma(\zeta)} (y_2^\rho - s^\rho)^{\zeta-2}.$

Now, we state the Lyapunov-type inequality for the FBVP (18).

THEOREM 20. *If a continuous nontrivial solution to the FBVP (18) exists, where r is a real and continuous function, then*

$$\int_{y_1}^{y_2} s^{\rho-1} (y_2^\rho - s^{\,\rho})^{\zeta-2} |r(s)| ds \geqslant \frac{\Gamma(\zeta)}{\rho^{1-\zeta}(y_2^\rho - y_1^\rho)}.$$

COROLLARY 4. *The FBVP (18) has no nontrivial solution if*

$$\int_{y_1}^{y_2} s^{\rho-1} (y_2^\rho - s^{\,\rho})^{\zeta-2} |r(s)| ds < \frac{\Gamma(\zeta)}{\rho^{1-\zeta}(y_2^\rho - y_2^\rho)}.$$

COROLLARY 5. *If λ is an eigenvalue of the boundary value problem:*

$$\begin{cases} D_{y_1+}^{\zeta,\rho} z(t) + \lambda z(t) = 0, \quad y_1 < t < y_2, \quad 1 < \zeta \leqslant 2, \\ z(y_1) = z'(y_2) = 0, \end{cases}$$

then

$$|\lambda| \geqslant (\zeta - 1)\Gamma(\zeta) \left(\frac{\rho}{y_2^\rho - y_1^\rho}\right)^{\zeta}.$$

9. Lyapunov-type inequalities for FBVP involving Hilfer fractional derivative

DEFINITION 12. [13] Let $y_1, y_2 \in \mathbb{R}$, $\zeta > 0$, $0 \leqslant \beta \leqslant 1$ and choose $n \in \mathbb{N}$ such that $n-1 < \zeta \leqslant n$. The Hilfer-type fractional derivative of order ζ and type β of a function $g : [y_1, y_2] \to \mathbb{R}$ is defined by

$$(D^{\zeta,\beta} g)(t) = (I_{y_1}^{\beta(n-\zeta)} D^n I^{(n-\zeta)(1-\beta)} g)(t), \ \ t > y_1,$$

if the right-hand side exists. Here $D^n = \dfrac{d^n}{dt^n}$ denotes the classical n^{th}-order differential operator.

In [16], Jonnalagadda established the Lyapunov-type inequalities for the FBVPs involving the Hilfer fractional differential operators:

$$\begin{cases} (D_{y_1+}^{\zeta,\beta} z)(t) + r(t)z(t) = 0, \ \ y_1 < t < y_2, \\ l(I_{y_1+}^{(2-\zeta)(1-\beta)} z)(y_1) - m(DI_{y_1+}^{2-\zeta)(1-\beta)} y)(y_1) = 0, \\ n(I_{y_1+}^{(2-\zeta)(1-\beta)} z)(y_2) + p(DI_{y_1+}^{2-\zeta)(1-\beta)} y)(y_2) = 0, \end{cases} \tag{19}$$

and

$$\begin{cases} (D_{y_1+}^{\zeta,\beta} z)(t) + r(t)z(t) = 0, \ \ y_1 < t < y_2, \\ (I_{y_1+}^{(2-\zeta)(1-\beta)} z)(y_1) + (I_{y_1+}^{2-\zeta)(1-\beta)} z)(y_2) = 0, \\ (DI_{y_1+}^{(2-\zeta)(1-\beta)} z)(y_1) + (DI_{y_1+}^{2-\zeta)(1-\beta)} z)(y_2) = 0, \end{cases} \tag{20}$$

where l, m, n, p are constants such that $l^2 + m^2 > 0$ and $n^2 + p^2 > 0$, $r : [y_1, y_2] \to \mathbb{R}$ is a continuous function, $D_{y_1+}^{\zeta,\beta}$ denotes the Hilfer fractional derivative operator of order $1 < \zeta \leqslant 2$ and type $0 \leqslant \beta \leqslant 1$ and D denotes the first order differential operator.

LEMMA 35. *If $mn + lp + ln(y_2 - y_1) \neq 0$, then the unique solution of the FBVP (19) is given by*

$$z(t) = \int_{y_1}^{y_2} G(t,s) r(s) z(s) ds, \ \ \ y_1 < t < y_2,$$

where

$$G(t,s) = \begin{cases} G_1(t,s), & y_1 < s \leqslant t < y_2, \\ G_2(t,s), & y_1 < t \leqslant s < y_2, \end{cases}$$

with

$$G_1(t,s) = G_2(t,s) - \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)},$$

and

$$\begin{aligned} G_2(t,s) = {} & (t-y_1)^{-(2-\zeta)(1-\beta)} (y_2 - s)^{-\beta(2-\zeta)} \\ & \times \frac{[l(t-y_1) + m(\zeta - 1 + \beta(2-\zeta))][n(y_2 - s) + p(1 - 2\beta + \zeta\beta)]}{[mn + lp + ln(y_2 - y_1)]\Gamma(2 - 2\beta + \zeta\beta)\Gamma(2 - (2-\zeta)(1-\beta))}. \end{aligned}$$

LEMMA 36. *The unique solution of the FBVP* (20) *is given by*

$$z(t)=\int_{y_1}^{y_2}\bar{G}(t,s)r(s)z(s)ds,\quad y_1<t<y_2,$$

where

$$\bar{G}(t,s)=\begin{cases}\bar{G}_1(t,s), & y_1<s\leqslant t<y_2,\\ \bar{G}_2(t,s), & y_1<t\leqslant s<y_2,\end{cases}$$

with

$$\bar{G}_1(t,s)=\bar{G}_2(t,s)-\frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)},$$

and

$$\begin{aligned}\bar{G}_2(t,s)=&\frac{1}{2\Gamma(1-2\beta+\zeta\beta)\Gamma(1-(2-\zeta)(1-\beta))}\Big\{(t-y_1)^{-(2-\zeta)(1-\beta)}\\ &\times(y_2-s)^{-\beta(2-\zeta)}\Big[\frac{t-y_1}{1-(2-\zeta)(1-\beta)}-\frac{y_2-s}{1-\beta(2-\zeta)}-\frac{y_2-y_1}{2}\Big]\Big\}.\end{aligned}$$

LEMMA 37. *Let* $l,m,n,p\geqslant 0$ *with* $mn+lp+ln(y_2-y_1)>0$ *and*

$$\begin{aligned}H(t,s)&=(t-y_1)^{(2-\zeta)(1-\beta)}(y_2-s)^{\beta(2-\zeta)}G(t,s)\\ &=\begin{cases}(t-y_1)^{(2-\zeta)(1-\beta)}(y_2-s)^{\beta(2-\zeta)}G_1(t,s), & y_1\leqslant s\leqslant t\leqslant y_2,\\ (t-y_1)^{(2-\zeta)(1-\beta)}(y_2-s)^{\beta(2-\zeta)}G_2(t,s), & y_1\leqslant t\leqslant s\leqslant y_2.\end{cases}\end{aligned}$$

Then

$$|H(t,s)|\leqslant\max\Big\{\Omega,\frac{y_2-y_1}{\Gamma(\zeta)}\Big\},\ \ (t,s)\in[y_1,y_2]\times[y_1,y_2],$$

where

$$\Omega=\frac{[l(y_2-y_1)+m(\zeta-1+\beta(2-\zeta))][n(y_2-y_1)+p(1-2\beta+\zeta\beta)]}{[mn+lp+ln(y_2-y_1)]\Gamma(2-2\beta+\zeta\beta)\Gamma(2-(2-\zeta)(1-\beta))}.$$

LEMMA 38. *Let*

$$\begin{aligned}\bar{H}(t,s)&=(t-y_1)^{(2-\zeta)(1-\beta)}(y_2-s)^{\beta(2-\zeta)}\bar{G}(t,s)\\ &=\begin{cases}(t-y_1)^{(2-\zeta)(1-\beta)}(y_2-s)^{\beta(2-\zeta)}\bar{G}_1(t,s), & y_1\leqslant s\leqslant t\leqslant y_2,\\ (t-y_1)^{(2-\zeta)(1-\beta)}(y_2-s)^{\beta(2-\zeta)}\bar{G}_2(t,s), & y_1\leqslant t\leqslant s\leqslant y_2.\end{cases}\end{aligned}$$

Then

$$|\bar{H}(t,s)|\leqslant\frac{y_2-y_1}{A}\Big[\frac{1}{1-(2-\zeta)(1-\beta)}+\frac{1}{1-\beta(2-\zeta)}-\frac{1}{2}\Big]+\frac{y_2-y_1}{\Gamma(\zeta)},$$

$(t,s)\in[y_1,y_2]\times[y_1,y_2]$, *where*

$$A=2\Gamma(1-2\beta+\zeta\beta)\Gamma(1-(2-\zeta)(1-\beta)).$$

Now, we are in a position to present the Lyapunov-type inequalities for the FBVPs (19) and (20).

THEOREM 21. *Assume that* $l,m,n,p \geqslant 0$ *and* $mn+lp+ln(y_2-y_1)>0$. *If the FBVP* (19) *has a nontrivial solution, then*

$$\int_{y_1}^{y_2}(s-y_2)^{(\zeta-2)(1-\beta)}(y_2-s)^{\beta(\zeta-2)}|r(s)|ds \geqslant \frac{1}{\Lambda},$$

where

$$\Lambda = \max\left\{\Omega, \frac{y_2-y_1}{\Gamma(\zeta)}\right\}.$$

THEOREM 22. *If the FBVP* (20) *has a nontrivial solution, then*

$$\int_{y_1}^{y_2}(s-y_1)^{(\zeta-2)(1-\beta)}(y_2-s)^{\beta(\zeta-2)}|r(s)|ds \geqslant \frac{1}{\Theta},$$

where

$$\Theta = \frac{y_2-y_1}{A}\left[\frac{1}{1-(2-\zeta)(1-\beta)}+\frac{1}{1-\beta(2-\zeta)}-\frac{1}{2}\right]+\frac{y_2-y_1}{\Gamma(\zeta)}.$$

Next, we give the Lyapunov-type inequalities for boundary value problems involving ψ-Hilfer fractional derivatives.

For $y_1<y_2$, let us define

$$H_+^1[y_1,y_2] = \{\psi \in C^1[y_1,y_2] : \psi'(t)>0 \text{ for all } t\in[y_1,y_2]\}.$$

DEFINITION 13. [43] For $\zeta>0$, $\psi\in H_+^1[y_1,y_2]$, and $g\in L^1[y_1,y_2]$, the fractional integral of a function g with respect to the function ψ is defined by

$$I_{y_1+}^{\zeta,\psi}g(t) = \frac{1}{\Gamma(\zeta)}\int_{y_1}^{t}\psi'(s)(\psi(t)-\psi(s))^{\zeta-1}g(s)ds.$$

DEFINITION 14. [43] For $n-1<\zeta\leqslant n$, and $g,\psi\in C^n[y_1,y_2]$ with $\psi'(t)>0$ for all $t\in[y_1,y_2]$, the left-sided ψ-Hilfer fractional derivative ${}^{H}D_{y_1+}^{\zeta,\beta,\psi}g$ of order ζ and type $0\leqslant\beta\leqslant 1$, is defined as

$${}^{H}D_{y_1+}^{\zeta,\beta,\psi}g(t) = I_{y_1+}^{\beta(n-\zeta),\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{y_1+}^{(1-\beta)(n-\zeta),\psi}g(t).$$

In [9], Dien and Nieto considered a nonlinear sequential FBVP in terms of the generalized ψ-Hilfer fractional derivatives given by

$$\begin{cases}\left({}^{H}D_{y_1+}^{\zeta_1,\beta_1,\psi}\,{}^{H}D_{y_1+}^{\zeta_2,\beta_2,\psi}z\right)(t)+g(t,z(t))=0, & y_1<t<y_2,\\ z(y_1)={}^{H}D_{y_1+}^{\zeta_3,\beta_3,\psi}z(y_2)=0,\end{cases} \tag{21}$$

where $0<\zeta_i\leqslant 1$, $0\leqslant\beta_i\leqslant 1$ $(i=1,2,3)$, $\zeta_1+\zeta_2>1$ and ${}^{H}D_{y_1+}^{\zeta_i,\beta_i,\psi}$ stands for the ψ-Hilfer fractional derivative operator of order ζ_i and type $0\leqslant\beta_i\leqslant 1$, $i=1,2,3$.

LEMMA 39. *Let* $0 < \zeta_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$ $(i = 1,2,3)$, $\zeta_1 + \zeta_2 > 1$ *and* $\psi \in H^1_+[y_1,y_2]$. *Let* z *be a solution to the problem* (21). *If* $g(\cdot,z(\cdot)) \in L^1[y_1,y_2]$, *then* z *is a solution of the integral equation*

$$z(t) = \int_{y_1}^{y_2} G(s,t)\psi'(s)(\psi(y_2)-\psi(s))^{\zeta_1+\zeta_2-\zeta_3-1} g(s,z(s))ds,$$

where

$$G(s,t) = \begin{cases} C_1(\psi(t)-\psi(y_1))^{\zeta_1+\gamma_1-1} \\ -C_2\dfrac{(\psi(t)-\psi(s))^{\zeta_1+\zeta_2-1}}{(\psi(y_2)-\psi(y_1))^{\zeta_1+\zeta_2-\zeta_3-1}}, & y_1 \leqslant s \leqslant t \leqslant y_2, \\ C_1(\psi(t)-\psi(y_1))^{\zeta_1+\gamma_1-1}, & y_1 \leqslant t \leqslant s \leqslant y_2, \end{cases}$$

with $\gamma_1 = \zeta_1 + \beta_1(1-\zeta_1)$ *and*

$$C_1 = \frac{\Gamma(\zeta_2+\gamma_1-\zeta_3)}{\Gamma(\zeta_2+\gamma_1)\Gamma(\zeta_1+\zeta_2-\zeta_3)(\psi(y_2)-\psi(y_1))^{\zeta_2+\gamma_1-\zeta_3-1}}, \ C_2 = \frac{1}{\Gamma(\zeta_1+\zeta_2)}.$$

Now, we state some properties of the Green's function given in the preceding lemma.

LEMMA 40. *Let* $0 < \zeta_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$ $(i = 1,2,3)$, $\zeta_1 + \zeta_2 > 1$, $\gamma_1 = \zeta_1 + \beta_1(1-\zeta_1)$ *and* $\psi \in H^1_+[y_1,y_2]$. *Then, the Green function* G *defined in Lemma* 39 *satisfies the properties:*

(i) *For any* $y_1 \leqslant t_1 \leqslant t_2 \leqslant y_2$, $|G(s,t_1) - G(s,t_2)| \leqslant C(\psi(t_2)-\psi(t_1))^{\sigma}$, *where* $\sigma = \min\{\zeta_1+\zeta_2-1, \zeta_3\}$ *and* $C = C_1(\psi(y_2)-\psi(y_1))^{\zeta_2+\gamma_1-\sigma-1} + C_2(\psi(y_2)-\psi(y_1))^{\zeta_3}$;

(ii) $\max_{y_1 \leqslant s,t \leqslant y_2} |G(s,t)| = C_{\max}$, *where*

$$C_{\max} = \max\left\{ \frac{\Gamma(\zeta_2+\gamma_1-\zeta_3)}{\Gamma(\zeta_2+\gamma_1)\Gamma(\zeta_1+\zeta_2-\zeta_3)}(\psi(y_2)-\psi(y_1))^{\zeta_3}, \right.$$
$$\left. \frac{\gamma_1-\zeta_1}{\zeta_2+\gamma_1-1}\frac{(\psi(y_2)-\psi(y_1))^{\zeta_3}}{\Gamma(\zeta_1+\zeta_2}\left(\frac{\Gamma(\zeta_2+\gamma_1-1)\Gamma(\zeta_1+\zeta_2-\zeta_3)}{\Gamma(\zeta_1+\zeta_2-1)\Gamma(\zeta_2+\gamma_1-\zeta_3)}\right)^{\frac{\zeta_1+\zeta_2-1}{\gamma_1-\zeta_1}} \right\},$$

with $\gamma_1 > \zeta_1$ *and*

$$C_{\max} = \frac{(\psi(y_2)-\psi(y_1))^{\zeta_3}}{\Gamma(\beta+\zeta_2)},$$

with $\gamma_1 = \zeta_1$.

Next, a generalized Lyapunov-type inequality for the FBVP (21) is presented.

THEOREM 23. *Let* $0 < \zeta_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$ $(i = 1,2,3)$, $\zeta_1 + \zeta_2 > 1$ *and* $\gamma_1 = \zeta_1 + \beta_1(1-\zeta_1)$. *Assume that:*

(A) *There exist* $\phi : (y_1,y_2) \to \mathbb{R}$ *and a positive, nondecreasing and concave function* $h : \mathbb{R} \to \mathbb{R}$ *such that*

$$|g(t,z)| \leqslant |\phi(t)|h(z)|, \ \textit{for any } t \in (y_1,y_2) \textit{ and } z \in \mathbb{R}.$$

If $\psi \in H_+^1[y_1,y_2]$ *and* $\psi'(\cdot)(\psi(y_2)-\psi(\cdot))^{\zeta_1+\zeta_2-\zeta_3-1}r(\cdot) \in L^1(y_1,y_2)$ *and the FBVP* (21) *has a nontrivial solution, then*

$$\int_{y_1}^{y_2} \psi'(s)(\psi(y_2)-\psi(s))^{\zeta_1+\zeta_2-\zeta_3-1}|\phi(s)|ds \geqslant \frac{1}{C_{\max}}\frac{\|z\|}{h(\|z\|)}.$$

COROLLARY 6. *Let* $0<\zeta_i \leqslant 1,\ 0 \leqslant \beta_i \leqslant 1\ \ (i=1,2,3)$ *and* $\psi \in H_+^1[y_1,y_2]$. *Suppose that* λ *is an eigenvalue of the FBVP*

$$\begin{cases} \left({}^HD_{y_1+}^{\zeta_1,\beta_1,\psi}\ {}^HD_{y_1+}^{\zeta_2,\beta_2,\psi} z\right)(t) = \lambda z(t), \ \ y_1<t<y_2, \\ z(y_1) = {}^HD_{y_1+}^{\zeta_3,\beta_3,\psi} z(y_2) = 0. \end{cases}$$

Then

$$|\lambda| \geqslant \frac{1}{C_{\max}}\frac{\zeta_1+\zeta_2-\zeta_3}{(\psi(y_2)-\psi(y_1))^{\zeta_1+\zeta_2-\zeta_3}}.$$

In [39], the authors studied the ψ-Hilfer fractional boundary value problem:

$$\begin{cases} {}^HD_{y_1+}^{\zeta,\beta,\psi} z(t) + r(t)z(t) = 0, \ \ y_1<t<y_2, \\ z(y_1) = z(y_2) = 0, \end{cases} \tag{22}$$

where $1<\zeta,\beta \leqslant 2,\ \zeta \leqslant \beta,\ r \in C([y_1,y_2],\mathbb{R})$, and ${}^HD_{y_1+}^{\zeta,\beta,\psi}$ denotes the ψ-Hilfer derivative operator of order ζ and type β. Let $\gamma = \zeta+\beta(2-\zeta)$.

LEMMA 41. *Let* $1<\zeta \leqslant \beta \leqslant 2,\ r \in C([y_1,y_2],\mathbb{R})$, *and* $\psi'(t)>0,\ t \in [y_1,y_2]$. *Then, the solution of the problem* (22) *is given by*

$$z(t) = \frac{1}{\Gamma(\zeta)}\int_{y_1}^{t} G(t,s)z(s)r(s)ds + \frac{1}{\Gamma(\zeta)}\int_{t}^{y_2} G(t,s)z(s)r(s)ds,$$

where $G(t,s)$ *is defined by*

$$\Gamma(\zeta)G(t,s) = \begin{cases} (\psi(t)-\psi(y_1))^{\gamma-1}\psi'(y_2)\left(\dfrac{\psi(y_2)-\psi(s)}{\psi(y_2)-\psi(y_1)}\right)^{\zeta-1} & \\ -\psi'(t)(\psi(t)-\psi(s))^{\zeta-1}, & y_1 \leqslant s \leqslant t, \\ (\psi(t)-\psi(y_1))^{\gamma-1}\psi'(y_2)\left(\dfrac{\psi(y_2)-\psi(s)}{\psi(y_2)-\psi(y_1)}\right)^{\zeta-1}, & t \leqslant s \leqslant y_2. \end{cases} \tag{23}$$

LEMMA 42. *The Green function* $G(t,s)$ *given in* (23) *satisfies the properties:*

(i) $G(t,s)$ *is positive for all* $(t,s) \in [y_1,y_2]\times[y_1,y_2]$;

(ii) $G(t,s) \leqslant G(s,s)$, *where*

$$G(s,s) = \left(\frac{\psi(s)-\psi(y_1)}{\psi(y_2)-\psi(y_1)}\right)^{\gamma-1} - \psi(y_2)-\psi(s))^{\zeta-1}, \ \ s \in (y_1,y_2);$$

(iii)

$$\max_{s\in[y_1,y_2]} G(s,s) = G\Big(\frac{(\gamma-1)\psi(y_1)+(\zeta-1)\psi(y_2)}{\gamma-\zeta-2}, \frac{(\gamma-1)\psi(y_1)+(\zeta-1)\psi(y_2)}{\gamma-\zeta-2}\Big).$$

Now, we state the Lyapunov-type inequalities for the problem (22).

THEOREM 24. *Assume that $r \in C([y_1,y_2],\mathbb{R})$ and $1<\zeta<\beta\leqslant 2$. If the FBVP (22) has a nontrivial solution, then*

$$\int_{y_1}^{y_2}\Big(\frac{\psi(s)-\psi(y_1)}{\psi(y_2)-\psi((y_1)}\Big)^{\gamma-1}(\psi((y_2)-\psi(y_1))^{\zeta-1}\psi'(s)|r(s)|ds \geqslant \Gamma(\zeta).$$

COROLLARY 7. *Assume that $r \in C([y_1,y_2],\mathbb{R})$ and $1<\zeta<\beta\leqslant 2$. If the FBVP (22) has a nontrivial solution, then*

$$\Big(\frac{\psi(y_2)-\psi(y_1)}{4}\Big)^{\zeta-1}\int_{y_1}^{y_2}\psi'(s)|r(s)|ds \geqslant \Gamma(\zeta).$$

10. Lyapunov-type inequalities for Katugampola-Hilfer type FBVPs

In this section, we present Lyapunov-type inequalities for Katugampola-Hilfer type FBVPs.

In 2023, Thabet and Kedim [45] considered the Katugampola-Hilfer type FBVPs:

$$\begin{cases} {}^{\rho,H}D^{\zeta,\beta}_{y_1+}z(t)+r(t)z(t)=0, \quad y_1<t<y_2, \\ z(y_1)=g(z), \quad z(y_2)=\displaystyle\int_{y_1}^{y_2}(hz)(u)du, \end{cases} \tag{24}$$

and

$$\begin{cases} {}^{\rho,H}D^{\zeta,\beta}_{y_1+}z(t)+r(t)z(t)=0, \quad y_1<t<y_2, \\ z(y_1)=g(z), \quad t^{1-\rho}\dfrac{d}{dt}z(t)|_{t=y_2}=\displaystyle\int_{y_1}^{y_2}(hz)(u)du, \end{cases} \tag{25}$$

where ${}^{\rho,H}D^{\zeta,\beta}_{y_1+}$ denotes the Katugampola-Hilfer derivative operator of fractional order $\zeta\in(1,2]$ and type $\beta\in[0,1]$ with $\rho>0$ (see Definition 18). Furthermore, $z,r,h:[y_1,y_2]\to\mathbb{R}$ are continuous functions, and the nonlocal function $g\in C([y_1,y_2],\mathbb{R})$ such that there exists a constant $K>0$ so that $|g(z)|\leqslant K$, $\forall z\in\mathbb{R}$.

LEMMA 43. *Let $\rho>0$, $\zeta\in(1,2]$, $\beta\in[0,1]$, $\gamma=\zeta+\beta(2-\zeta)$, $\zeta,\gamma\in(1,2]$, $z,h\in C((y_1,y_2),\mathbb{R})$ and $r:[y_1,y_2]\to\mathbb{R}$. Then, the solution of Katugampola-Hilfer type FBVP (24) is*

$$\begin{aligned} z(t) = &\int_{y_1}^{y_2}H(t,s)r(s)z(s)ds+\Big(1-\frac{(t^\rho-y_1^\rho)^{\gamma-1}}{(y_2^\rho-y_1^\rho)^{\gamma-1}}\Big)g(z) \\ &+\frac{(t^\rho-y_1^\rho)^{\gamma-1}}{(y_2^\rho-y_1^\rho)^{\gamma-1}}\frac{1}{Q}\int_{y_1}^{y_2}\Big(1-\frac{(\sigma^\rho-y_1^\rho)^{\gamma-1}}{(y_2^\rho-y_1^\rho)^{\gamma-1}}\Big)g(z)h(\sigma)d\sigma, \end{aligned} \tag{26}$$

where $H(t,s) = H_1(t,s) + H_2(t,s)$ *with*

$$H_1(t,s) = \frac{\rho^{1-\zeta} s^{\rho-1}}{\Gamma(\zeta)(y_2^\rho - y_1^\rho)^{\gamma-1}} \begin{cases} (t^\rho - y_1^\rho)^{\gamma-1}(y_2^\rho - s^\rho)^{\zeta-1}, & y_1 \leqslant t \leqslant s \leqslant y_2, \\ (t^\rho - y_1^\rho)^{\gamma-1}(y_2^\rho - s^\rho)^{\zeta-1} \\ -(y_2^\rho - y_1^\rho)^{\gamma-1}(t^\rho - s^\rho)^{\zeta-1}, & y_1 \leqslant s \leqslant t \leqslant y_2, \end{cases}$$

and

$$H_2(t,s) = \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(y_2^\rho - y_1^\rho)^{\gamma-1}} \frac{1}{Q} \int_{y_1}^{y_2} H_1(\sigma,s) h(\sigma) d\sigma,$$

$$Q = 1 - \int_{y_1}^{y_2} \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(y_2^\rho - y_1^\rho)^{\gamma-1}} h(t) dt > 0.$$

LEMMA 44. *Let* $\rho > 0$, $\zeta \in (1,2]$, $\beta \in [0,1]$, $\gamma = \zeta + \beta(2-\zeta)$, $\zeta, \gamma \in (1,2]$, $z, h \in C((y_1,y_2),\mathbb{R})$ *and* $r : [y_1,y_2] \to \mathbb{R}$. *Then, the solution of Katugampola-Hilfer type FBVP* (25) *is*

$$z(t) = \int_{y_1}^{y_2} G(t,s) r(s) z(s) ds + g(z) + \frac{1}{\rho(\gamma-1)} \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(y_2^\rho - y_1^\rho)^{\gamma-1}} \frac{1}{S} \int_{y_1}^{y_2} g(z) h(\sigma) d\sigma, \quad (27)$$

where $G(t,s) = G_1(t,s) + G_2(t,s)$ *with*

$$G_1(t,s) = \frac{\rho^{1-\zeta} s^{\rho-1} (y_2^\rho - s^\rho)^{\zeta-2}}{\Gamma(\zeta)(\gamma-1)} \begin{cases} (\zeta-1)(t^\rho - y_1^\rho)^{\gamma-1}(y_2^\rho - y_1^\rho)^{2-\gamma}, & y_1 \leqslant t \leqslant s \leqslant y_2, \\ (\zeta-1)(t^\rho - y_1^\rho)^{\gamma-1}(y_2^\rho - y_1^\rho)^{2-\gamma} \\ -(\gamma-1)\frac{(t^\rho - s^\rho)^{\zeta-1}}{(y_2^\rho - s^\rho)^{\zeta-2}}, & y_1 \leqslant s \leqslant t \leqslant y_2, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{\rho(\gamma-1)} \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(y_2^\rho - y_1^\rho)^{\gamma-1}} \frac{1}{S} \int_{y_1}^{y_2} G_1(\sigma,s) h(\sigma) d\sigma,$$

$$S = 1 - \int_{y_1}^{y_2} \frac{1}{\rho(\gamma-1)} \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(y_2^\rho - y_1^\rho)^{\gamma-2}} h(t) dt > 0.$$

In the next lemma, for the functions $H(t,s)$ and $G(t,s)$ defined in Lemmas 43 and 44, respectively, we give some properties.

LEMMA 45. *Let* $\rho > 0$, $\zeta \in (1,2]$, $\beta \in [0,1]$, $\gamma = \zeta + \beta(2-\zeta)$, $\zeta, \gamma \in (1,2]$. *Then, the Green's functions* $H(t,s)$ *and* $G(t,s)$ *given in Lemmas* 43 *and* 44*, respectively satisfy the properties:*

(i) $H(t,s)$ *and* $G(t,s)$ *are continuous functions for all* $(t,s) \in [y_1,y_2] \times [y_1,y_2]$;

(ii) $$|H(t,s)| < \Big(\frac{\gamma-1}{\gamma+\zeta-2}\Big)^{\gamma-1} \Big(\frac{(y_2^\rho - y_1^\rho)(\zeta-1)}{\gamma+\zeta-2}\Big)^{\zeta-1} \frac{\rho^{1-\zeta} s^{\rho-1}}{\Gamma(\zeta)} \Big[1 + \frac{1}{Q} \int_{y_1}^{y_2} |h(\sigma)| d\sigma\Big];$$

(iii) $|G(t,s)|$
$$< \frac{(y_2^\rho - y_1^\rho)(y_2^\rho - s^\rho)^{\zeta-2}}{(\gamma-1)\Gamma(\zeta)\rho^{\zeta-1}s^{1-\rho}} \max\{\gamma-\zeta, \zeta-1\}\left[1 + \frac{y_2-y_1}{\rho(\gamma-1)S}\int_{y_1}^{y_2}|h(\sigma)|d\sigma\right].$$

10.1. Lyapunov-type inequalities for the FBVP (24)

In this section, we present Lyapunov-type inequalities for the FBVP (24).

THEOREM 25. *Assume that the Katugampola-Hilfer type FBVP* (24) *possesses a nontrivial solution $z \in C((y_1,y_2),\mathbb{R})$. Then, the following inequality holds:*

$$\frac{1}{1+\frac{1}{Q}\int_{y_1}^{y_2}|h(\sigma)|d\sigma}$$
$$< 2K + \left(\frac{\gamma-1}{\gamma+\zeta-2}\right)^{\gamma-1}\left(\frac{(y_2^\rho - y_1^\rho)(\zeta-1)}{\gamma+\zeta-2}\right)^{\zeta-1}\frac{\max\{y_1^{\rho-1}, y_2^{\rho-1}\}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_{y_1}^{y_2}|r(s)|ds.$$

Using the above theorem, we present a condition on the nonexistence of nontrivial solutions to the FBVP (24).

COROLLARY 8. *The Katugampola-Hilfer type FBVP* (24) *has no nontrivial solution if*

$$\frac{1}{1+\frac{1}{Q}\int_{y_1}^{y_2}|h(\sigma)|d\sigma}$$
$$\geqslant 2K + \left(\frac{\gamma-1}{\gamma+\zeta-2}\right)^{\gamma-1}\left(\frac{(y_2^\rho - y_1^\rho)(\zeta-1)}{\gamma+\zeta-2}\right)^{\zeta-1}\frac{\max\{y_1^{\rho-1}, y_2^{\rho-1}\}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_{y_1}^{y_2}|r(s)|ds.$$

COROLLARY 9. *Assume that the Hadamard-Hilfer FBVP:*

$$\begin{cases} {}^{H,H}D_{y_1+}^{\zeta,\beta}z(t) + r(t)z(t) = 0, \quad 0 < y_1 < t < y_2, \\ z(y_1) = g(z), \quad z(y_2) = \int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z \in C((y_1,y_2),\mathbb{R})$. Then, the following inequality holds:

$$\frac{1}{1+\frac{1}{Q_0}\int_{y_1}^{y_2}|h(\sigma)|d\sigma}$$
$$< 2K + \left(\frac{\gamma-1}{\gamma+\zeta-2}\right)^{\gamma-1}\left(\frac{(\ln y_2 - \ln y_1)(\zeta-1)}{\gamma+\zeta-2}\right)^{\zeta-1}\frac{1}{y_1\Gamma(\zeta)}\int_{y_1}^{y_2}|r(s)|ds,$$

where $Q_0 = 1 - \int_{y_1}^{y_2}\frac{(\ln t - \ln y_1)^{\gamma-1}}{(\ln y_2 - \ln y_1)^{\gamma-1}}h(t)dt > 0.$

COROLLARY 10. *Assume that the Hilfer FBVP:*

$$\begin{cases} {}^{H}D^{\zeta,\beta}_{y_1+}z(t)+r(t)z(t)=0, & y_1<t<y_2,\\ z(y_1)=g(z), \quad z(y_2)=\int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z\in C((y_1,y_2),\mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{1}{Q_1}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < 2K+\Big(\frac{\gamma-1}{\gamma+\zeta-2}\Big)^{\gamma-1}\Big(\frac{(y_2-y_1)(\zeta-1)}{\gamma+\zeta-2}\Big)^{\zeta-1}\frac{1}{\Gamma(\zeta)}\int_{y_1}^{y_2}|r(s)|ds,$$

where $Q_1=1-\int_{y_1}^{y_2}\frac{(t-y_1)^{\gamma-1}}{(y_2-y_1)^{\gamma-1}}h(t)dt>0.$

COROLLARY 11. *Assume that the Katugampola-Riemann FBVP:*

$$\begin{cases} {}^{\rho,R}D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0, & y_1<t<y_2,\\ z(y_1)=g(z), \quad z(y_2)=\int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z\in C((y_1,y_2),\mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{1}{Q_2}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < 2K+\Big(\frac{(y_2^\rho-y_1^\rho)}{4}\Big)^{\zeta-1}\frac{\rho^{1-\zeta}\max\{y_1^{\rho-1},y_2^{\rho-1}\}}{\Gamma(\zeta)}\int_{y_1}^{y_2}|r(s)|ds,$$

where $Q_2=1-\int_{y_1}^{y_2}\frac{(t^\rho-y_1^\rho)^{\gamma-1}}{(y_2-y_1)^{\gamma-1}}h(t)dt>0.$

COROLLARY 12. *Assume that the Katugampola-Caputo type FBVP:*

$$\begin{cases} {}^{\rho,C}D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0, & y_1<t<y_2,\\ z(y_1)=g(z), \quad z(y_2)=\int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z\in C((y_1,y_2),\mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{1}{Q_3}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < 2K+\Big(\frac{1}{\zeta}\Big)^{\zeta}((y_2^\rho-y_1^\rho)(\zeta-1))^{\zeta-1}\frac{\rho^{1-\zeta}\max\{y_1^{\rho-1},y_2^{\rho-1}\}}{\Gamma(\zeta)}\int_{y_1}^{y_2}|r(s)|ds,$$

where $Q_3=1-\int_{y_1}^{y_2}\frac{(t^\rho-y_1^\rho)}{(y_2-y_1)}h(t)dt>0.$

10.2. Lyapunov-type inequalities for the FBVP (25)

Now, we describe the Lyapunov-type inequalities for the FBVP (25).

THEOREM 26. *Assume that the Katugampola-Hilfer type FBVP* (25) *possesses a nontrivial solution* $z \in C((y_1, y_2), \mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{y_2^\rho - y_1^\rho}{\rho(\gamma-1)S}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < K + \frac{(y_2^\rho - y_1^\rho)\rho^{1-\zeta}}{(\gamma-1)\Gamma(\zeta)}\max\{\gamma-\zeta, \zeta-1\}\int_{y_1}^{y_2} s^{\rho-1}(y_2^\rho - s^\rho)^{\zeta-2}|r(s)|ds.$$

COROLLARY 13. *The Katugampola-Hilfer type FBVP* (25) *has no nontrivial solution if*

$$\frac{1}{1+\frac{y_2^\rho - y_1^\rho}{\rho(\gamma-1)S}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} \geqslant K + \frac{(y_2^\rho - y_1^\rho)\rho^{1-\zeta}}{(\gamma-1)\Gamma(\zeta)}\max\{\gamma-\zeta, \zeta-1\}\int_{y_1}^{y_2} s^{\rho-1}(y_2^\rho - s^\rho)^{\zeta-2}|r(s)|ds.$$

COROLLARY 14. *Assume that the Hadamard-Hilfer type FBVP:*

$$\begin{cases} {}^{H,H}D_{y_1+}^{\zeta,\beta}z(t) + r(t)z(t) = 0, \quad y_1 < t < y_2, \\ z(y_1) = g(z), \quad t^{1-\rho}\dfrac{d}{dt}z(t)|_{t=y_2} = \displaystyle\int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possess a nontrivial solution $z \in C((y_1, y_2), \mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{1}{S_0}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < K + \frac{(\ln y_2 - \ln y_1)}{(\gamma-1)}\frac{\max\{\gamma-\zeta, \zeta-1\}}{\Gamma(\zeta)}\int_{y_1}^{y_2}\frac{1}{s}(\ln y_2 - \ln s)^{\zeta-2}|r(s)|ds,$$

where $S_0 = \dfrac{1}{\ln y_2 - \ln y_1} - \displaystyle\int_{y_1}^{y_2}\frac{(\ln t - \ln y_1)^{\gamma-1}}{(\ln y_2 - \ln y_1)^{\gamma-1}}h(t)dt > 0.$

COROLLARY 15. *Assume that the Hilfer FBVP:*

$$\begin{cases} {}^{H}D_{y_1+}^{\zeta,\beta}z(t) + r(t)z(t) = 0, \quad y_1 < t < y_2, \\ z(y_1) = g(z), \quad t^{1-\rho}\dfrac{d}{dt}z(t)|_{t=y_2} = \displaystyle\int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z \in C((y_1, y_2), \mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{y_2-y_1}{(\gamma-1)S_1}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < K + \frac{(y_2-y_1)}{(\gamma-1)\Gamma(\zeta)}\max\{\gamma-\zeta, \zeta-1\}\int_{y_1}^{y_2}(y_2-s)^{\zeta-2}|r(s)|ds,$$

where $S_1 = 1 - \int_{y_1}^{y_2} \frac{1}{\gamma-1} \frac{(t-y_1)^{\gamma-1}}{(y_2-y_1)^{\gamma-1}} h(t)dt > 0.$

COROLLARY 16. *Assume that the Katugampola-Riemann FBVP:*

$$\begin{cases} {}^{\rho,R}D^{\zeta}_{y_1+}z(t) + r(t)z(t) = 0, \quad y_1 < t < y_2, \\ z(y_1) = g(z), \quad t^{1-\rho}\frac{d}{dt}z(t)|_{t=y_2} = \int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z \in C((y_1,y_2),\mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{y_2-y_1}{\rho(\zeta-1)S_2}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < K + \frac{\rho^{1-\zeta}(y_2^{\rho}-y_1^{\rho})}{\Gamma(\zeta)}\int_{y_1}^{y_2} s^{\rho-1}(y_2^{\rho}-s^{\rho})^{\zeta-2}|r(s)|ds,$$

where $S_2 = 1 - \int_{y_1}^{y_2} \frac{1}{\rho(\zeta-1)} \frac{(t^{\rho}-y_1^{\rho})^{\zeta-1}}{(y_2^{\rho}-y_1^{\rho})^{\zeta-1}} h(t)dt > 0.$

COROLLARY 17. *Assume that the Katugampola-Caputo type FBVP:*

$$\begin{cases} {}^{\rho,C}D^{\zeta}_{y_1+}z(t) + r(t)z(t) = 0, \quad y_1 < t < y_2, \\ z(y_1) = g(z), \quad t^{1-\rho}\frac{d}{dt}z(t)|_{t=y_2} = \int_{y_1}^{y_2}(hz)(u)du, \end{cases}$$

possesses a nontrivial solution $z \in C((y_1,y_2),\mathbb{R})$. *Then, the following inequality holds:*

$$\frac{1}{1+\frac{y_2-y_1}{\rho S_3}\int_{y_1}^{y_2}|h(\sigma)|d\sigma} < K + \frac{(y_2^{\rho}-y_1^{\rho})}{\rho^{\zeta-1}\Gamma(\zeta)}\max\{2-\zeta,\zeta-1\}\int_{y_1}^{y_2}\frac{(y_2^{\rho}-s^{\rho})^{\zeta-2}}{s^{1-\rho}}|r(s)|ds,$$

where $S_3 = 1 - \int_{y_1}^{y_2}(t^{\rho}-y_1^{\rho})h(t)dt > 0.$

11. Lyapunov-type inequalities for FBVP involving proportional fractional derivative

DEFINITION 15. [15] Let $\rho \in [0,1]$. The local fractional proportional integral of order ρ for the function g is defined by $({}_{y_1}I^0 g)(t) = g(t)$ and

$$({}^{\rho}_{y_1}I^1 g)(t) = \frac{1}{\rho}\int_{y_1}^{t} e^{\frac{\rho-1}{\rho}(t-s)}g(s)ds, \text{ for } \rho \in (0,1], \ t \in [y_1,y_2].$$

In [23], Zaadjal et al. studied the sequential local fractional proportional boundary value problem:

$$\begin{cases} ({}^{\rho_1}_{y_1}D^{\sigma}\,{}^{\rho_2}_{y_1}D^{\sigma})z(t) + r(t)z(t) = 0, \ t \in [y_1,y_2], \\ z(y_1) = 0, \quad z(y_2) = 0, \end{cases} \tag{28}$$

where ${}_{y_1}\rho D^{\sigma}$ denotes the local fractional proportional differential operator of order $\sigma \in \{\rho_1,\rho_2\}$ with $0 < \rho_1, \ \rho_2 < 1, \ 1 < \rho_1+\rho_2 < 2, \ \rho_1 \neq \rho_2$, and $r : [y_1,y_2] \to \mathbb{R}$ is a continuous function.

LEMMA 46. *The unique integral solution of the FBVP* (28) *is given by*

$$z(t)=\int_{y_1}^{y_2}G(t,s)r(s)z(s)ds,$$

where

$$G(t,s)=\begin{cases}\dfrac{[e^{\gamma_1(t-y_1)}-e^{\gamma_2(t-y_1)}][e^{\gamma_1(y_2-s)}-e^{\gamma_2(y_2-s)}]}{\delta[e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}]}\\ -\dfrac{1}{\delta}[e^{\gamma_1(t-s)}-e^{\gamma_2(t-s)}], & y_1\leqslant s\leqslant t\leqslant y_2,\\ \dfrac{[e^{\gamma_1(t-y_1)}-e^{\gamma_2(t-y_1)}][e^{\gamma_1(y_2-s)}-e^{\gamma_2(y_2-s)}]}{\delta[e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}]}, & y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

with $\gamma_1=\frac{\rho_1-1}{\rho_1}$, $\gamma_2=\frac{\rho_2-1}{\rho_2}$ *and* $\delta=\rho_1\rho_2(\gamma_1-\gamma_2)$.

LEMMA 47. *The Green's function defined in Lemma* 46 *satisfies the properties:*

(i) $G(t,s)\geqslant 0$ *for all* $t,s\in[y_1,y_2]$;

(ii) $G(t,s)\leqslant \dfrac{\varepsilon[e^{\gamma_1(y_2-s)}-e^{\gamma_2(y_2-s)}]}{|\delta|[e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}]}$ *for all* $t,s\in[y_1,y_2]$;

(iii) $G(t,s)\leqslant \dfrac{\varepsilon^2}{\delta[e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}]}$ *for all* $t,s\in[y_1,y_2]$, *where*

$$\varepsilon=\begin{cases}\max\left\{\left|e^{\gamma_1\xi}-e^{\gamma_2\xi}\right|,\left|e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}\right|\right\}, & \xi\leqslant y_2-y_1,\\ \left|e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}\right|, & \xi\geqslant y_2-y_1,\end{cases}$$

with $\xi=\dfrac{1}{\gamma_1-\gamma_2}\ln\dfrac{\gamma_2}{\gamma_1}$;

(iv) $\max_{t\in[y_1,y_2]}\int_{y_1}^{y_2}|G(t,s)|ds=\dfrac{1}{(\rho_1-1)(\rho_2-1)}$.

Now, we present the Hartman-Wintner-type inequality for the FBVP (28).

THEOREM 27. *If a continuous nontrivial solution to the problem* (28) *exists, then*

$$\int_{y_1}^{y_2}sign(\delta)[e^{\gamma_1(y_2-s)}-e^{\gamma_2(y_2-s)}]|r(s)|ds\geqslant\frac{\delta}{\varepsilon}[e^{\gamma_1(y_2-y_1)}-e^{\gamma_2(y_2-y_1)}],$$

where $sign(\delta)=\begin{cases}1, & \delta\geqslant 0,\\ -1, & \delta<0.\end{cases}$

We have the following Lyapunov-type inequality for the problem (28).

THEOREM 28. *If a continuous nontrivial solution to the FBVP (28) exists, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant \frac{\delta}{\varepsilon^2}[e^{\gamma_1(y_2-y_1)} - e^{\gamma_2(y_2-y_1)}].$$

Consider the local fractional proportional boundary value problem:

$$\begin{cases} ({}^{\rho}_{y_1}D^{\alpha})z(t) + r(t)z(t) = 0, \ \ t \in [y_1, y_2], \\ z(y_1) = 0, \quad z(y_2) = 0, \end{cases} \tag{29}$$

where ${}_{y_1}D^{\rho}$ denotes the local fractional proportional derivative operator of order ρ with $0 < \rho < 1,\ 1 < \alpha < 2$ and $r : [y_1, y_2] \to \mathbb{R}$ is a continuous function.

LEMMA 48. *Let* $\sigma = \dfrac{\rho - 2}{\rho - 1}$. *Then, the FBVP (29) has a unique solution given by*

$$z(t) = \int_{y_1}^{y_2} \tilde{G}(t,s)r(s)z(s)ds,$$

where

$$\tilde{G}(t,s) = \frac{1}{2-\rho}\begin{cases} \dfrac{[1-e^{\sigma(t-y_1)}][1-e^{\sigma(y_2-s)}]}{1-e^{\sigma(y_2-y_1)}} - [1-e^{\sigma(t-s)}], & y_1 \leqslant s \leqslant t \leqslant y_2, \\ \dfrac{[1-e^{\sigma(t-y_1)}][1-e^{\sigma(y_2-s)}]}{1-e^{\sigma(y_2-y_1)}}, & y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

LEMMA 49. *The Green's function defined in Lemma 48 satisfies the properties:*

(i) $\tilde{G}(t,s) \geqslant 0$ *for all* $t, s \in [y_1, y_2]$;

(ii) $\max\limits_{t \in [y_1,y_2]} |\tilde{G}(t,s)| = \tilde{G}(s,s)$ *for all* $s \in [y_1, y_2]$;

(iii) $\max\limits_{s \in [y_1,y_2]} |\tilde{G}(s,s)| = \tilde{G}\Big(\dfrac{y_1+y_2}{2}, \dfrac{y_1+y_2}{2}\Big) = \dfrac{\Big(1-e^{\sigma\left(\frac{y_2-y_1}{2}\right)}\Big)^2}{(2-\rho)[1-e^{\sigma(y_2-y_1)}]}$;

(iv) $\max\limits_{t \in [y_1,y_2]} \displaystyle\int_{y_1}^{y_2} |\tilde{G}(t,s)|ds = \frac{1}{2-\rho}\Big[\frac{1}{\sigma} - \frac{y_2-y_1}{e^{\sigma(y_2-y_1)}-1} - \frac{1}{\sigma}\ln\frac{\Big(e^{\sigma(y_2-y_1)}-1\Big)}{\sigma(y_2-y_1)}\Big]$.

Next, we present the Hartman-Wintner-type inequality for the FBVP (29).

THEOREM 29. *If a continuous nontrivial solution to the problem (29) exists, then*

$$\int_{y_1}^{y_2} [1-e^{\sigma(s-y_2)}][1-e^{\sigma(y_2-s)}]|r(s)|ds \geqslant (2-\rho)[1-e^{\sigma(y_2-y_1)}].$$

We have the following Lyapunov-type inequality for the FBVP (29).

THEOREM 30. *If a continuous nontrivial solution to the FBVP (29) exists, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant \frac{(2-\rho)[1-e^{\sigma(y_2-y_1)}]}{\Big(1-e^{\sigma\left(\frac{y_2-y_1}{2}\right)}\Big)^2}.$$

12. Lyapunov-type inequality for a pantograph FBVP involving a variable order Hadamard fractional derivative

DEFINITION 16. [41] Let $1 \leqslant y_1 < y_2 < \infty$ and $\psi : [y_1, y_2] \to (0, \infty)$. The left-sided Hadamard fractional integral of variable order $\alpha(t)$ for a function g is given by

$$^{H}I_{y_1+}^{\alpha(t)} g(t) = \frac{1}{\Gamma(\psi(t))} \int_{y_1}^{t} \left(\ln \frac{t}{s}\right)^{\alpha(t)-1} \frac{g(s)}{s} ds, \ \ t > y_1.$$

DEFINITION 17. [41] Let $n \in \mathbb{N}$ and $\alpha : [y_1, y_2] \to (n-1, n)$. The left-sided Hadamard derivative of variable order $\alpha(t)$ for a function g is given by

$$^{H}\mathscr{D}_{y_1+}^{\alpha(t)} g(t) = \frac{t^n}{\Gamma(n-\alpha(t))} \frac{d^n}{dt^n} \Big[\int_{y_1}^{t} \left(\ln \frac{t}{s}\right)^{n-1-\alpha(t)} \frac{g(s)}{s} ds \Big], \ \ t > y_1.$$

In 2023, Graef et al. [12] studied the nonlinear pantograph FBVP containing a Hadamard fractional derivative operator of variable order:

$$\begin{cases} ^{H}D_{1+}^{\alpha(t)} z(t) = g(t, z(t), z(\lambda t)), \ \ t \in [1, T], \\ z(1) = z(T) = 0, \end{cases} \tag{30}$$

where $1 < \alpha(t) < 2$, $0 < \lambda < 1$, $g : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $^{H}D_{1+}^{\alpha(t)}$ is the left-sided Hadamard fractional derivative operator of variable order $\alpha(t)$. Let $P = [1, t_1], (t_1, t_2], (t_2, t_3], \ldots, (t_n, T]$ be a partition of the interval $[1, T]$, and let $\alpha(t) : [1, T] \to (1, 2)$ be the piecewise constant function with respect to P given by $\alpha(t) = \sum_{i=1}^{n} \alpha_i(t) \chi_i(t)$, $t \in [1, T]$, where $1 < \alpha_i < 2$, $i = 1, 2, \ldots, n$ are constants, and χ_i denote the characteristic function for the interval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$, that is,

$$\chi_i(t) = \begin{cases} 1, & t \in [t_{i-1}, t_i], \\ 0, & \text{otherwise}. \end{cases}$$

Then, the FBVP can be written as

$$\begin{cases} ^{H}D_{t_{i-1}+}^{\alpha_i(t)} \hat{y}(t) = \dfrac{t^2}{\Gamma(2-\psi_i)} \dfrac{d^2}{dt^2} \Big[\displaystyle\int_{t_{i-1}}^{t} \left(\ln \frac{t}{s}\right)^{1-\alpha_i(t)} \frac{\hat{y}(s)}{s} ds \Big] = g(t, \hat{y}(t), \hat{y}(\lambda t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t, \lambda t) \in [t_{i-1}, t_i], \\ \hat{y}(t_{i-1}) = \hat{y}(t_i) = 0. \end{cases} \tag{31}$$

LEMMA 50. *The Green's function for the FBVP* (31) *is given by*

$$\Phi_i(s,t) = \begin{cases} \Phi_{1,i}(s,t), & t_{i-1} \leqslant s \leqslant t \leqslant t_i, \\ \Phi_{2,i}(s,t), & t_{i-1} \leqslant t \leqslant s \leqslant t_i, \end{cases}$$

where

$$\Phi_{1,i}(s,t) = \frac{1}{s\Gamma(\alpha_i)} \Big[\left(\ln \frac{t}{s}\right)^{\alpha_i - 1} - \left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\alpha_i - 1} \left(\ln \frac{t_i}{s}\right)^{\alpha_i - 1} \Big],$$

and

$$\Phi_{2,i}(s,t) = \frac{-1}{s\Gamma(\alpha_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{t}{t_{i-1}}\right)^{\alpha_i-1}\left(\ln\frac{t_i}{s}\right)^{\alpha_i-1}, \ i\in\{1,2,\dots,n\}.$$

LEMMA 51. *Let Φ_i, $i\in\{1,2,\dots,n\}$, be the Green's function defined in Lemma 50. Then, we have*

$$\max_{t\in[t_{i-1},t_i]}|\Phi_i(s,t)| \leqslant \frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{1-\alpha_i}[(\mu_i-\ln t_{i-1})(\ln t_i-\mu_i)]^{\alpha_i-1}\exp(-\mu_i),$$

where

$$\mu_i = \frac{1}{2}\left[2\alpha_i-2+\ln t_it_{i-1}-\sqrt{(2\alpha_i-2+\ln t_it_{i-1})^2-4[(\alpha_i-1)\ln t_it_{i-1}+\ln t_i\ln t_{i-1}]}\right].$$

The Lyapunov inequality for the FBVP (30) is given in the following result.

THEOREM 31. *Suppose that there exists $h\in C([1,T],\mathbb{R}^+)$ such that*

$$|g(t,z(t),z(\lambda t))| \leqslant h(t)|z(t)|+|z(\lambda t)|, \ 1\leqslant t\leqslant T.$$

If the FBVP (30) has a nontrivial solution, then

$$\int_1^T h(s)ds > \sum_{i=1}^n \frac{\Gamma(\alpha_i)}{2}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\alpha_i-1}[(\mu_i-\ln t_{i-1})(\ln t_i-\mu_i)]^{1-\alpha_i}\exp(\mu_i).$$

13. Lyapunov-type inequalities for partial fractional differential equations

This section is devoted to the Lyapunov-type inequalities for problems involving partial fractional derivatives studied by Odzijewicz [34] in 2023.

13.1. Partial differential equation of the first type

In this subsection, we present a Lyapunov-type inequality for problems involving right Caputo and the left Riemann-Liouville partial fractional derivatives.

Suppose that $\zeta,\beta\in(0,1)$, $\zeta+\beta\in(1,2]$, $\gamma=\delta/2$, $\delta\in(0,2]$, $K\in\mathbb{R}$, and $w\in C([y_1,y_2],\mathbb{R})$. We consider the following equation

$${}^CD^{\zeta}_{y_2-,t}\left(D^{\beta}_{y_1+,t}u(t,x)\right)-(1-x)^{\gamma}(1+x)^{\gamma}D^{\gamma}_{1-,x}(K\,{}^CD^{\gamma}_{-1+,x}u(t,x)) = w(t)u(t,x) \quad (32)$$

for $(t,x)\in(y_1,y_2)\times(-1,1)$, subject to boundary conditions

$$u(t,-1)=0, \quad I^{1-\gamma}_{1-,x}(K\,{}^CD^{\gamma}_{-1+,x}u(t,x))\Big|_{x=1}=0, \ t\in(y_1,y_2), \quad (33)$$

$$u(y_1,x)=D^{\beta}_{y_1+,t}u(y_2,x)=0, \ x\in(-1,1). \quad (34)$$

LEMMA 52. *Let us consider the following FBVP with mixed fractional derivatives*

$$ {}^{C}D^{\zeta}_{y_2-}\left(D^{\beta}_{y_1+}v(t)\right)+k(t)v(t)=0, \quad t\in(y_1,y_2), \tag{35}$$

$$ v(y_1)=D^{\beta}_{y_1+}v(y_2)=0, \tag{36}$$

where $\zeta,\beta\in(0,1),\ \zeta+\beta\in(1,2]$, *and*

$$ k(t)=-\left(w(t)+\frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\right),\ t\in[y_1,y_2]. $$

If u *is a solution to the problem* (32)–(34) *which is not identically equal to zero, then the function*

$$ v(t)=\int_{-1}^{1}(1-x)^{-\gamma}u(t,x)dx $$

is a solution to the problem (35)–(36).

THEOREM 32. *If* u *is a positive solution to the problem* (32)–(34)*, which is not identically equal to zero, then the following Lyapunov-type inequality is satisfied*

$$ \int_{y_1}^{y_2}\left|w(s)+\frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\right|ds \geqslant \frac{(\zeta+\beta-1)\Gamma(\zeta)\Gamma(\beta)}{(y_2-y_1)^{\zeta+\beta-1}}. $$

13.2. Partial differential equation of the second type

Let us consider a partial differential equation with mixed fractional derivatives defined on the set $(0,1)\times(-1,1)$:

$$ D^{\zeta}_{0-,t}\left({}^{C}D^{\zeta}_{1-,t}u(t,x)\right)-(1-x)^{\beta}(1+x)^{\beta}\ D^{\beta}_{1-,x}(K\,{}^{C}D^{\beta}_{-1+,x}u(t,x))=w(t)u(t,x), \tag{37}$$

with boundary conditions

$$ u(t,-1)=0,\quad I^{1-\beta}_{1-,x}(K\,{}^{C}D^{\beta}_{-1+,x}u(t,x))\Big|_{x=1}=0,\quad t\in(0,1), \tag{38}$$

$$ u(0,x)=u(1,x)=0,\ \ x\in(-1,1), \tag{39}$$

where $\zeta\in\left(\frac{1}{2},1\right),\ \beta=\delta/2\in(0,1],\ \delta\in(0,2],\ K\in\mathbb{R}_+$, and $w\in C[0,1]$.

LEMMA 53. *If* u *is a solution to the problem* (37)–(39) *such that* $u\neq 0$, *then the function*

$$ v(t)=\int_{-1}^{1}(1-x)^{-\beta}u(t,x)dx,\ \ t\in[0,1] $$

is a solution to the FBVP with mixed fractional derivatives:

$$ D^{\zeta}_{0+}\left({}^{C}D^{\zeta}_{1-}v(t)\right)-k(t)v(t)=0),\quad t\in(0,1), \tag{40}$$

$$ v(0)=v(1)=0, \tag{41}$$

where

$$k(t)=-\Big(w(t)+\frac{K\Gamma(1+\beta)}{\Gamma(1-\beta)}\Big),\ \ t\in[0,1].$$

THEOREM 33. *Let* $\zeta\in\left(\frac{1}{2},1\right)$, $\beta=\delta/2\in(0,1]$, $\delta\in(0,2]$ *and* w *be continuous on* $[0,1]$. *If* u *is a positive solution to the problem* (37)–(39), *such that* $u\neq 0$, *then*

$$\int_0^1\left|w(s)+\frac{K\Gamma(1+\beta)}{\Gamma(1-\beta)}\right|ds\geqslant\frac{(2\zeta-1)\Gamma^2(\zeta)}{h},$$

where

$$h=\sup_{0<x<1}\Big[(1-x)^{2\zeta-1}-(1-x^{2\zeta-1})^2\Big].$$

13.3. Lyapunov-type inequalities for fractional elliptic boundary value problem

Let $\Omega\subset\mathbb{R}^N$, $N>2$, be an open bounded domain with smooth boundary and (y_1,y_2), $-\infty<y_1<y_2<+\infty$, be an interval. In cylindrical domain $D=(y_1,y_2)\times\Omega$, we consider the boundary value problem

$$\begin{cases}D^{\zeta}_{y_1+,x}\mathscr{D}^{\zeta}_{y_2-,x}u(x,y)+(-\Delta)^s_y u(x,y)=r(x)u(x,y), & (x,y)\in D,\\ u(y_1,y)=u(y_2,y)=0, & y\in\Omega,\\ u(x,y)=0, & y\in\mathbb{R}^N\setminus\{\Omega\},\end{cases}\tag{42}$$

where $1/2<\zeta\leqslant 1$, $s\in(0,1)$, $D^{\zeta}_{y_1+,x}u(x,y)=\partial_x\, I^{1-\zeta}_{y_1+,s}u(x,y)$ and $\mathscr{D}^{\zeta}_{y_2-,x}u(x,y)=I^{1-\zeta}_{y_2-,x}u(x,y)$ are respectively the left Riemann-Liouville and right Caputo fractional derivatives of order $0<\zeta\leqslant 1$ and $(-\Delta)^s$ is the fractional Laplacian of order $s\in(0,1)$ defined by

$$(-\Delta)^s_y u(x,y)=C_{N,s}\int_{\mathbb{R}^N}\frac{u(x,y)-u(x,\xi)}{|y-\xi|^{N+2s}}d\xi,\ \ y\in\mathbb{R}^N,$$

$C_{N,s}$ is some normalization constant and r is a real-valued continuous function.

Now, we present a Lyapunov type inequality for the elliptic FBVP (42) studied in [20] by Kassymov.

We first consider the problem:

$$\begin{cases}D^{\zeta}_{y_1+}\mathscr{D}^{\zeta}_{y_2-}u(x)-r(x)u(x)=0, & (x)\in(y_1,y_2),\\ u(y_1)=u(y_2)=0.\end{cases}\tag{43}$$

LEMMA 54. *Assume that* $\zeta\in\left(\frac{1}{2},1\right]$ *and* u *is a solution to the problem* (43). *Then,* u *satisfies the integral equation:*

$$u(x)=\int_{y_1}^{y_2}G(x,t)r(t)u(t)dt,$$

where

$$G(x,t) = K(x,t) - \frac{K(y_1,t)K(x,y_1)}{K(y_1,y_2)},$$

and

$$K(x,t) = \frac{1}{\Gamma^2(\zeta)} \int_{\max\{x,t\}}^{y_2} (s-x)^{\zeta-1}(s-t)^{\zeta-1} dt.$$

LEMMA 55. *Let* $\zeta \in \left(\frac{1}{2}, 1\right]$*, then*

$$\sup_{y_1<x<y_2} G(x,t) = G(x,x), \quad y_1 < t < x < y_2.$$

Next, let us consider the elliptic FBVP:

$$\begin{cases} D^{\zeta}_{y_1+,x} \mathscr{D}^{\zeta}_{y_2-,x} u(x,y) = r(x)u(x,y), & (x,y) \in D, \\ u(y_1,y) = u(y_2,y) = 0, & y \in \Omega, \\ u(x,y) = 0, & y \in \mathbb{R}^N \setminus \{\Omega\}. \end{cases} \tag{44}$$

We now present the Lyapunov-type inequality for the fractional elliptic FBVP (44).

THEOREM 34. *Let* $\zeta \in \left(\frac{1}{2}, 1\right]$*,* $s \in (0,1)$ *and* $r \in C([y_1,y_2],\mathbb{R})$*. Then, for* (44)*, we have that*

$$\int_{y_1}^{y_2} |r(x) - \lambda_1(\Omega)| dx \geqslant \left(\sup_{y_1<x<y_2} G(x,x) \right)^{-1},$$

where $\lambda_1(\Omega)$ *is the first eigenvalue of the problem:*

$$\begin{cases} (-\Delta)^s_y \phi_1(y) = \lambda_1(\Omega)\phi_1(y), & y \in \Omega, \\ \phi_1(y) = 0, & y \in \mathbb{R}^N \setminus \{\Omega\}. \end{cases}$$

14. Lyapunov-type inequalities for systems of Riemann-Liouville fractional differential equations with multi-point coupled boundary conditions

In 2023, Zhou and Cui [57] established some Lyapunov-type inequalities for a system of Riemann-Liouville fractional differential equations equipped with multipoint coupled boundary conditions given by

$$\begin{cases} D^{\beta_1}_{y_1+} u(t) + g_1(t,u(t),v(t)) = 0, & t \in (y_1,y_2), \\ D^{\beta_2}_{y_1+} v(t) + g_2(t,u(t),v(t)) = 0, & t \in (y_1,y_2), \\ u(y_1) = 0, \quad u(y_2) = \sum_{i=1}^{n} a_{1i} u(\xi_i) + \sum_{j=1}^{n} a_{2j} v(\eta_j), \\ v(y_1) = 0, \quad v(y_2) = \sum_{i=1}^{n} a_{3i} u(\xi_i) + \sum_{j=1}^{n} a_{4j} v(\eta_j), \end{cases} \tag{45}$$

where $y_1, y_2 \in \mathbb{R},\ 0 < y_1 < y_2,\ y_1 < \xi_1 < \xi_2 < \ldots < \xi_n < y_2,\ y_1 < \eta_1 < \eta_2 < \ldots < \eta_n < y_2,\ a_{ij} \geqslant 0\ (i = 1,2,3,4; j = 1,2,\ldots,n),\ 1 < \beta_i \leqslant 2\ (i = 1,2),\ D_{y_1+}^{\beta_i}\ (i = 1,2)$ is the Riemann-Liouville fractional derivative of order β_i and $g_1, g_2 : [y_1, y_2] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

The following assumptions are used in the subsequent results.

(H0) $a_{ij} \geqslant 0\ \ (i = 1,2,3,4;\ j = 1,2,\ldots,n),\ \kappa_{ij} \geqslant 0\ \ (i,j = 1,2)$ and $\kappa = \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} > 0,$ where

$$\kappa_{11} = 1 - \sum_{i=1}^{n} \frac{a_{1i}(\xi_i - y_2)^{\beta_1 - 1}}{(y_2 - y_1)^{\beta_1 - 1}}, \qquad \kappa_{12} = \sum_{i=1}^{n} \frac{a_{2j}(\eta_j - y_2)^{\beta_2 - 1}}{(y_2 - y_1)^{\beta_2 - 1}},$$
$$\kappa_{21} = \sum_{i=1}^{n} \frac{a_{3i}(\xi_i - y_2)^{\beta_1 - 1}}{(y_2 - y_1)^{\beta_1 - 1}}, \qquad \kappa_{22} = 1 - \sum_{i=1}^{n} \frac{a_{4ij}(\eta_j - y_2)^{\beta_2 - 1}}{(y_2 - y_1)^{\beta_2 - 1}}.$$

(H1) $g_1, g_2 : [y_1, y_2] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H2) There exist positive functions $p_{11}, p_{12} \in C([y_1, y_2], \mathbb{R})$ such that

$$|g_1(t,x,y)| \leqslant p_{11}(t)|x| + p_{12}|y|,\ \ t \in [y_1, y_2],\ \ x,y \in \mathbb{R}.$$

(H3) There exist positive functions $p_{21}, p_{22} \in C([y_1, y_2], \mathbb{R})$ such that

$$|g_2(t,x,y)| \leqslant p_{21}(t)|x| + p_{22}|y|,\ \ t \in [y_1, y_2],\ \ x,y \in \mathbb{R}.$$

LEMMA 56. *Let $\phi_1, \phi_2 \in C([y_1, y_2], \mathbb{R})$. Then, (u,v) is a solution of the system*

$$\begin{cases} D_{y_1+}^{\beta_1} u(t) + \phi_1(t) = 0,\ \ t \in (y_1, y_2), \\ D_{y_1+}^{\beta_2} v(t) + \phi_2(t) = 0,\ \ t \in (y_1, y_2), \\ u(y_1) = 0,\ \ u(y_2) = \displaystyle\sum_{i=1}^{n} a_{1i} u(\xi_i) + \sum_{j=1}^{n} a_{2j} v(\eta_j), \\ v(y_1) = 0,\ \ v(y_2) = \displaystyle\sum_{i=1}^{n} a_{3i} u(\xi_i) + \sum_{j=1}^{n} a_{4j} v(\eta_j), \end{cases} \tag{46}$$

if and only if (u,v) is a solution of the system of integral equations

$$\begin{cases} u(t) = \displaystyle\int_{y_1}^{y_2} G_{11}(t,s)\phi_1(s)ds + \int_{y_1}^{y_2} G_{12}\phi_2(s)ds, \\ v(t) = \displaystyle\int_{y_1}^{y_2} G_{21}(t,s)\phi_1(s)ds + \int_{y_1}^{y_2} G_{22}\phi_2(s)ds, \end{cases}$$

where

$$G_{11}(t,s) = G_{\beta_1}(t,s) + \frac{(t-y_1)^{\beta_1-1}}{\kappa(y_2-y_1)^{\beta_1-1}} \sum_{i=1}^{n} (\kappa_{22}a_{1i} + \kappa_{12}a_{3i}) G_{\beta_1}(\xi_i, s),$$
$$G_{12}(t,s) = \frac{(t-y_1)^{\beta_1-1}}{\kappa(y_2-y_1)^{\beta_1-1}} \sum_{j=1}^{n} (\kappa_{22}a_{2j} + \kappa_{12}a_{4j}) G_{\beta_2}(\eta_j, s),$$
$$G_{21}(t,s) = \frac{(t-y_1)^{\beta_2-1}}{\kappa(y_2-y_1)^{\beta_2-1}} \sum_{i=1}^{n} (\kappa_{21}a_{1i} + \kappa_{11}a_{3i}) G_{\beta_1}(\xi_i, s),$$
$$G_{22}(t,s) = G_{\beta_2}(t,s) + \frac{(t-y_1)^{\beta_2-1}}{\kappa(y_2-y_1)^{\beta_2-1}} \sum_{j=1}^{n} (\kappa_{21}a_{2j} + \kappa_{11}a_{4j}) G_{\beta_2}(\eta_j, s),$$

and

$$G_{\beta_i}(t,s) = \frac{1}{\Gamma(\beta_i)} \begin{cases} \dfrac{(t-y_1)^{\beta_i-1}}{(y_2-y_1)^{\beta_i-1}} (y_2-s)^{\beta_1-1} - (t-s)^{\beta_1-1}, & y_1 \leqslant s \leqslant t \leqslant y_2, \\ \dfrac{(t-y_1)^{\beta_i-1}}{(y_2-y_1)^{\beta_i-1}} (y_2-s)^{\beta_1-1}, & y_1 \leqslant t \leqslant s \leqslant y_2. \end{cases}$$

LEMMA 57. *The Green functions $G_{\beta_i}(t,s)$ defined in Lemma 56 satisfies the properties:*

(1) $G_{\beta_i}(t,s) \geqslant 0$ *for all* $t,s \in [y_1,y_2]$;

(2) $\max_{s\in[y_1,y_2]} G_{\beta_i}(s,s) = G_{\beta_i}(s,s), s \in [y_1,y_2]$;

(3) $G_{\beta_i}(s,s) = G_{\beta_i}\left(\frac{y_1+b}{2}, \frac{y_1+b}{2}\right) = \frac{1}{\Gamma(\beta_i)}\left(\frac{y_2-y_1}{4}\right)^{\beta_i-1}$;

(4) $G_{\beta_i}(t,s) \leqslant \frac{1}{\Gamma(\beta_i)} \frac{(t-y_1)^{\beta_i-1}}{(y_2-y_1)^{\beta_i-1}} (y_2-s)^{\beta_i-1}$ *for all* $t,s \in [y_1,y_2]$.

LEMMA 58. *For $y_1 < \xi_i < y_2$, we have*

$$\max_{s\in[y_1,y_2]} G_{\beta_i}(\xi_i, s) = G_{\beta_i}(\xi_i, \xi_i) = \frac{1}{\Gamma(\beta_i)} \frac{(\xi_i - y_2)^{\beta_i-1}}{(y_2-y_1)^{\beta_i-1}} (y_2 - \xi_i)^{\beta_i-1}.$$

LEMMA 59. *The functions G_{ij} $(i,j=1,2)$ defined in Lemma 56 satisfy the properties:*

(i) $G_{ij}(t,s) \leqslant \lambda_{ij}$ *for all* $y_1 \leqslant t,s \leqslant y_2$;

(ii) $G_{ij}(t,s) \leqslant \mu_{ij}(t-y_1)^{\beta_i-1}(y_2-s)^{\beta_j-1}$ *for all* $y_1 \leqslant t,s \leqslant y_2$, *where* λ_{ij}, μ_{ij} $(i,j=1,2)$ *are given by*

$$\begin{aligned}
\lambda_{11} &= \frac{1}{\Gamma(\beta_1)}\Big(\frac{y_2-y_1}{4}\Big)^{\beta_1-1} + \frac{1}{\kappa\Gamma(\beta_1)}\sum_{i=1}^{n}\frac{(\kappa_{22}a_{1i}+\kappa_{12}a_{3i})(\xi_i-y_2)^{\beta_1-1}}{(y_2-y_1)^{\beta_1-1}}(y_2-\xi_i)^{\beta_1-1},\\
\lambda_{12} &= \frac{1}{\kappa\Gamma(\beta_2)(y_2-y_1)^{\beta_1-1}}\sum_{j=1}^{n}(\kappa_{22}a_{2j}+\kappa_{12}a_{4j})(\eta_j-y_2)^{\beta_2-1}(y_2-\eta_j)^{\beta_2-1},\\
\lambda_{21} &= \frac{1}{\kappa\Gamma(\beta_1)(y_2-y_1)^{\beta_2-1}}\sum_{i=1}^{n}(\kappa_{21}a_{1i}+\kappa_{11}a_{3i})(\xi_i-y_2)^{\beta_1-1}(y_2-\xi_i)^{\beta_1-1},\\
\lambda_{22} &= \frac{1}{\Gamma(\beta_2)}\Big(\frac{y_2-y_1}{4}\Big)^{\beta_2-1} + \frac{1}{\kappa\Gamma(\beta_2)}\sum_{j=1}^{n}\frac{(\kappa_{21}a_{2j}+\kappa_{11}a_{4j})(\eta_j-y_2)^{\beta_2-1}}{(y_2-y_1)^{\beta_2-1}}(y_2-\eta_j)^{\beta_2-1},\\
\mu_{11} &= \frac{\kappa_{22}}{\kappa\Gamma(\beta_1)(y_2-y_1)^{\beta_1-1}}, \quad \mu_{12} = \frac{1}{\kappa\Gamma(\beta_2)(y_2-y_1)^{\beta_1-1}},\\
\mu_{21} &= \frac{1}{\kappa\Gamma(\beta_1)(y_2-y_1)^{\beta_2-1}}, \quad \mu_{22} = \frac{\kappa_{11}}{\kappa\Gamma(\beta_2)(y_2-y_1)^{\beta_2-1}}.
\end{aligned}$$

Now we present our main results on Lyapunov-type inequalities for the system (45).

For $p_{ij} \in C([y_1,y_2],\mathbb{R})$ $(i,j=1,2)$, let

$$J_{ij}(p_{1j},p_{2j}) = \lambda_{i1}\int_{y_1}^{y_2} p_{1j}(s)ds + \lambda_{i2}\int_{y_1}^{y_2} p_{2j}(s)ds, \quad i,j=1.2.$$

THEOREM 35. *Suppose that* $(H0)$–$(H3)$ *are satisfied. If there exists a nontrivial solution to the problem* (45)*, then*

$$\begin{aligned}
&J_{11}(p_{11},p_{21})+J_{22}(p_{12},p_{22})\\
&+\sqrt{[J_{11}(p_{11},p_{21})-J_{22}(p_{12},p_{22})]^2+4J_{12}(p_{12},p_{22})J_{21}(p_{11},p_{2.1}} \geqslant 2.
\end{aligned}$$

For $p_{ij} \in C([y_1,y_2],\mathbb{R})$ $(i,j=1,2)$, let

$$\begin{aligned}
I_{ij}(p_{1j},p_{2j}) = {}& \mu_{i1}\int_{y_1}^{y_2} p_{1j}(s)(y_2-s)^{\beta_1-1}(s-y_2)^{\beta_j-1}ds\\
&+\mu_{i2}\int_{y_1}^{y_2} p_{2j}(s)(y_2-s)^{\beta_2-1}(s-y_2)^{\beta_j-1}ds, \quad i,j=1.2.
\end{aligned}$$

THEOREM 36. *Suppose that* $(H0)$–$(H3)$ *are satisfied. If the problem* (45) *has a nontrivial solution, then*

$$\begin{aligned}
&I_{11}(p_{11},p_{21})+I_{22}(p_{12},p_{22})\\
&+\sqrt{[I_{11}(p_{11},p_{21})-I_{22}(p_{12},p_{22})]^2+4I_{12}(p_{12},p_{22})I_{21}(p_{11},p_{2.1}} \geqslant 2.
\end{aligned}$$

15. Lyapunov-type inequalities for FBVP involving bi-ordinal Hilfer fractional derivative

In this section, we present Lyapunov-type inequalities for FBVP with multi-point boundary conditions in the framework of bi-ordinal Hilfer-Katugampola and ψ-Hilfer fractional derivatives.

15.1. Lyapunov-type inequalities for bi-ordinal Hilfer-Katugampola fractional derivative

DEFINITION 18. [36] Let $\zeta > 0$, $n = [\zeta] + 1$ and $\rho > 0$. The left-sided Hilfer-Katugampola fractional derivative ${}^{\rho}D_{y_1+}^{\zeta,\beta}g$ of order ζ and type β, $0 \leqslant \beta \leqslant 1$, of a function g is defined by

$$({}^{\rho}D_{y_1+}^{\zeta,\beta}g)(t) = ({}^{\rho}I_{y_1+}^{\beta(n-\zeta)}(t^{1-\rho}\frac{d}{dt})^n\, {}^{\rho}I_{y_1+}^{(1-\beta)(n-\zeta)}g)(t).$$

DEFINITION 19. [19] Let $\zeta > 0$, $n-1 < \zeta, \beta \leqslant n$ and $\rho > 0$. The bi-ordinal Hilfer-Katugampola fractional derivative ${}^{\rho}D_{y_1+}^{(\zeta,\beta)\mu}g$ of order ζ,β and type μ, $0 \leqslant \mu \leqslant 1$, of a function g is defined by

$$({}^{\rho}D_{y_1+}^{(\zeta,\beta)\mu}g)(t) = ({}^{\rho}I_{y_1+}^{\mu(n-\zeta)}(t^{1-\rho}\frac{d}{dt})^n\, {}^{\rho}I_{y_1+}^{(1-\mu)(n-\beta)}g)(t).$$

In [7], Chen et al. established Lyapunov-type inequalities for the multipoint boundary value problems:

$$\begin{cases} {}^{\rho}D_{y_1+}^{(\zeta,\beta)\mu}z(t) + r(t)z(t) = 0, \quad 1 < \zeta, \beta < 2,\ \rho > 0, \quad y_1 < t < y_2, \\ z(y_1) = 0, \quad z(y_2) = \sum_{i=1}^{m-2} \omega_i z(\phi_i), \end{cases} \tag{47}$$

and

$$\begin{cases} {}^{\rho}D_{y_1+}^{(\zeta,\beta)\mu}z(t) + r(t)z(t) = 0, \quad 1 < \zeta, \beta < 2,\ \rho > 0, \quad y_1 < t < y_2, \\ z(y_1) = 0, \quad t^{1-\rho}\frac{d}{dt}z(t)|_{t=y_2} = \sum_{i=1}^{m-2} \lambda_i z(\eta_i), \end{cases} \tag{48}$$

where $r \in C([y_1,y_2],\mathbb{R})$, ${}^{\rho}D_{y_1+}^{(\zeta,\beta)\mu}$ is bi-ordinal Hilfer-Katugampola fractional derivative of order ζ,β and type μ $(0 \leqslant \mu \leqslant 1)$, $\omega_i, \lambda_i \geqslant 0$, $y_1 < \phi_i, \eta_i < y_2$, $i = 1,2,\ldots,m-2$, $y_1 < \phi_1 < \phi_2 < \ldots < \phi_{m-2} < y_2$, and $y_1 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < y_2$. Let $\gamma = \beta + \mu(2-\beta)$ and $\delta = \beta + \mu(\zeta - \beta)$.

In the sequel, we assume the following hypotheses:

(A) $\sum_{i=1}^{m-2} \omega_i(\phi_i^{\rho} - y_1^{\rho})^{\gamma-1} < (y_2^{\rho} - y_1^{\rho})^{\gamma-1}$;

(B) $\sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho - y_1^\rho)^{\gamma-1} < (\gamma-1)\rho(y_2^\rho - y_1^\rho)^{\gamma-2}$.

LEMMA 60. *Assume that* (A) *holds. A function* $z \in C[(y_1, y_2], \mathbb{R})$ *is a solution to the FBVP* (47) *if and only if it satisfies the integral equation*

$$z(t) = \int_{y_1}^{y_2} G(t,s)r(s)z(s)ds + M(t)\sum_{i=1}^{m-2}\omega_i\int_{y_1}^{y_2} G(\phi_i,s)r(s)z(s)ds, \ \ t \in [y_1, y_2],$$

where

$$M(t) = \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(y_2^\rho - y_1^\rho)^{\gamma-1} - \sum_{i=1}^{m-2}\omega_i(\phi_i^\rho - y_1^\rho)^{\gamma-1}}, \ \ t \in [y_1, y_2],$$

and the Green's function $G(t,s)$ *is given by*

$$G(t,s) = \frac{\rho^{1-\delta}s^{\rho-1}}{\Gamma(\delta)(y_2^\rho - y_1^\rho)^{\gamma-1}}\begin{cases} g_1(t,s), & y_1 \leqslant s \leqslant t \leqslant y_2, \\ g_2(t,s), & y_1 \leqslant t \leqslant s \leqslant y_2, \end{cases}$$

with

$$\begin{aligned} g_1(t,s) &= (t^\rho - y_1^\rho)^{\gamma-1}(y_2^\rho - s^\rho)^{\delta-1} - (y_2^\rho - y_1^\rho)^{\gamma-1}(t^\rho - s^\rho)^{\delta-1}, \\ g_2(t,s) &= (t^\rho - y_1^\rho)^{\gamma-1}(y_2^\rho - s^\rho)^{\delta-1}. \end{aligned}$$

LEMMA 61. *Assume that* (B) *holds. A function* $z \in C[(y_1, y_2], \mathbb{R})$ *is a solution to the FBVP* (48) *if and only if*

$$z(t) = \int_{y_1}^{y_2} z(t,s)r(s)z(s)ds + L(t)\sum_{i=1}^{m-2}\lambda_i\int_{y_1}^{y_2} Y(\eta_i,s)r(s)z(s)ds, \ \ t \in [y_1, y_2],$$

where

$$L(t) = \frac{(t^\rho - y_1^\rho)^{\gamma-1}}{(\gamma-1)\rho(y_2^\rho - y_1^\rho)^{\gamma-2} - \sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho - y_1^\rho)^{\gamma-1}}, \ \ t \in [y_1, y_2],$$

and the Green's function $Y(t,s)$ *is given by*

$$Y(t,s) = \frac{(y_2^\rho - s^\rho)^{\delta-2}\rho^{1-\delta}s^{\rho-1}}{(\gamma-1)\Gamma(\zeta)}\begin{cases} h_1(t,s), & y_1 \leqslant s \leqslant t \leqslant y_2, \\ h_2(t,s), & y_1 \leqslant t \leqslant s \leqslant y_2, \end{cases}$$

with

$$\begin{aligned} h_1(t,s) &= (\delta-1)(y_2^\rho - y_1^\rho)^{2-\gamma}(t^\rho - y_1^\rho)^{\gamma-1} - (\gamma-1)\frac{(t^\rho - s^\rho)^{\delta-1}}{(y_2^\rho - s^\rho)^{\delta-2}}, \\ h_2(t,s) &= (\delta-1)(y_2^\rho - y_1^\rho)^{2-\gamma}(t^\rho - y_1^\rho)^{\gamma-1}. \end{aligned}$$

LEMMA 62. *The Green's functions $G(t,s)$ and $Y(t,s)$ defined in Lemmas 60 and 61, respectively, satisfy the properties:*

(i) $G(t,s)$ *and* $Y(t,s)$ *are two continuous functions for any* $(t,s) \in [y_1,y_2] \times [y_1,y_2]$;

(ii) for any $(t,s) \in [y_1,y_2] \times [y_1,y_2]$,

$$|G(t,s)| \leqslant \frac{(\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}}{\Gamma(\zeta)(\delta+\gamma-2)^{\delta+\gamma-2}} \rho^{1-\delta} s^{\rho-1} (y_2^\rho - y_1^\rho)^{\delta-1};$$

(iii) for any $(t,s) \in [y_1,y_2] \times [y_1,y_2]$,

$$|Y(t,s)| \leqslant \frac{\rho^{1-\delta} s^{\rho-1} (y_2^\rho - s^\rho)^{\delta-2}}{\Gamma(\zeta)(\gamma-1)} \max\{\gamma-\delta, \delta-1\}.$$

We are now in a position to state the Lyapunov-type inequalities for the boundary value problems (47) and (48).

THEOREM 37. *Let* (A) *hold and* $r \in C([y_1,y_2],\mathbb{R})$. *If* z *is a nontrivial continuous solution to the FBVP* (47)*, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant \frac{\Gamma(\zeta)\rho^{\delta-1}(\delta+\gamma-2)^{\delta+\gamma-2}}{\Lambda_1[1+M(y_2)\sum_{i=1}^{m-2}\omega_i] \max\{y_1^{\rho-1}, y_2^{\rho-1}\}},$$

where

$$\Lambda_1 = (\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}(y_2^\rho - y_i^\rho)^{\delta-1}.$$

Theorem 37 with $\zeta = \beta$, $\gamma = \beta + \mu(2-\zeta)$ and $\delta = \zeta$ reduces to the following corollary (see [55, Theorem 4.1]).

COROLLARY 18. *Consider the following Hilfer-Katugampola fractional* m*-point boundary value problem*

$$\begin{cases} {}^\rho D_{y_1+}^{\zeta,\mu} z(t) + r(t)z(t) = 0, \quad 1<\zeta<2, \quad \rho>0, \quad y_1<t<y_2, \\ z(y_1) = 0, \quad z(y_2) = \displaystyle\sum_{i=1}^{m-2} \omega_i z(\phi_i), \end{cases} \tag{49}$$

where $r \in C([y_1,y_2],\mathbb{R})$ *and* ${}^\rho D_{y_1+}^{\zeta\mu}$ *denotes the Hilfer-Katugampola fractional derivative of order* ζ *and type* $\mu, 0 \leqslant \mu \leqslant 1$. *If the problem* (49) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant \frac{[2(\zeta-1)+\mu(2-\zeta)]^{2(\zeta-1)+\mu(2-\zeta)}\Gamma(\zeta)\rho^{\zeta-1}}{\Lambda_1[1+M(y_2)\sum_{i=1}^{m-2}\omega_i] \max\{y_1^{\rho-1}, y_2^{\rho-1}\}}.$$

Theorem 37 with $\zeta = \beta$ and $\rho \to 1$ can be expressed in the form of the following corollary ([51, Theorem 3.1]).

COROLLARY 19. *Consider the following Hilfer fractional m-point boundary value problem*

$$\begin{cases} D^{\zeta,\mu}_{y_1+}z(t)+r(t)z(t)=0, \ \ y_1<t<y_2, \ \ 1<\zeta\leqslant 2, \ \ 0\leqslant\mu\leqslant 1, \\ z(y_1)=0, \ \ z(y_2)=\displaystyle\sum_{i=1}^{m-2}\omega_i z(\phi_i), \end{cases} \tag{50}$$

where $r\in C([y_1,y_2],\mathbb{R})$ *and* $D^{\zeta,\mu}_{y_1+}$ *denotes the Hilfer fractional derivative of order* ζ *and type* μ. *If the problem* (50) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2}|r(s)|ds\geqslant\frac{\Gamma(\zeta)}{\Delta_1}\frac{1}{1+\displaystyle\sum_{i=1}^{m-2}\omega_i T(y_2)},$$

where

$$\begin{aligned}\Delta_1 &= \frac{(\zeta-1)^{\zeta-1}(\zeta-1+2\mu-\zeta\mu)^{\zeta-1+2\mu-\zeta\mu}(y_2-y_1)^{\zeta-1}}{(2\zeta-2+2\mu-\zeta\mu)^{2\zeta-2+2\mu-\zeta\mu}},\\ T(y_2) &= \frac{(y_2-y_1)^{1-(2-\zeta)(1-\mu)}}{(y_2-y_1)^{1-(2-\zeta)(1-\mu)}-\displaystyle\sum_{i=1}^{m-2}\omega_i(\phi_i-y_1)^{1-(2-\zeta)(1-\mu)}}.\end{aligned}$$

The next corollary is obtained from Theorem 37 by taking $\zeta=\beta$, $\mu=0$ and $\omega_i=0$ ([28, Theorem 5]).

COROLLARY 20. *Consider the following Katugampola fractional Dirichlet boundary value problem:*

$$\begin{cases} {}^{\rho}D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0, \ \ y_1<t<y_2, \ \ 1<\zeta\leqslant 2, \\ z(y_1)=0, \ \ z(y_2)=0, \end{cases} \tag{51}$$

where $r\in C([y_1,y_2],\mathbb{R})$ *and* ${}^{\rho}D^{\zeta}_{y_1+}$ *denotes the Katugampola fractional derivative of order* ζ. *If the problem* (51) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2}|r(s)|ds\geqslant\frac{\Gamma(\zeta)}{\max\{y_1^{\rho-1},y_2^{\rho-1}\}}\left(\frac{4\rho}{y_2^{\rho}-y_1^{\rho}}\right)^{\zeta-1}.$$

Theorem 37 with $\zeta=\beta$, $\mu=1$ and $\rho\to 0^+$ reduces to the subsequent corollary ([52, Theorem 3.7]).

COROLLARY 21. *Consider the following Caputo-Hadamard fractional m-point boundary value problem:*

$$\begin{cases} {}^{C}_{H}D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0, \ \ y_1<t<y_2, \ \ 1<\zeta<2, \\ z(y_1)=0, \ \ z(y_2)=\displaystyle\sum_{i=1}^{m-2}\omega_i z(\phi_i), \end{cases} \tag{52}$$

where $r \in C([y_1,y_2],\mathbb{R})$ *and* ${}^{C}_{H}D^{\zeta}_{y_1+}$ *denotes the Caputo-Hadamard fractional derivative of order* ζ. *If the problem* (52) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant \frac{y_1\zeta^{\zeta}\Gamma(\zeta)}{[(\zeta-1)(\ln y_2-\ln y_1)]^{\zeta-1}} \cdot \frac{\ln\frac{y_2}{y_1} - \sum_{i=1}^{m-2}\omega_i \ln\frac{\omega_i}{y_1}}{\ln\frac{y_2}{y_1} + \sum_{i=1}^{m-2}\omega_i \ln\frac{\omega_i}{y_1}}.$$

Finally, Theorem 37 with $\zeta=\beta,\ \mu=0,\ \omega_i=0$ and $\rho\to 0^+$ takes the form of the following corollary ([25, Theorem 2]).

COROLLARY 22. *Consider the following Hadamard fractional Dirichlet boundary value problem:*

$$\begin{cases} {}^{H}D^{\zeta}_{y_1+}z(t)+r(t)z(t)=0,\ \ y_1<t<y_2,\ \ 1<\zeta<2, \\ z(y_1)=0,\ \ z(y_2)=0, \end{cases} \tag{53}$$

where $r \in C([y_1,y_2],\mathbb{R})$ *and* ${}^{H}D^{\zeta}_{y_1+}$ *denotes the Hadamard fractional derivative of order* ζ. *If the problem* (53) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2} |r(s)|ds \geqslant 4^{(\zeta-1)}y_1\Gamma(\zeta)\left(\ln\frac{y_2}{y_1}\right)^{1-\zeta}.$$

THEOREM 38. *Let* (B) *hold and* $r\in C([y_1,y_2],\mathbb{R})$. *If* z *is a nontrivial continuous solution to the FBVP* (48), *then*

$$\int_{y_1}^{y_2} (y_2^{\rho}-s^{\rho})^{\delta-2}|r(s)|ds \geqslant \frac{(\gamma-1)\rho^{\delta-1}\Gamma(\zeta)}{\Lambda_2[1+L(y_2)\sum\limits_{i=1}^{m-2}\lambda_i]},$$

where

$$\Lambda_2=(y_2^{\rho}-y_1^{\rho})\max\{\gamma-\delta,\delta-1\}\max\{y_1^{\rho-1},y_2^{\rho-1}\}.$$

Theorem 38 with $\zeta=\beta$ and $\mu=0$ reduces to the following corollaries ([56, Theorem 4.2] and [56, Theorem 4.5] respectively).

COROLLARY 23. *Consider the Hilfer-Katugampola fractional* m*-point boundary value problem:*

$$\begin{cases} {}^{\rho}D^{\zeta,\mu}_{y_1+}z(t)+r(t)z(t)=0,\ \ y_1<t<y_2,\ \ 1<\zeta<2,\ \rho>0, \\ z(y_1)=0,\ \ t^{1-\rho}\dfrac{d}{dt}z(t)|_{t=y_2}=\sum\limits_{i=1}^{m-2}\lambda_i z(\eta_i), \end{cases} \tag{54}$$

where $r\in C([y_1,y_2],\mathbb{R})$ *and* ${}^{\rho}D^{\zeta,\mu}_{y_1+}$ *denotes the Hilfer-Katugampola fractional derivative of order* ζ *and type* μ. *If the problem* (54) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2} (y_2^{\rho}-s^{\rho})^{\zeta-2}|r(s)|ds \geqslant \frac{[1-(2-\zeta)(1-\mu)]\rho^{\zeta-1}\Gamma(\zeta)}{\Delta_2[1+R(y_2)\sum_{i=1}^{m-2}\lambda_i]},$$

where

$$\Delta_2 = (y_2^\rho - y_1^\rho)\max\{\mu(2-\zeta), \zeta-1\}\max\{y_1^{\rho-1}, y_2^{\rho-1}\},$$
$$R(y_2) = \frac{(y_2^\rho - y_1^\rho)^{1-(2-\zeta)(1-\mu)}}{[1-(2-\zeta)(1-\mu)]\rho(y_2^\rho - y_1^\rho)^{-(2-\zeta)(1-\mu)} - \sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho - y_2^\rho)^{[1-(2-\zeta)(1-\mu)]}}.$$

COROLLARY 24. *Consider the Katugampola fractional m-point boundary value problem:*

$$\begin{cases} {}^\rho D_{y_1+}^\zeta z(t) + r(t)z(t) = 0, \ \ y_1 < t < y_2, \ \ 1 < \zeta < 2, \ \ \rho > 0, \\ z(y_1) = 0, \ \ t^{1-\rho}\dfrac{d}{dt}z(t)|_{t=y_2} = \displaystyle\sum_{i=1}^{m-2}\lambda_i z(\eta_i), \end{cases} \tag{55}$$

where $r \in C([y_1, y_2], \mathbb{R})$ and ${}^\rho D_{y_1+}^\zeta$ denotes the Katugampola fractional derivative of order ζ. If the problem (55) *has a nontrivial continuous solution, then*

$$\int_{y_1}^{y_2}(y_2^\rho - s^\rho)^{\zeta-2}|r(s)|ds \geqslant \frac{\rho^{\zeta-1}\Gamma(\zeta)}{\Delta_3(1 + H(y_2)\sum_{i=1}^{m-2}\lambda_i)},$$

where

$$\Delta_3 = (y_2^\rho - y_1^\rho)\max\{y_1^{\rho-1}, y_2^{\rho-1}\},$$
$$H(y_2) = \frac{(y_2^\rho - y_1^\rho)^{\zeta-1}}{(\zeta-1)\rho(y_2^\rho - y_1^\rho)^{\zeta-2} - \sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho - y_1^\rho)^{\zeta-1}}.$$

15.2. Lyapunov-type inequalities for FBVP involving bi-ordinal ψ-Hilfer fractional derivative

DEFINITION 20. ([53]) Let $n-1 < \zeta, \beta \leqslant n$ with $n \in \mathbb{N}$, $I = [y_1, y_2]$ be the interval such that $-\infty \leqslant y_1 < y_2 \leqslant \infty$ and $x, \psi \in C^n([y_1, y_2], \mathbb{R})$, where ψ is an increasing function such that $\psi'(t) \neq 0$, for all $t \in I$. The bi-ordinal ψ-Hilfer fractional derivative (left-sided) ${}^H D_{y_1+}^{(\zeta,\beta)\mu,\psi}x$ of order (ζ, β) and type μ $(0 \leqslant \mu \leqslant 1)$ is defined by

$$({}^H D_{y_1+}^{(\zeta,\beta)\mu,\psi}x)(t) = (I_{y_1+}^{\mu(n-\zeta),\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{y_1+}^{(1-\mu)(n-\beta),\psi}x)(t).$$

In 2023, Wang et al. [53] considered the bi-ordinal ψ-Hilfer boundary value problems with m-point boundary conditions:

$$\begin{cases} ({}^H D_{y_1+}^{(\zeta,\beta)\mu,\psi}z)(t) + r(t)z(t) = 0, \quad y_1 < t < y_2, \\ z(y_1) = 0, \quad z(y_2) = \displaystyle\sum_{i=1}^{m-2}\sigma_i z(\eta_i), \end{cases} \tag{56}$$

and

$$\begin{cases} ({}^{H}D_{y_1+}^{(\zeta,\beta)\mu,\psi}z)(t)+r(t)z(t)=0, & y_1<t<y_2, \\ z(y_1)=0, \quad \dfrac{1}{\psi'(t)}\dfrac{d}{dt}z(t)|_{t=y_2}=\displaystyle\sum_{i=1}^{m-2}\lambda_i z(\xi_i), \end{cases} \tag{57}$$

where $r\in C([y_1,y_2],\mathbb{R})$, $\psi\in C^2([y_1,y_2],\mathbb{R})$, $\psi'(t)>0$, ${}^{H}D_{y_1+}^{(\zeta,\beta)\mu,\psi}$ is bi-ordinal ψ-Hilfer fractional derivative operator of order $(\zeta,\ \beta)$, $1<\zeta<2$, $0\leqslant\beta\leqslant 1$ and type μ, $0\leqslant\mu\leqslant 1$, $\sigma_i,\lambda_i\geqslant 0$, $y_1<\eta_i,\xi_i<y_2$, $i=1,2,\ldots,m-2$ for $y_1<\eta_1<\eta_2<\ldots<\eta_{m-2}<y_2$, $y_1<\xi_1<\xi_2<\xi_{m-2}<y_2$.

LEMMA 63. *Assume that:*

$$(C_1)\quad (\psi(y_2)-\psi(y_1))^{\omega-1}>\sum_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(y_1))^{\omega-1}.$$

The function $z\in C([y_1,y_2],\mathbb{R})$ is the solution of the FBVP (56) if and only if it satisfies the integral equation

$$z(t)=\int_{y_1}^{y_2}H(t,s)\psi'(s)r(s)z(s)ds+R(t)\sum_{i=1}^{m-2}\sigma_i\int_{y_1}^{y_2}H(\eta_i,s)\psi'(s)r(s)z(s)ds, \tag{58}$$

where

$$R(t)=\frac{(\psi(t)-\psi(y_1))^{\omega-1}}{(\psi(y_2)-\psi(y_1))^{\omega-1}-\displaystyle\sum_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(y_1))^{\omega-1}},$$

and the Green's function $H(t,s)$ is given by

$$H(t,s)=\frac{1}{\Gamma(\delta)}(\psi(y_2)-\psi(y_1))^{\omega-1}\begin{cases}(\psi(t)-\psi(y_1))^{\omega-1}(\psi(y_2)-\psi(y_1))^{\delta-1}\\ -(\psi(y_2)-\psi(y_1))^{\omega-1}(\psi(t)-\psi(y_1))^{\delta-1}, \\ \qquad y_1\leqslant s\leqslant t\leqslant y_2, \\ (\psi(t)-\psi(y_1))^{\omega-1}(\psi(y_2)-\psi(y_1))^{\delta-1}, \\ \qquad y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

with $\delta=\beta+\mu(\zeta-\beta)$.

LEMMA 64. *Assume that:*

$$(C_2)\quad (\omega-1)(\psi(y_2)-\psi(y_1))^{\omega-2}>\sum_{i=1}^{m-2}\lambda_i(\psi(\xi_i)-\psi(y_1))^{\omega-1}.$$

The function $z\in C([y_1,y_2],\mathbb{R})$ is the solution of the FBVP (57) if and only if it satisfies the integral equation

$$z(t)=\int_{y_1}^{y_2}G(t,s)\psi'(s)r(s)z(s)ds+Q(t)\sum_{i=1}^{m-2}\lambda_i\int_{y_1}^{y_2}G(\xi_i,s)\psi'(s)r(s)z(s)ds, \tag{59}$$

where

$$Q(t)=\frac{(\psi(t)-\psi(y_1))^{\omega-1}}{(\omega-1)(\psi(y_2)-\psi(y_1))^{\omega-2}-\sum_{i=1}^{m-2}\lambda_i(\psi(\xi_i)-\psi(y_1))^{\omega-1}},$$

and the Green's function $G(t,s)$ *is given by*

$$G(t,s)=\frac{\psi(y_2)-\psi(y_1))^{\delta-2}}{\Gamma(\delta)(\omega-1)}\begin{cases}(\delta-1)(\psi(y_2)-\psi(y_1))^{2-\omega}(\psi(t)-\psi(y_1))^{\omega-1}\\ -(\omega-1)\dfrac{(\psi(t)-\psi(s))^{\delta-1}}{(\psi(y_2)-\psi(s))^{\delta-2}}, \\ \qquad y_1\leqslant s\leqslant t\leqslant y_2,\\ (\delta-1)(\psi(y_2)-\psi(y_1))^{2-\omega}(\psi(t)-\psi(y_1))^{\omega-1},\\ \qquad y_1\leqslant t\leqslant s\leqslant y_2,\end{cases}$$

with $\delta=\beta+\mu(\zeta-\beta)$.

LEMMA 65. *The Green's functions* $H(t,s)$ *and* $G(t,s)$ *defined in Lemmas 63 and 64 respectively, satisfy the properties:*

(i) $H(t,s)$ *and* $G(t,s)$ *are continuous functions in* $[y_1,y_2]\times[y_1,y_2]$;

(ii) For any $(t,s)\in[y_1,y_2]\times[y_1,y_2]$, *we have*

$$|H(t,s)|\leqslant\frac{(\delta-1)^{\delta-1}(\omega-1)^{\omega-1}(\psi(y_2)-\psi(y_1))^{\delta-1}}{\Gamma(\delta)(\omega+\delta-2)^{\omega+\delta-2}};$$

(iii) For any $(t,s)\in[y_1,y_2]\times[y_1,y_2]$, *we have*

$$|G(t,s)|\leqslant\frac{(\psi(y_2)-\psi(s))^{\delta-2}(\psi(y_2)-\psi(y_1))}{(\omega-1)\Gamma(\delta)}\max\{\omega-\delta,\delta-1\}.$$

Now, we present the Lyapunov-type inequalities for problems (56) and (57).

THEOREM 39. *If the FBVP (56) has a nontrivial solution in* $C([y_1,y_2],\mathbb{R})$ *and* $r\in C([y_1,y_2],\mathbb{R})$ *is a real and continuous function, then*

$$\int_{y_1}^{y_2}\psi'(s)|r(s)|ds\geqslant\frac{\Gamma(\delta)(\delta+\omega-2)^{\delta+\omega-2}}{\left[1+R(y_2)\sum_{i=1}^{m-2}\sigma_i\right](\delta-1)^{\delta-1}(\omega-1)^{\omega-1}(\psi(y_2)-\psi(y_1))^{\delta-1}}.$$

THEOREM 40. *If the FBVP (57) has a nontrivial solution in* $C([y_1,y_2],\mathbb{R})$ *and* $r\in C([y_1,y_2],\mathbb{R})$ *is a real and continuous function, then*

$$\int_{y_1}^{y_2}(\psi(y_2)-\psi(s))^{\delta-2}\psi'(s)|r(s)|ds$$
$$\geqslant\frac{(\omega-1)\Gamma(\delta)}{(\psi(y_2)-\psi(y_1))\max\{\omega-\delta,\delta-1\}\left[1+Q(y_2)\sum_{i=1}^{m-2}\lambda_i\right]}.$$

16. Lyapunov-type inequalities for discrete FBVP

In this section, we present Lyapunov-type inequalities for a discrete FBVP. We start this section with some basic definitions. The falling factorial function is defined as

$$t^{\underline{\zeta}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\zeta)},$$

for any t and ζ for which the right hand side is defined. Conventionally, if $t+1-\zeta$ is pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\zeta}} = 0$.

DEFINITION 21. [11] The ζ-th order fractional sum of a function g defined on $\mathbb{N}_{y_1} := \{y_1, y_1+1, y_1+2, \ldots\}$, $y_1 \in \mathbb{R}$ and for $\zeta > 0$, is defined as

$$\Delta_{y_1}^{-\zeta} g(t) = \frac{1}{\Gamma(\zeta)} \sum_{s=y_1}^{t-\zeta} (t-s-1)^{\underline{\zeta-1}} g(s), \quad t \in \mathbb{N}_{y_1+\zeta}.$$

In [1], the authors studied the discrete FBVP of fractional difference equations:

$$\begin{cases} -\Delta^{\zeta} z(t) = \lambda h(t+\zeta-1) g(z(t+\zeta-1)), & t \in [0, y_2]_{\mathbb{N}_0}, \\ z(\zeta-2) = 0, \quad \Delta z(\zeta-2) = \Delta z(\beta+y_2-1), \end{cases} \tag{60}$$

where $g : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing, $h : [\zeta-1, \beta+n]_{\mathbb{N}_0} \to [0,\infty)$, $1 < \zeta \leqslant 2$, and λ is a positive parameter.

LEMMA 66. *The function z is a solution to the FBVP (60) if and only if*

$$z(t) = -\frac{\lambda}{\Gamma(\zeta)} \sum_{s=0}^{y_2} G(t,s) h(s+\zeta-1) g(z(s+\zeta-1)),$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\underline{\zeta-1}} (y_1+y_2-s-2)^{\underline{\zeta-2}}}{\Gamma(\zeta-1) - (\beta+y_2-1)^{\underline{\zeta-2}}} + (t-s-1)^{\underline{\zeta-1}}, & 0 \leqslant s < t-\zeta \leqslant y_2, \\ \dfrac{t^{\underline{\zeta-1}} (y_1+y_2-s-2)^{\underline{\zeta-2}}}{\Gamma(\zeta-1) - (\beta+y_2-1)^{\underline{\zeta-2}}}, & 0 \leqslant t-\zeta < s \leqslant y_2. \end{cases}$$

LEMMA 67. *The Green's function $G(t,s)$ given in Lemma 66 satisfies the properties:*

(1) $G(t,s) > 0$ *for all* $t \in [\zeta-2, \beta+b]_{\mathbb{N}_{\zeta-2}}$ *and* $s \in [0, y_2]_{\mathbb{N}_0}$;

(2) $\max_{t \in [\zeta-2, \beta+y_2]_{\mathbb{N}_{\zeta-2}}} G(t,s) = G(s+\zeta-2, s), \quad s \in [0, y_2]_{\mathbb{N}_0}$;

(3) The function $G(s+\zeta-2, s)$ *has a unique maximum given by*

$$\max_{s \in [0,y_2]_{\mathbb{N}_0}} G(s+\zeta-2, s) = \frac{\Gamma(y_2+\zeta-1)\Gamma(\zeta-1)\Gamma(y_2+2)}{\Gamma(y_2)[\Gamma(\zeta-1)\Gamma(y_2+2) - \Gamma(\zeta+y_2)]}.$$

The Lyapunov-type inequality for the discrete FBVP (60) is given in the following result.

THEOREM 41. *Let $h : [\zeta-1, \beta+y_2]_{\mathbb{N}_{\zeta-1}} \to [0,\infty)$ be a nontrivial function. Assume that $g \in C(\mathbb{R}^+,\mathbb{R}^+)$ is a nondecreasing function. If the discrete FBVP (60) has a nontrivial solution, then*

$$\sum_{s=0}^{y_2} |h(s+\zeta-1)| \geqslant \frac{\Gamma(\zeta)\Gamma(y_2)[\Gamma(\zeta-1)\Gamma(y_2+2)-\Gamma(\zeta+y_2)]\eta}{\Gamma(y_2+\zeta-1)\Gamma(\zeta-1)\Gamma(y_2+2)g(\eta)},$$

where $\eta = \max_{[\zeta-1,\beta+y_2]_{\mathbb{N}_{\zeta-1}}} z(s+\zeta-1)$.

In [35], Oguz et al. considered the higher-order discrete FBVP

$$\begin{cases} \Delta^{\zeta}_{\zeta-n_0} z(t) + r(t)z(t+\zeta-1), & t \in \mathbb{N}_0^{y_2+m_0}, \\ \Delta^i z(\zeta-n_0) = 0, & i \in \mathbb{N}_0^{n_0-2}, \\ \Delta^{\beta}_{\zeta-n_0} z(y_2+m_0+\zeta-\beta) = 0, \end{cases} \tag{61}$$

where r is a real valued continuous function defined on $\mathbb{N}_0^{y_2+m_0}$, $y_2, n_0, m_0 \in \mathbb{N}$, $n_0 - 1 < \zeta \leqslant n_0$, $m_0 - 1 < \beta \leqslant m_0$ and $1 \leqslant \beta < \zeta$ when $\zeta \geqslant 2$ and $\beta = 1$ when $1 < \zeta < 2$.

LEMMA 68. *If g is defined on $\mathbb{N}_0^{y_2+m_0}$, then the solution of the discrete FBVP*

$$\begin{cases} -\Delta^{\zeta}_{\zeta-n_0} z(t) = g(t), & t \in \mathbb{N}_0^{y_2+m_0}, \\ \Delta^i z(\zeta-n_0) = 0, & i \in \mathbb{N}_0^{n_0-2}, \\ \Delta^{\beta}_{\zeta-n_0} z(y_2+m_0+\zeta-\beta) = 0, \end{cases}$$

can be represented by

$$z(t) = \sum_{s=0}^{y_2+m_0} G(t,s)g(s), \quad t \in \mathbb{N}_{\zeta-n_0}^{y_2+m_0+\zeta},$$

where

$$G(t,s) = \frac{1}{\Gamma(\zeta)} \begin{cases} \dfrac{[y_2+m_0+\zeta-\beta-s-1]^{\underline{\zeta-\beta-1}}}{(y_2+m_0+\zeta-\beta)^{\underline{\zeta-\beta-1}}} t^{\underline{\zeta-1}}, \\ \qquad\qquad 0 \leqslant t-\beta+1 \leqslant s \leqslant y_2+m_0, \\ \dfrac{[y_2+m_0+\zeta-\beta-s-1]^{\underline{\zeta-\beta-1}}}{(y_2+m_0+\zeta-\beta)^{\underline{\zeta-\beta-1}}} t^{\underline{\zeta-1}} - [t-s-1]^{\underline{\zeta-1}}, \\ \qquad\qquad 0 \leqslant s \leqslant t-\zeta \leqslant y_2+m_0, \\ 0, \qquad \zeta-n_0 \leqslant t \leqslant \zeta-2, 0 \leqslant s \leqslant y_2+m_0. \end{cases}$$

LEMMA 69. *The Green's function $G(t,s)$ given in Lemma 68 satisfies the inequality:*

$$G(t,s) \leqslant \frac{1}{\Gamma(\zeta)}[y_2+m_0+\zeta-\beta-s-1]^{\underline{\zeta-\beta-1}}\Big\{(y_2+m_0+\zeta)^{\beta}-[y_2+m_0+\zeta-s-s]^{\underline{\beta}}\Big\},$$

for $(t,s) \in \mathbb{N}_{\zeta-n_0}^{y_2+m_0+\zeta} \times \mathbb{N}_0^{y_2+m_0}$.

Now, we state the Lyapunov-type inequality for the discrete FBVP (61).

THEOREM 42. *Let z be a nontrivial solution to the discrete FBVP (61). If $z(t+\zeta-1) \neq 0$ for $t \in \mathbb{N}_0^{y_2+m_0}$, then the following inequality holds:*

$$\sum_{s=0}^{y_2+m_0}|r(s)| \geqslant \frac{1}{M_0}\Gamma(\zeta-\beta)\Big[\frac{(y_2+m_0+\zeta)^{\underline{y_2+m_0-1}}}{(y_2+m_0+\zeta-\beta)^{\underline{y_2+m_0+1}}}-1\Big]^{-1},$$

where

$$M_0 = \max\Big\{\Gamma(\zeta-\beta), \frac{\Gamma(y_2+m_0+\zeta-\beta)}{\Gamma(y_2+m_0+1)}\Big\}.$$

COROLLARY 25. *Let $\zeta = n_0 \in \mathbb{N}_2$ and $\beta = 1$. If z is a nontrivial solution to the discrete FBVP (61) and $z(t+n_0-1) \neq 0$ for $t \in \mathbb{N}_0^{y_2+1}$, then the following inequality holds:*

$$\sum_{s=0}^{y_2+1}|r(s)| \geqslant \frac{(n_0-1)!}{y_2+2}\prod_{p=2}^{n_0-1}(p+y_2)^{-1}.$$

In 2023, Vianny et al. [49] discussed the discrete fractional order boundary value problem:

$$\begin{cases} {}^{RL}\Delta^{\zeta}z(t)+r(t+\zeta-1)z(t+\zeta-1)=0, & t\in\mathbb{N}_0^{\ell}, \\ z(\zeta-3)=0, \ \ \Delta z(\zeta-3)=0, \ \ z(\beta+\ell)=0, \end{cases} \tag{62}$$

where $r : \mathbb{N}_{\zeta-2}^{\beta+\ell-2} \to [0,\infty)$, ${}^{RL}\Delta^{\zeta}$ is the Riemann-Liouville fractional differential operator of order $\zeta \in (2,3]$ and $\ell \in \mathbb{N}_2$. An integral inequality of Lyapunov-type is obtained for the problem (62).

LEMMA 70. *Let $2 < \zeta \leqslant 3$. Then the discrete FBVP (70) has a unique solution*

$$z(t) = \frac{1}{\Gamma(\zeta)}\sum_{s=0}^{\ell}G(t,s)r(s+\zeta-1)z(s+\zeta-1),$$

where $G(t,s) : \mathbb{N}_{\zeta-3}^{\beta+\ell} \times \mathbb{N}_0^{\ell} \to \mathbb{R}$ is defined by

$$G(t,s) = \begin{cases} \dfrac{(\beta+\ell-s-1)^{\underline{\zeta-1}}}{(\beta+\ell)^{\underline{\zeta-1}}}t^{\underline{\zeta-1}}-(t-s-1)^{\underline{\zeta-1}}, & 0\leqslant s<t-\beta+1\leqslant \ell, \\ \dfrac{(\beta+\ell-s-1)^{\underline{\zeta-1}}}{(\beta+\ell)^{\underline{\zeta-1}}}t^{\underline{\zeta-1}}, & 0\leqslant t-\beta+1\leqslant s\leqslant \ell. \end{cases}$$

LEMMA 71. *The Green's function $G(t,s)$ given in Lemma 70 satisfies the properties:*

(1) $G(t,s) > 0$ *for all* $t \in \mathbb{N}_{\zeta-3}^{\beta+\ell}$ *and* $s \in \mathbb{N}_0^{\ell}$;

(2) $\max_{t\in\mathbb{N}_{\zeta-3}^{\beta+\ell}} G(t,s) = G(s+\zeta-1,s),\ \ s \in \mathbb{N}_0^{\ell}$;

(3) The function $G(s+\zeta-1,s)$ *has a unique maximum given by*

$$\max_{s\in\mathbb{N}_0^{\ell}} G(s+\zeta-1,s) = \frac{\Gamma(\zeta)\Gamma(\beta+\ell)}{\Gamma(\ell+1)\Gamma(\beta+\ell)^{\underline{\zeta-1}}}.$$

Now, we present the Lyapunov-type inequality for the discrete FBVP (62).

THEOREM 43. *Let* $r : \mathbb{N}_{\zeta-2}^{\beta+\ell-2} \to [0,\infty)$ *be a nonzero function. If the discrete FBVP* (62) *has a nontrivial solution, then*

$$\sum_{s=0}^{\ell} |r(s+\zeta-1)| \geqslant \frac{\Gamma(\ell+1)}{\Gamma(\beta+\ell)}(\beta+\ell)^{\underline{\zeta-1}}.$$

17. Conclusions

In this paper, we presented an extensive review of the most recent results on Lyapunov-type inequalities for FBVP involving a variety of fractional derivative operators and boundary conditions. Fractional derivative operators include Riemann–Liouville, Caputo, mixed Riemann–Liouville and Caputo, Riesz-Caputo, ψ-Caputo, Hadamard, Katugampola, Hilfer, ψ-Hilfer, proportional, variable order Hadamard, partial, systems of Riemann-Liouville, bi-ordinal Hilfer-Katugampola and ψ-Hilfer. In each section/subsection of the present paper, we first described the fractional integral operators, related to the results collected for Lyapunov-type fractional integral inequalities. We provided comprehensive details (without proof), for the convenience of the reader, of all Lyapunov-type inequalities presented in this survey. This survey paper together with the published survey papers [31]–[33] serves as an excellent platform for the researchers who wish to initiate/develop new work on such inequalities.

REFERENCES

[1] N. G. ABUJ AND D. B. PACHPATTE, *Lyapunov type inequalities for discrete fractional boundary value problems*, South East Asian J. Math. Math. Sci. **2023**, 19, 63–72.

[2] R. P. AGARWAL, M. BOHNER AND A. OZBEKLER, *Lyapunov Inequalities and Applications*, Springer, Cham, 2021.

[3] B. AHMAD, A. ALSAEDI, S. K. NTOUYAS AND J. TARIBOON, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, Switzerland, 2017.

[4] R. A. ALMEIDA, *Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul. **2017**, 44, 460–481.

[5] P. BENNER, T. DAMM, C. RODRIGUEZ AND R. YOLANDA, *Dual pairs of generalized Lyapunov inequalities and balanced truncation of stochastic linear systems*, IEEE Trans. Automat. Control **2017**, 62, 782–791.

[6] M. CAPPELLI, (Ed.) *Instrumentation and Control Systems for Nuclear Power Plants*, (Enhanced Edition), Elsevier, 2023.

[7] G. CHEN, J. NI, H. DONG AND W. ZHANG, *Lyapunov-type inequalities for fractional multi-point boundary value problems using a new generalized fractional derivative*, J. Nonlinear Func. Anal. **2023** 2023, Article ID 28, pp. 1–16.

[8] S. DHAR AND J. T. NEUGEBAUER, *Lyapunov-type inequalities for a fractional boundary value problem with fractional boundary condition*, Nonlinear Dyn. Syst. Theory **2022**, 22, 133–143.

[9] N. M. DIEN AND J. J. NIETO, *Lyapunov-type inequalities for a nonlinear sequential fractional FBVP in the frame of generalized Hilfer derivatives*, Math. Inequal. Appl. **2022**, 25, 851–867.

[10] P. ELOE AND B. K. M. BODDU, *Lyapunov-type inequalities for (m,p)-type nonlinear fractional boundary value problem*, Turk. J. Math. **2023**, 47, 816–829.

[11] C. GOODRICH AND A. PETERSON, *Discrete Fractional Calculus*, Springer International Publishing, Switzerland, 2015.

[12] J. R. GRAEF, K. MAAZOUZ AND M. D. A. ZAAK, *A generalized Lyapunov inequality for a pantograph FBVP involving a variable order Hadamard fractional derivative*, Mathematics **2023**, 11, 2984.

[13] R. HILFER, Y. LUCHKO AND Z. TOMOVSKI, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. **2009**, 12, 299–318.

[14] Y. HOSOE AND T. HAGIWARA, *Equivalent stability notions, Lyapunov inequality, and its application in discrete-time linear systems with stochastic dynamics determined by an i.i.d. process*, IEEE Trans. Autom. Control. **2019**, 64, 4764–4771.

[15] F. JARAD, T. ABDELJAWAD AND J. ALZABUT, *Generalized fractional derivatives generated by a class of local proportional derivatives*, Eur. Phys. J. Special Topics **2017**, 226, 3457–3471.

[16] J. M. JONNALAGADDA, *A short note on Lyapunov-type inequalities for Hilfer fractional boundary value problems*, Appl. Math. E-Notes, 2 **24** (2024), 35–45.

[17] J. M. JONNALAGADDA, D. BASUA AND D. K. SATPATHI, *Lyapunov type inequality for an anti-periodic conformable boundary value problem*, Kragujevac J. Math. **2021**, 45, 289–298.

[18] R. M. JUNGERS, A. A. AHMADI, P. A. PARRILO AND M. ROOZBEHANI, *A characterization of Lyapunov inequalities for stability of switched systems*, IEEE Trans. Automat. Control **2017**, 62, 3062–3067.

[19] E. KARIMOV AND B. TOSHTEMIROV, *Non-local boundary value problem for a mixed-type equation involving the bi-ordinal Hilfer fractional differential operators*, Uzbek. Math. J. **2021**, 65, 61–67.

[20] A. KASSYMOV, M. RUZHANSKY AND B. T. TOREBEK, *Rayleigh-Faber-Krahn, Lyapunov and Hartmann-Wintner inequalities for fractional elliptic problems*, Mediterr. J. Math. **2023**, 20, 119.

[21] U. N. KATUGAMPOLA, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. App. **2014**, 6, 1–15.

[22] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of the Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier: Amsterdam, The Netherlands, 2006, vol. 204.

[23] Z. LAADJAL, T. ABDELJAWAD AND F. JARAD, *Some results for two classes of two-point local fractional proportional boundary value problems*, Filomat **2023**, 37, 7199–7216.

[24] Z. LAADJAL AND Q. MA, *Lyapunov-type inequalities for fractional Langevin differential equations*, J. Math. Inequal. **2023**, 17, 67–82.

[25] Z. LAADJAL, N. ADJEROUD AND Q. H. MA, *Lyapunov-type inequality for the Hadamard fractional boundary value problem on a general interval $[a,b]$*, J. Math. Inequal. **2019**, **13**, 789–799.

[26] Y. LIU AND Q. LI, *Existence and uniqueness of solutions and Lyapunov-type inequality for a mixed fractional boundary value problem*, J. Nonlinear Model. Anal. **2022**, 4, 207–219.

[27] L. Q. LONG, *A Lyapunov-type inequality for a fractional differential equation under multi-point boundary conditions*, Thu Dau Mot University Journal of Science **2022**, 4, 135–141.

[28] B. LUPINSKA AND T. ODZIJEWICZ, *A Lyapunov-type inequality with the Katugampola fractional derivative*, Math. Method. Appl. Sci. **2018**, 41, 8985–8996.

[29] B. LUPINSKA, *Existence and nonexistence results for fractional mixed boundary value problems via a Lyapunov-type inequality*, Period. Math. Hungar. **88** (2024), 118–126.

[30] A. LYAPUNOV, *Probleme general de la stabilite du mouvement*, Ann. Fac. Sci. Univ. Toulouse 2, **1907**, 203–407.

[31] S. K. NTOUYAS, B. AHMAD AND T. P. HORIKIS, *Recent developments of Lyapunov-type inequalities for fractional differential equations*, In Differential and Integral Inequalities; Andrica, D., Rassias, T. M., Eds.; Springer Optimization and Its Applications; Springer: Berlin/Heidelberg, Germany, 2019, pp. 619–686.

[32] S. K. NTOUYAS AND B. AHMAD, *Lyapunov-type inequalities for fractional differential equations: A survey*, Surv. Math. Appl. **2021**, 16, 43–93.

[33] S. K. NTOUYAS, B. AHMAD AND J. TARIBOON, *A survey on recent results on Lyapunov-type inequalities for fractional differential equations*, Fractal Fract. **2022**, 6, 273.

[34] T. ODZIJEWICZ, *Lyapunov inequalities for two dimensional fractional boundary-value problems with mixed fractional derivatives*, Axioms **2023**, 12, 301.

[35] A. D. OGUZ, J. ALZABUT, A. ÖZBEKLER AND J. M. JONNALAGADDA, *Lyapunov and Hartman-type inequalities for higher-order discrete fractional boundary value problems*, Miskolc Math. Notes **2023**, 24, 953–963.

[36] D. S. OLIVEIRA AND E. C. DE OLIVEIRA, *Hilfer-Katugampola fractional derivatives*, Comput. Appl. Math. **2018**, 37, 3672–3690.

[37] J. P. PINASCO, *Lyapunov-type inequalities. With Applications to Eigenvalue Problems*, Springer Briefs in Mathematics, Springer, New York, NY, 2013.

[38] I. PODLUBNY, *Fractional Differential Equations*, Academic Press: San Diego, CA, USA, 1999.

[39] L. RAGOUB, J. F. GÓMEZ-AGUILAR, E. PŔEZ-CARETA AND D. BALEANU, *On a class of Lyapunov's inequality involving λ-Hilfer Hadamard fractional derivative*, AIMS Math. **2024**, 9, 4907–4924.

[40] S. REZAPOUR, S. ETEMAD, R. P. AGARWAL AND K. NONLAOPON, *On a Lyapunov-type inequality for control of a ψ-model thermostat and the existence of its solutions*, Mathematics **2022**, 10, 4023.

[41] S. REZAPOUR, Z. BOUAZZA, M. S. SOUID, S. ETEMAD AND M. K. A. KAABAR, *Darbo fixed point criterion on solutions of a Hadamard nonlinear variable order problem and Ulam-Hyers-Rassias stability*, J. Funct. Spaces **2022**, 2022, 1769359.

[42] A. S. SILVA, *A Lyapunov type inequality for a class of fractional boundary value problems with Riemann-Liouville derivative*, Krag. J. Math. **2026**, 50, 347–356.

[43] J. V. SOUSA AND E. C. DE OLIVEIRA, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simulat. **2018**, 60, 72–91.

[44] S. N. SRIVASTAVA, S. PATI, S. PADHI AND A. DOMOSHNITSKY, *Lyapunov inequality for a Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions*, Math. Methods Appl. Sci.

[45] S. T. M. THABET AND I. KEDIM, *An investigation of a new Lyapunov-type inequality for Katugampola-Hilfer fractional BVP with nonlocal and integral boundary conditions*, J. Ineq. Appl. **2023**, 2023, 162.

[46] S. TOPRAKSEVEN, *Hartman-Wintner and Lyapunov-type inequalities for high order fractional boundary value problem*, Filomat **2020**, 34, 2273–2281.

[47] S. TOPRAKSEVEN, R. BENGI AND Y. ZEREN, *Lyapunov-type inequality for an anti-periodic fractional boundary value problem of the Riesz-Caputo derivative*, Sigma J Eng Nat Sci. **2023**, 41, 807–811.

[48] S. TOPRAKSEVEN, *Lyapunov-type inequality for a class of higher-order fractional boundary value problems*, J. Math. Inequal. **2023**, 17, 435–445.

[49] A. A. VIANNY, R. DHINESHBABU AND A. G. M. SELVAN, *Lyapunov type inequalities and their applications on an eigenvalue problem for discrete fractional order equation with a class of boundary conditions*, Adv. Differ. Equ. Control Processes **2022**, 28, 55–71.

[50] Y. WANG AND Q. WANG, *Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions*, J. Ineq. Appl. **2019**, 13, 611–619.

[51] Y. WANG AND Q. WANG, *Lyapunov-type inequalities for nonlinear fractional differential equation with Hilfer fractional derivative under multi-point boundary conditions*, Frac. Calc. Appl. Anal. **2018**, 21, 833–843.

[52] Y. WANG, Y. H. WU AND Z. CAO, *Lyapunov-type inequalities for differential equation with Caputo-Hadamard fractional derivative under multi-point boundary conditions*, J. Inequal. Appl. **2021**, 2021, 1–12.

[53] L. WANG, H. DONG AND G. CHEN, *New Lyapunov-type inequalities for fractional differential equations with bi-ordinal psi-Hilfer fractional derivative involving multi-point boundary conditions*, Engineering Letters **2023**, 31, 648–655.

[54] W. ZHANG, J. ZHANG AND J. NI, *Lyapunov-type inequalities for fractional Langevin-type equations involving Caputo-Hadamard fractional derivative*, J. Inequal. App. **2022**, 48.

[55] W. ZHANG, J. ZHANG AND J. NI, *New Lyapunov-type inequalities for fractional multi-point boundary value problems involving Hilfer-Katugampola fractional derivative*, AIMS Math. **2022**, 7, 1074–1094.

[56] W. ZHANG, J. ZHANG AND J. NI, *New Lyapunov-type inequalities for fractional multi-point boundary value problems involving Hilfer-Katugampola fractional derivative*, AIMS Math. **2022**, 7, 1074–1094.

[57] Y. ZOU AND Y. CUI, *Lyapunov-type inequalities for systems of Riemann-Liouville fractional differential equations with multi-point coupled boundary conditions*, Fractal Fract. **2023**, 7, 454.

(Received May 14, 2024)

Sotiris K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr

Bashir Ahmad
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group
Department of Mathematics, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com

Jessada Tariboon
Intelligent and Nonlinear Dynamic Innovations
Department of Mathematics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok
Bangkok 10800, Thailand
e-mail: jessada.t@sci.kmutnb.ac.th