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DIRECT AND INVERSE PROBLEMS FOR A FRACTIONAL PARABOLIC EQUATION WITH MULTIPLE INVOLUTION

BATIRKHAN TURMETOV*, MAIRA KOSHANOVA,
MOLDIR MURATBEKOVA AND ISABEK ORAZOV

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Abstract. In this paper, the concept of a nonlocal analogue of the Laplace operator is introduced. For the nonlocal parabolic equation with a fractional derivative in a cylindrical domain, the solvability of direct and inverse problems is studied. The problems are solved using the Fourier method. Theorems on the existence and uniqueness of solutions to the studied problems are proved.

1. Introduction

Inverse problems in the theory of equations of mathematical physics are the problems in which, along with the solution of the equation, it is required to find the right side or coefficients of the equation, initial or boundary functions. Applications of inverse problems in modern science are described in detail in [12, 13]. Various versions of inverse problems for the classical equations of mathematical physics have been studied in the works of numerous authors [5, 24, 28].

Recently, the attention of researchers has been turned to the study of direct and inverse problems for differential equations with involution [1, 2, 3, 4, 15, 21, 22, 26]. Note that in these papers, the case of one spatial variable was mainly studied. Inverse problems in the case of two spatial variables are studied in [9, 15, 20]. It should be noted that in these papers, the problems under consideration were studied for classical equations, i.e., for equations without involutive transformations.

In the case of many spatial variables, we can note the works [6, 7, 18, 19, 27], where inverse problems on finding the right side depending on the spatial and temporal variables were studied. In this paper, differential equations with involutively transformed arguments are considered in the multidimensional case. For such equations, direct problems with initial-boundary conditions and inverse problems of finding the right side depending on the spatial variable are studied.

Let us consider the problem statement. Let $Q = \Omega \times (0,T)$, Ω is a unit ball in R^n, $n \geqslant 2$, $\partial\Omega$ is a unit sphere. Let $S_1, S_2, \dots, S_l$, $l \leqslant n$, also be a set of real

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* Corresponding author.

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symmetric commutative matrices $S_iS_j = S_jS_i$ such that $S_j^2 = I$. As an example, consider the mapping matrix $S_1x = (-x_1, x_2, \dots, x_n)$. If we consider all possible products of mappings S_j, $1 \leqslant j \leqslant l$, then the total number of such mappings, taking into account the mapping $S_0x = x$, will be 2^l. To enumerate these mappings, we will use the notation of an integer in the binary system. If i is the index, then in addition to the usual notation, we will also use the notation of this number in the binary system $i = (i_l \dots i_1)_2$, where $i_k = 0$ or $i_k = 1$. Then we can consider mappings of the form $S_l^{i_l} \dots S_1^{i_1} x$. Let us introduce the operator

$$Lv(x) = \sum_{i=0}^{2^l-1} a_i \Delta v \left(S_l^{i_l} \dots S_1^{i_1} x \right).$$

Let $0 < \alpha \leqslant 1$ and let us denote the derivative of order α in the Riemann-Liouville sense as $D^\alpha u(t)$, i.e.

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} u(\tau) d\tau, \ \ 0 < \alpha < 1$$

and

$$\partial^\alpha u(t) = D^\alpha \left[u(t) - u(0) \right], \ \ 0 < \alpha < 1, \ \ \partial^\alpha u(t) = \frac{du(t)}{dt}$$

derivative of order α in Caputo's sense.

In the domain Q we can consider the equation

$$\partial_t^\alpha u(t,x) = L_x u(t,x) + F(t,x), \ \ (t,x) \in Q, \tag{1}$$

with the initial

$$u(0,x) = \varphi(x), \ \ x \in \bar{\Omega}, \tag{2}$$

and boundary condition

$$u(t,x) = 0, 0 \leqslant t \leqslant T, \ \ x \in \partial\Omega. \tag{3}$$

Here $F(t,x), \varphi(x)$ are prescribed functions.

A solution to problem (1)–(3) is a function $u(t,x)$ continuous in a closed domain $\bar{Q}$, having derivatives of all orders in equation (1), which are continuous in Q, and satisfying conditions (1)–(3) in the classical sense.

Let $F(x,t) = f(x)g(t)$. Along with problem (1)–(3), we will also study the inverse problem of determining the right-hand side of equation (1).

Inverse problem. Find a pair of functions $\{u(t,x), f(x)\}$ that satisfies conditions (1)–(3) and the additional condition

$$u(t_0,x) = \psi(x), \ \ 0 < t_0 \leqslant T, \ \ x \in \partial\Omega, \tag{4}$$

where $g(t)$ and $\psi(x)$ are prescribed functions. In case of inverse problem, we will seek for a solution in the class of functions:

$$f(x) \in C\left(\bar{\Omega}\right), \ \ u(t,x) \in C\left(\bar{Q}\right), \ \ \partial^\alpha u(t,x) \in C\left(\bar{Q}\right), \ \ L_x u(t,x) \in C\left(\bar{Q}\right).$$

It should be noted that the problems considered in this paper were studied for the case $n=2$ of an integer and fractional parabolic equation in a rectangular domain in [10, 25] and for an equation with involution in [29]. We especially note the work [8], where similar problems were studied in an arbitrary bounded domain for the subdiffusion equation with the Caputo operator.

2. Auxiliary statements

This section provides information about the convergence of Fourier series with respect to the system of eigenfunctions of the Dirichlet problem for the operator L_x.

Let us assume that $w_m(x)$ and μ_m, $m=1,2,\ldots,$ respectively, are eigenfunctions and eigenvalues of the classical Dirichlet problem

$$\Delta w(x)+\mu w(x)=0,\ \ x\in\Omega,\ \ w(x)=0,\ \ x\in\partial\Omega. \tag{5}$$

Let $k\in\left\{0,1,\ldots,2^l-1\right\}$. Let us introduce the functions

$$v_m^k(x)\equiv v_m^{(k_l\ldots k_1)_2}(x)=\frac{1}{2^l}\sum_{q=0}^{2^l-1}(-1)^{k\otimes q}w_m\left(S_l^{q_l}\ldots S_1^{q_1}x\right), \tag{6}$$

where $k\otimes q=k_1q_1+k_2q_2+\ldots+k_lq_l$.

Obviously, if $x\in\partial\Omega$, then for any $0\leqslant q\leqslant 2^l-1$ points $S_l^{q_l}\ldots S_1^{q_1}x$ also belong to $\partial\Omega$. Therefore, the condition $v_m^k(x)\big|_{\partial\Omega}=0$ is satisfied. Moreover, from the equality $(-\Delta)\,w_m\left(S_l^{q_l}\ldots S_1^{q_1}x\right)=\mu_m w_m\left(S_l^{q_l}\ldots S_1^{q_1}x\right)$, $x\in\Omega$ it follows that

$$\begin{aligned}-\Delta v_m^k(x)&=\frac{1}{2^l}\sum_{q=0}^{2^l-1}(-1)^{k\otimes q}(-\Delta)\,w_m\left(S_l^{q_l}\ldots S_1^{q_1}x\right)\\&=\frac{\mu_m}{2^l}\sum_{q=0}^{2^l-1}(-1)^{k\otimes q}w_m\left(S_l^{q_l}\ldots S_1^{q_1}x\right)=\mu_m v_m^k(x).\end{aligned}$$

Therefore, $v_m^k(x)$ is also an eigenfunction of the Dirichlet problem (5). On the other hand, for a system of functions $v_m^k(x)$ the following assertion was proved [30].

LEMMA 1. *The elements of the system* $\left\{v_m^k(x)\right\}_{m=0}^{\infty}$, $0\leqslant k\leqslant 2^l-1$ *are eigenfunctions of the problem*

$$Lv(x)+\lambda v(x)=0,\ \ x\in\Omega,\ \ v(x)=0,\ \ x\in\partial\Omega, \tag{7}$$

and their eigenvalues are determined by the formula

$$\lambda_m^k=\mu_m\sum_{i=0}^{2^l-1}(-1)^{k\otimes i}a_i\equiv\mu_m\varepsilon_k,\quad \varepsilon_k=\sum_{i=0}^{2^l-1}(-1)^{k\otimes i}a_i.$$

Moreover, $\left\{v_m^k(x)\right\}_{m=1}^{\infty}$, $k=0,1,\ldots,2^l-1$ *is a complete and orthonormal system in the space* $L_2(\Omega)$.

In what follows, we will assume that the inequalities $\varepsilon_k = \sum_{i=0}^{2^l-1} (-1)^{k\otimes i} a_i > 0$ are valid for all $k \in \{0,1,\dots,2^l-1\}$.

Let us express the eigenfunctions $v_m^k(x)$ and eigenvalues λ_m^k as follows

$$v_{2^l m-k}(x) = \frac{1}{2^l} \sum_{q=0}^{2^l-1} (-1)^{k\otimes q} w_{2^l m-k}\left(S_l^{q_l} \dots S_1^{q_1} x\right),$$

$$\lambda_{2^l m-k} = \varepsilon_k \mu_{2^l m-k}, \;\; m = 1,2,\dots, \;\; k = 0,1,\dots,2^l-1.$$

EXAMPLE 1. Let $n \geqslant 2$ and mapping $S_1 x = -x$ is given. Then $l = 1$ and

$$v_{2m}(x) = \frac{1}{2}\left[w_{2m}(x) + w_{2m-1}(-x)\right], \quad v_{2m-1}(x) = \frac{1}{2}\left[w_{2m-1}(x) - w_{2m-1}(-x)\right],$$

$$\lambda_{2m} = (a_0 + a_1)\mu_{2m}, \quad \lambda_{2m-1} = (a_0 - a_1)\mu_{2m-1}.$$

EXAMPLE 2. Let $n = 2$ and $S_1 x = (-x_1, x_2)$, $S_2 x = (x_1, -x_2)$. Then $l = 2$ and

$$\begin{aligned} v_{4m}(x) &= \frac{1}{4}\sum_{q=0}^{3} (-1)^{0\otimes q} w_{4m}\left(S_2^{q_2} S_1^{q_1} x\right) \\ &= \frac{1}{4}\left[w_{4m}(x) + w_{4m}(S_1 x) + w_{4m}(S_2 x) + w_{4m}(S_2 S_1 x)\right], \end{aligned}$$

$$\begin{aligned} v_{4m-1}(x) &= \frac{1}{4}\sum_{q=0}^{3} (-1)^{1\otimes q} w_{4m-1}\left(S_2^{q_2} S_1^{q_1} x\right) \\ &= \frac{1}{4}\left[w_{4m-1}(x) - w_{4m-1}(S_1 x) + w_{4m-1}(S_2 x) - w_{4m-1}(S_2 S_1 x)\right], \end{aligned}$$

$$\begin{aligned} v_{4m-2}(x) &= \frac{1}{4}\sum_{q=0}^{3} (-1)^{2\otimes q} w_{4m-2}\left(S_2^{q_2} S_1^{q_1} x\right) \\ &= \frac{1}{4}\left[w_{4m-2}(x) + w_{4m-2}(S_1 x) - w_{4m-2}(S_2 x) + w_{4m-2}(S_2 S_1 x)\right], \end{aligned}$$

$$\begin{aligned} v_{4m-3}(x) &= \frac{1}{4}\sum_{q=0}^{3} (-1)^{3\otimes q} w_{4m-3}\left(S_2^{q_2} S_1^{q_1} x\right) \\ &= \frac{1}{4}\left[w_{4m-3}(x) - w_{4m-3}(S_1 x) - w_{4m-3}(S_2 x) + w_{4m-3}(S_2 S_1 x)\right]. \end{aligned}$$

Let us present some assertions about eigenfunctions $w_m(x)$ and eigenvalues μ_m proved by V. A. Il'in [11].

LEMMA 2. ([11], Lemma 1) *For the system* $\{w_m(x)\}_{m=1}^{\infty}$ *the following statements are valid:*

1) the series $\sum\limits_{m=1}^{\infty} \mu_m^{-\left(\left[\frac{n}{2}\right]+1\right)} w_m^2(x)$ *converges uniformly in a closed domain* $\overline{\Omega}$*;*

2) series $\sum\limits_{m=1}^{\infty} \mu_m^{-\left(\left[\frac{n}{2}\right]+2\right)} \left[\frac{\partial w_m(x)}{\partial x_i}\right]^2$ *and* $\sum\limits_{m=1}^{\infty} \mu_m^{-\left(\left[\frac{n}{2}\right]+3\right)} \left[\frac{\partial^2 w_m(x)}{\partial x_i \partial x_j}\right]^2$ *converge uniformly in an arbitrary closed subdomain* $\overline{\Omega}_0$ *located strictly inside* Ω.

LEMMA 3. ([11], Lemma 5) *Let the function* $g(x)$ *satisfy the conditions*

1) $g(x) \in C^p\left(\overline{\Omega}\right),\ \frac{\partial^{p+1} g(x)}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}} \in L_2(\Omega),\ p_1+\ldots+p_n = p+1,\ p \geqslant 1,$

2) $g(x)|_{\partial\Omega} = \Delta g(x)|_{\partial\Omega} = \ldots = \Delta^{\left[\frac{p}{2}\right]} g(x)\Big|_{\partial\Omega} = 0.$

Then the number series $\sum\limits_{k=1}^{\infty} g_k^2 \mu_k^{p+1}$ *converges, where* $g_k = \langle g, w_k \rangle$.

LEMMA 4. ([11], Lemma 7) *Let the function* $f(t,x)$ *satisfy the conditions:*

1) $f(t,x) \in C_{t,x}^{0,\left[\frac{n}{2}\right]+1}\left(\bar{Q}\right),\ \frac{\partial^p f(t,x)}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}} \in L_2(Q),\ p = (p_1, p_2, \ldots, p_n),\ p = \left[\frac{n}{2}\right]+2;$

2) $f(t,x)|_{\partial\Omega\times[0,T]} = \Delta f(t,x)|_{\partial\Omega\times[0,T]} = \ldots = \Delta^{\left[\frac{n+2}{4}\right]} f(t,x)\Big|_{\partial\Omega\times[0,T]} = 0.$

Then the series

$$\sum_{m=1}^{\infty} f_m(t) w_m(x)$$

converges uniformly in a closed cylinder $\bar{Q}$*, where* $f_m(t) = \langle f, w_m \rangle$.

As $\left\{v_{2^l m-k}(x)\right\}_{m=1}^{\infty},\ k = 0,1,\ldots,2^l-1,$ are eigenfunctions of problem (5) and form a complete orthonormal system in $L_2(\Omega)$, then Lemma 2 implies the following assertions.

COROLLARY 1. *For the system* $\left\{v_{2^l m-k}(x)\right\}_{m=1}^{\infty},\ k = 0,1,\ldots,2^l-1$ *the following statements are valid:*

1) the series $\sum\limits_{k=0}^{2^l-1} \sum\limits_{m=1}^{\infty} \mu_{2^n m-k}^{-\left(\left[\frac{n}{2}\right]+1\right)} v_{2^n m-k}^2(x)$ *converges uniformly in a closed domain* $\overline{\Omega}$;

2) series $\sum\limits_{k=0}^{2^l-1} \sum\limits_{m=1}^{\infty} \mu_{2^n m-k}^{-\left(\left[\frac{n}{2}\right]+2\right)} \left[\frac{\partial v_{2^n m-k}(x)}{\partial x_i}\right]^2$ *and* $\sum\limits_{k=0}^{2^l-1} \sum\limits_{m=1}^{\infty} \mu_{2^n m-k}^{-\left(\left[\frac{n}{2}\right]+3\right)} \left[\frac{\partial^2 v_{2^n m-k}(x)}{\partial x_i \partial x_j}\right]^2$ *converges uniformly in an arbitrary strictly closed subdomain* $\overline{\Omega}_0$ *located inside* Ω.

By assumption $\left(\sum\limits_{i=0}^{2^l-1} (-1)^{k\otimes i} a_i\right) > 0$, and then there exist constants C_1 and C_2 such that the estimates hold

$$C_1 \mu_{2^l m-k} \leqslant \lambda_{2^l m-k} \leqslant C_2 \mu_{2^l m-k} \tag{8}$$

Hence, we obtain the following assertion.

COROLLARY 2. *For the system* $\{v_{2^n m-k}(x)\}_{m=1}^{\infty}$, $k=0,1,\ldots,2^l-1$ *the following statements are valid:*

1) the series $\sum\limits_{m=1}^{\infty}\sum\limits_{k=0}^{2^l-1}\lambda_{2^n m-k}^{-([\frac{n}{2}]+1)}v_{2^n m-k}^2(x)$ *converges uniformly in a closed domain* $\bar{\Omega}$;

2) series $\sum\limits_{m=1}^{\infty}\sum\limits_{k=0}^{2^l-1}\lambda_{2^n m-k}^{-([\frac{n}{2}]+2)}\left[\frac{\partial v_{2^n m-k}(x)}{\partial x_i}\right]^2$ *and* $\sum\limits_{m=1}^{\infty}\sum\limits_{k=0}^{2^l-1}\lambda_{2^n m-k}^{-([\frac{n}{2}]+3)}\left[\frac{\partial^2 v_{2^n m-k}(x)}{\partial x_i \partial x_j}\right]^2$ *converge uniformly in an arbitrary strictly closed subdomain* $\overline{\Omega_0}$ *of the domain* Ω.

COROLLARY 3. *Let the function* $g(x)$ *satisfy the conditions of Lemma 3 with* $p \geqslant 1$. *Then the number series* $\sum\limits_{k=0}^{2^l-1}\sum\limits_{m=1}^{\infty}\left|g_{2^l m-k}\right|^2\mu_{2^l m-k}^{p+1}$ *converges.*

COROLLARY 4. *Let the function* $f(t,x)$ *satisfy the conditions of Lemma 2.4. Then the series*

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} f_{2^l m-k}(t)\cdot v_{2^l m-k}(x)$$

converges uniformly in a closed cylinder $\bar{Q}$.

3. Direct problem

According to the Fourier method, we seek the function $u(t,x)$ as a formal series

$$u(t,x)=\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} u_{2^l m-k}(t)v_{2^l m-k}(x). \tag{9}$$

Here the functions $u_{2^l m-k}(t)$ are a solution of the Cauchy problem

$$\partial^{\alpha}u_{2^l m-k}(t)+\lambda_{k,j}u_{2^l m-k}(t)=f_{2^l m-k}(t),\ \ 0<t<T, \tag{10}$$

$$u_{2^l m-k}(0)=\varphi_{2^l m-k}, \tag{11}$$

where $f_{2^l m-k}(t)=\langle f, v_{2^l m-k}\rangle$, $\varphi_{2^l m-k}=\langle \varphi, v_{2^l m-k}\rangle$.

In [14], Th 4.3, 231 p. it is proved that if $f_{2^l m-k}(\tau)\in C[0,T]$, problem (10)–(11) has a unique solution, which can be written as:

$$u_{2^l m-k}(t)=\varphi_{2^l m-k}E_{\alpha,1}\left(-\lambda_{2^l m-k}t^{\alpha}\right)+\int\limits_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}(t-\tau)^{\alpha}\right)f_{2^l m-k}(\tau)d\tau.$$

In this case the functions $u_{2^l m-k}(t)$ and $\partial^{\alpha}u_{2^l m-k}(t)$ belong to the class $C[0,T]$. Here $E_{\alpha,\beta}(z)=\sum\limits_{j=0}^{\infty}\frac{z^k}{\Gamma(\alpha j+\beta)}$ is a Mittag-Leffler type function.

Thus, the formal solution to problem (1)–(3) has the form:

$$u(t,x) = \sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \varphi_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t^{\alpha}\right) v_{2^l m-k}(x) \tag{12}$$

$$+\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left[\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}(t-\tau)^{\alpha}\right) f_{2^l m-k}(\tau)d\tau\right] v_{2^l m-k}(x).$$

Here we use the notation:

$$u_1(t,x) = \sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \varphi_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t^{\alpha}\right) v_{2^l m-k}(x), \tag{13}$$

$$u_2(t,x) = \sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} F_{2^l m-k}(t) v_{2^l m-k}(x), \tag{14}$$

where

$$F_{2^l m-k}(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}(t-\tau)^{\alpha}\right) f_{2^l m-k}(\tau)d\tau.$$

Let us estimate the series representing the functions $u_j(t,x)$, $\partial_t^{\alpha} u_j(t,x)$ and $L_x u_j(t,x)$, $j = 1,2$. In what follows, the symbol C will denote a positive constant, not necessarily the same one.

To do this, we use the following property of the function $E_{\alpha,\beta}(z)$:

1) for $\alpha \in (0,2)$, $\gamma \leqslant |\arg z| \leqslant \pi$, $\beta \in R$, $\gamma \in (\pi\alpha/2; \min\{\pi; \pi\alpha\})$ satisfies the estimate (see, for example, [23], 35 p.),

$$\left|E_{\alpha,\beta}(z)\right| \leqslant \frac{C}{1+|z|}, \quad \mu \leqslant |\arg(z)| \leqslant \pi. \tag{15}$$

2) if $Re(\alpha) > 0$, $\beta \notin -N_0$, then (see, for example, [17], 274 p.),

$$E_{\alpha,\alpha+\beta}(z) = zE_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} \tag{16}$$

Using estimate (15), as well as the Cauchy-Bunyakovsky inequalities for the function $u_1(t,x)$, we obtain

$$|u_1(t,x)| \leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \left|\varphi_{2^l m-k}\right| \left|v_{2^l m-k}(x)\right|$$

$$\leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \left|\varphi_{2^l m-k}\right| \mu_{2^l m-k}^{\left(\left[\frac{n}{2}\right]+1\right)} \mu_{2^l m-k}^{-\left(\left[\frac{n}{2}\right]+1\right)} \left|v_{2^l m-k}(x)\right|$$

$$\leqslant C\sqrt{\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \left|\varphi_{2^l m-k}\right|^2 \mu_{2^l m-k}^{\left(\left[\frac{n}{2}\right]+1\right)}} \sqrt{\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \mu_{2^l m-k}^{-\left(\left[\frac{n}{2}\right]+1\right)} \left|v_{2^l m-k}(x)\right|^2}.$$

Further, as the function $\left|-\lambda_{2^l m-k} t^\alpha E_{2^l m-k}\left(-\lambda_{2^l m-k} t^\alpha\right)\right|$ is limited in the domain $0<\delta \leqslant t \leqslant T$, $x \in \bar{\Omega}$, then in this domain for the function $\partial_t^\alpha u_1(t,x)$ we get:

$$
\begin{aligned}
\left|\partial_t^\alpha u_1(t,x)\right| &= \left|-\sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \lambda_{2^l m-k} \varphi_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right) v_{2^l m-k}(x)\right| \\
&\leqslant \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} t^{-\alpha}\left|\varphi_{2^l m-k}\right|\left|-\lambda_{2^l m-k} t^\alpha E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right)\right|\left|v_{2^l m-k}(x)\right| \\
&\leqslant C \sqrt{\sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1}\left|\varphi_{2^l m-k}\right|^2 \mu_{2^l m-k}^{\left(\left[\frac{n}{2}\right]+1\right)}} \sqrt{\sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \mu_{2^l m-k}^{-\left(\left[\frac{n}{2}\right]+1\right)}\left|v_{2^l m-k}(x)\right|^2}.
\end{aligned}
$$

Now, if the function $\varphi(x)$ satisfies the conditions of Lemma 3 with the exponent $p=\left[\frac{n}{2}\right]$, then according to Corollary 3 the number series $\sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1}\left|\varphi_{2^l m-k}\right|^2 \mu_{2^l m-k}^{\left(\left[\frac{n}{2}\right]+1\right)}$ converges. In addition, according to Corollary 1 the series $\sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \mu_{2^l m-k}^{-\left(\left[\frac{n}{2}\right]+1\right)}\left|v_{2^l m-k}(x)\right|$ converges uniformly in a closed domain $\bar{\Omega}$. Then, the series representing the function $u_1(t,x)$ converges uniformly in the closed domain $\bar{Q}$, and the series representing the function $\partial_t^\alpha u_1(t,x)$ converges uniformly in any closed subdomain $\bar{Q}_\delta$ of the domain $\bar{Q}_\delta$. Hence, $u_1(t,x) \in C\left(\bar{Q}\right)$ and $\partial^\alpha u_1(t,x) \in C(Q)$. The inclusion $L_x u_1(t,x) \in C(Q)$ is proved in a similar way.

Further, we consider the smoothness of the function $u_2(t,x)$.

Let us assume that the function $f(t,x)$ belongs to the class $C_{t,x}^{0,p}\left(\bar{Q}\right)$ for some $p \geqslant 1$. Then, $f_{2^l m-k}(\tau) \in C[0,T]$ and, hence, $\partial^\alpha F_{2^l m-k}(t)$ exists and belongs to the class $C[0,T]$. Denote

$$
\left|f_{2^l m-k}(t_M)\right| = \max_{0 \leqslant t \leqslant T}\left|f_{2^l m-k}(t)\right|, \quad t_M \in [0,T].
$$

Then, the function $F_{2^l m-k}(t)$ satisfies the following estimate

$$
\left|F_{2^l m-k}(t)\right| \leqslant \left|f_{2^l m-k}(t_M)\right| t^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t^\alpha\right) \tag{17}
$$

Hence, as the function $t^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t^\alpha\right)$ is limited, for $u_2(t,x)$ we get

$$
\left|u_2(t,x)\right| \leqslant \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1}\left[\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}(t-\tau)^\alpha\right)\left|f_{2^l m-k}(\tau)\right| d\tau\right]\left|v_{2^l m-k}(x)\right|.
$$

As $\partial^\alpha F_{2^l m-k}(t) = f_{2^l m-k}(t) - \lambda_{2^l m-k} F_{2^l m-k}(t)$, then for $\partial^\alpha u_2(t,x)$ we obtain

$$
\left|\partial^\alpha u_2(t,x)\right| \leqslant \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1}\left|f_{2^l m-k}(t) v_{2^l m-k}(x)\right| + \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \lambda_{2^l m-k}\left|F_{2^l m-k}(t) v_{2^l m-k}(x)\right|.
$$

From estimates (15) and (17) for the second series in the last expression, for all $0 < \delta \leqslant t \leqslant T$, $x \in \bar{\Omega}$ we get

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\lambda_{2^l m-k}\left|F_{2^l m-k}(t)v_{2^l m-k}(x)\right|$$
$$\leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left|f_{2^l m-k}(t_M)\right|t^{\alpha}\lambda_{2^l m-k}E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k}t^{\alpha}\right)\left|v_{2^l m-k}(x)\right|$$
$$\leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left|f_{2^l m-k}(t_M)\right|\left|v_{2^l m-k}(x)\right|.$$

Similarly, for the function $L_x u_2(t,x)$ we get

$$|L_x u_2(t,x)| = \left|\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\lambda_{2^l m-k}F_{2^l m-k}(t)v_{2^l m-k}(x)\right| \leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left|f_{2^l m-k}(t_M)\right|\left|v_{2^l m-k}(x)\right|.$$

Thus, the series $\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left|f_{2^l m-k}(t)\right|\left|v_{2^l m-k}(x)\right|$ is majorant with respect to series representing functions $u_2(t,x)$, $\partial_t^{\alpha}u_2(t,x)$ and $L_x u_2(t,x)$.

If the function $f(t,x)$ satisfies the conditions of Lemma 4, then from Corollary 4 we obtain that the series

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left|f_{2^l m-k}(t)\right|\left|v_{2^l m-k}(x)\right|$$

converges uniformly in a closed cylinder $\bar{Q}$. Then the series (14) also converges uniformly in a closed cylinder, and the series

$$\partial^{\alpha}u_2(t,x) = \sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left[f_{2^l m-k}(t) - \lambda_{2^l m-k}F_{2^l m-k}(t)\right]v_{2^l m-k}(x),$$

$$L_x u_2(t,x) = -\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\lambda_{2^l m-k}F_{2^l m-k}(t)v_{2^l m-k}(x)$$

converge uniformly in any closed subdomain $\bar{Q}_\delta$ of the domain Q. Hence, $u_2(t,x) \in C\left(\bar{Q}\right)$, and functions $\partial_t^{\alpha}u_2(t,x)$ and $L_x u_2(t,x)$ belong to class $C(Q)$.

Let us investigate the uniqueness of the solution. Assume the opposite, i.e. let problem (1)–(3) have two solutions $u_1(t,x)$ and $u_2(t,x)$. Let us prove that $u(t,x) = u_1(t,x) - u_2(t,x) \equiv 0$. As the problem is linear, for $u(t,x)$ we get a homogeneous problem (1)–(3). By definition of the solution, the function $u(t,x)$ has the following properties: $u(t,x) \in C\left(\bar{Q}\right)$, $\partial_t^{\alpha}u(t,x) \in C(Q)$ and $L_x u(t,x) \in C(Q)$. Then due to continuity, the function $u_{2^l m-k}(t) = \left(u(t,x), v_{2^l m-k}(x)\right)$ satisfies the homogeneous condition (11). If δ is an arbitrary small number, then according to the condition $\partial_t^\alpha u(t,x)$, $L_x u(t,x) \in C\left(\bar{Q}_\delta\right)$, $\bar{Q}_\delta \subset Q$ for all $t \geqslant \delta > 0$, the equality

$$\partial^\alpha u_{2^l m-k}(t) + \lambda_{k,j} u_{2^l m-k}(t) = 0 \tag{18}$$

is satisfied. As δ is an arbitrary number, equation (18) is satisfied in the domain $0 < t < T$. Thus, the function $u_{2^l m-k}(t) = \left\langle u(t,x), v_{2^l m-k}(x)\right\rangle$ is a solution to the homogeneous problem (10), (11). Then, due to the uniqueness of the solution to this problem, we get $\left\langle u(t,x), v_{2^l m-k}(x)\right\rangle = 0$, i.e. the function $u(t,x)$ is orthogonal to all elements of the system $\{v_{2^l m-k}(x)\}$. Therefore, $u(t,x) = 0$ for almost all $t \in [0,T]$. As $u(t,x) \in C\left(\bar{Q}\right)$, then $u(t,x) \equiv 0$, $(t,x) \in \bar{Q}$.

Thus, we proved the following assertion.

THEOREM 1. *Let the coefficients a_i, $0 \leqslant i \leqslant 2^l - 1$ in Problem (1)–(3) be such that the conditions $\varepsilon_k = \sum_{i=0}^{2^l-1} (-1)^{k \otimes i} a_i > 0$, $k = 0,1,\ldots,2^l-1$ are satisfied, the function $\varphi(x)$ satisfies the conditions of Lemma 3 with the exponent $p = \left[\frac{n}{2}\right]$, and the function $f(t,x)$ satisfies the conditions of Lemma 4 with the exponent $p = \left[\frac{n}{2}\right] + 1$. Then a solution to Problem (1)–(3) exists, is unique, and can be represented as (12).*

4. Inverse problem for the case $g(t) = 1$

In this section, the inverse problem is studied. Let us first consider the problem for the case of $g(t) = 1$. The following assertion is valid.

THEOREM 2. *Let the conditions $\varepsilon_i > 0$, $i = 1,2\ldots,2^l-1$, $g(t) = 1$ be satisfied and functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 3 with the exponent $p = \left[\frac{n}{2}\right] + 2$. Then the solution to Problem (1)–(4) exists, is unique, and is represented in the form of series*

$$f(x) = \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \left[\frac{1}{1 - E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \psi_{2^l m-k} - \frac{E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)}{1 - E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \varphi_{2^l m-k} \right] \times \lambda_{2^l m-k} v_{2^l m-k}(x), \tag{19}$$

$$u(t,x) = \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \left[\frac{E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right) - E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)}{1 - E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \varphi_{2^l m-k} + \frac{1 - E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right)}{1 - E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \psi_{2^l m-k} \right] v_{2^l m-k}(x). \tag{20}$$

Proof. Suppose that the function $u(t,x)$ is a solution to Problem (1)–(4) for $g(t) = 1$. Consider the functions

$$u_{2^l m-k}(t) = \int_\Omega u(t,x) v_{2^l m-k}(x) dx, \quad k = 0,1,\ldots,2^l - 1.$$

Using the conditions of Problem (1)–(4) for the function $u_{2^l m-k}(t)$, we obtain

$$\begin{aligned} D_t^\alpha u_{2^l m-k}(t) &= \int_\Omega D_t^\alpha u(t,x) v_{2^l m-k}(x) dx = \int_\Omega [Lu(t,x) + f(x)] v_{2^l m-k}(x) dx \\ &= \int_\Omega u(t,x) L v_{2^l m-k}(x) dx + \int_\Omega f(x) v_{2^l m-k}(x) dx \\ &= -\lambda_{2^l m-k} \int_\Omega u(t,x) v_{2^l m-k}(x) dx + f_{2^l m-k} = -\lambda_{2^l m-k} v_{2^l m-k}(x) + f_{2^l m-k}, \end{aligned}$$

where $\lambda_{2^l m-k} = \varepsilon_k \mu_{2^l m-k}$.

Further, from conditions (3) and (4) for the function $u_{2^l m-k}(t)$ we get

$$u_{2^l m-k}(0) = \int_\Omega u(0,x) v_{2^l m-k}(x) dx = \int_\Omega \varphi(x) v_{2^l m-k}(x) dx = \varphi_{2^l m-k},$$

$$u_{2^l m-k}(t_0) = \int_\Omega u(t_0,x) v_{2^l m-k}(x) dx = \int_\Omega \varphi(x) v_{2^l m-k}(x) dx = \psi_{2^l m-k}.$$

Thus, the function $u_{2^l m-k}(t)$ satisfies the conditions of the following problem

$$D_t^\alpha u_{2^l m-k}(t) + \lambda_{2^l m-k} u_{2^l m-k}(t) = f_{2^l m-k}, \quad 0 < t < T, \tag{21}$$

$$u_{2^l m-k}(0) = \varphi_{2^l m-k}, \quad u_{2^l m-k}(t_0) = \psi_{2^l m-k}. \tag{22}$$

The general solution of equation (21) has the form

$$u_{2^l m-k}(t) = C_{2^l m-k} \cdot E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right) + f_{2^l m-k} t^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t^\alpha\right), \tag{23}$$

where $C_{2^l m-k}$ are arbitrary constants. Substituting function (23) into conditions (22), we obtain

$$C_{2^l m-k} = \varphi_{2^l m-k},$$

and

$$C_{2^l m-k} \cdot E_\alpha\left(-\lambda_{2^l m-k} t_0^\alpha\right) + f_{2^l m-k} t_0^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t_0^\alpha\right) = \psi_{2^l m-k}.$$

Then we find

$$f_{2^l m-k} = \frac{\psi_{2^l m-k} - \varphi_{2^l m-k} \cdot E_\alpha\left(-\lambda_{2^l m-k} t_0^\alpha\right)}{t_0^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)}.$$

From equality (16) for the function $t_0^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)$ we get

$$t_0^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k} t_0^\alpha\right) = -\frac{1}{\lambda_{2^l m-k}}\left[E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right) - 1\right].$$

Then the coefficients $f_{2^l m-k}$ and functions $u_{2^l m-k}(t)$ are represented as

$$f_{2^l m-k} = \frac{\lambda_{2^l m-k}}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}\psi_{2^l m-k} - \frac{\lambda_{2^l m-k}E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}\varphi_{2^l m-k}, \qquad (24)$$

$$u_{2^l m-k}(t) = \frac{E_{\alpha,1}\left(-\lambda_{2^l m-k}t^\alpha\right) - E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}\varphi_{2^l m-k} + \frac{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t^\alpha\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}\psi_{2^l m-k}. \qquad (25)$$

Note that if under the conditions of Problem (1)–(4) the equalities $\varphi(x) \equiv 0$ and $\psi(x) \equiv 0$ are satisfied, then from (24) and (25) it follows that $u_{2^l m-k}(t) = 0$, $f_{2^l m-k} = 0$ for all $m \geqslant 1$, $k = 0,1,\ldots,2^l-1$..

Hence, $\langle u, v_{2^l m-k}\rangle = 0$, $\langle f, v_{2^l m-k}\rangle = 0$, $m \geqslant 1$, $k = 0,1,\ldots,2^l-1$. As $v_{2^l m-k}(x)$ is a complete orthonormal system, then $f(x) \equiv 0$, $x \in \bar{\Omega}$ and $u(t,x) \equiv 0$, $x \in \bar{\Omega}$ for almost all $t \in [0,T]$. By assumption, $u(t,x)$ is a continuous function in a closed domain $\bar{Q}$. Then $u(t,x) \equiv 0, (t,x) \in \bar{Q}$. This implies that Problem (1)–(4) is unique. Indeed, if there are two solutions $\{u_1(t,x), f_1(x)\}$ and $\{u_2(t,x), f_2(x)\}$, then for the functions $u(t,x) = u_1(t,x) - u_2(t,x)$ and $f(x) = f_1(x) - f_2(x)$ we obtain Problem (1)–(4) with homogeneous conditions (3) and (4). Hence, in this case $f(x) \equiv 0$, $x \in \bar{\Omega}$ and $u(t,x) \equiv 0, (t,x) \in \bar{Q}$, i.e., $u_1(t,x) \equiv u_2(t,x)$, $f_1(x) \equiv f_2(x)$.

Let us study the existence of a solution to Problem (1)–(4). Under certain conditions for the functions $\varphi(x)$ and $\psi(x)$ we will show that the function $f(x)$ from equality (19) and the function $u(t,x)$ from equality (20) are a solution to Problem (1)–(4). By construction, these functions formally satisfy all the conditions of Problem (1)–(4). Now we have to investigate smoothness of these functions.

To do this, it suffices to show that the series (19) and (20), respectively, converge uniformly in the closed domain $\bar{\Omega}$ and $\bar{Q}$, while the series obtained by applying the operators D_t^α and L_x to the series (20) converge in any strictly closed subdomain $\bar{Q}_0$ of the domain Q. If we denote $\Delta_{2^l m-k} = 1 - E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)$, then from the conditions $E_{\alpha,1}(0) = 1$, $\lambda_{2^l m-k} > 0$ it follows that there exists such $\delta > 0$ that $|\Delta_{nk}| \geqslant \delta > 0$. Then the following functions

$$\frac{1}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)}, \quad E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)$$

are bounded for all $m \in N$, $k = 0,1,\ldots,2^l-1$. Therefore, for the series (19), taking into account estimate (8), we obtain

$$|f(x)| \leqslant C\left(\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\lambda_{2^l m-k}\left|\psi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right| + \sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\lambda_{2^l m-k}\left|\varphi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right|\right)$$
$$\leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}\left|\psi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right| + C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}\left|\varphi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right|.$$

Let us consider the convergence of the series

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}\left|\psi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right| \tag{26}$$

and

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}\left|\varphi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right|. \tag{27}$$

Using the Bessel inequality for the series (26), we obtain the following estimates

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}\left|\psi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right|$$

$$=\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}\left(\sqrt{\mu_{2^l m-k}}\right)^{\left[\frac{n}{2}\right]+1}\left|\psi_{2^l m-k}\right|\left(\sqrt{\mu_{2^l m-k}}\right)^{-\left(\left[\frac{n}{2}\right]+1\right)}\left|v_{2^l m-k}(x)\right|$$

$$\leqslant\sqrt{\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}^{\left[\frac{n}{2}\right]+3}\left|\psi_{2^l m-k}\right|^2}\cdot\sqrt{\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\frac{\left|v_{2^l m-k}(x)\right|^2}{\mu_{2^l m-k}^{\left[\frac{n}{2}\right]+1}}}.$$

According to Lemma 2, the series

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}^{-([n/2]+1)}\left|v_{2^l m-k}(x)\right|^2$$

converges uniformly in a closed domain $\bar{\Omega}$. Further, as the function $\psi(x)$ satisfies the conditions of Lemma 3 for $p=\left[\frac{n}{2}\right]+2$, the numerical series

$$\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\mu_{2^l m-k}^{\left[\frac{n}{2}\right]+3}\left|\psi_{2^l m-k}\right|^2$$

converges. This implies the uniform convergence of the series (26) in the closed domain $\bar{\Omega}$. The uniform convergence of the series (27) in the closed domain $\bar{\Omega}$ is proved in a similar way. Then the series on the right-hand side of equality (19) converges uniformly in a closed domain $\bar{Q}$ and its sum $f(x)$ belongs to the class $C\left(\bar{Q}\right)$.

Let us study the convergence of the series (20). As the functions $\frac{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t^{\alpha}\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^{\alpha}\right)}$ and $\frac{E_{\alpha,1}\left(-\lambda_{2^l m-k}t^{\alpha}\right)-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^{\alpha}\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^{\alpha}\right)}$ are bounded on the interval $0\leqslant t\leqslant T$, then we can use the estimate

$$|u(t,x)|\leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left(\left|\varphi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right|+\left|\psi_{2^l m-k}\right|\left|v_{2^l m-k}(x)\right|\right).$$

As we have already noticed, the series on the right side of the last inequality converge uniformly in a closed domain $\bar{\Omega}$. This implies that the functional series (20) representing the function $u(t,x)$ converges uniformly in a closed domain $\bar{Q}$ and therefore $u(t,x) \in C\left(\bar{Q}\right)$.

Further, we study the smoothness of the functions $D_t^\alpha u(t,x)$ and $L_x u(t,x)$. Formally applying the operators $D_t^\alpha u(t,x)$ and $L_x u(t,x)$ to the function (20), we obtain

$$D_t^\alpha u(t,x) = -\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \lambda_{2^l m-k} \frac{E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \left[\varphi_{2^l m-k} - \psi_{2^l m-k}\right] v_{2^l m-k}(x),$$

$$L_x u(t,x) = -\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \lambda_{2^l m-k} \left[\frac{E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right) - E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \varphi_{2^l m-k} + \frac{1-E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right)}{1-E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^\alpha\right)} \psi_{2^l m-k}\right] v_{2^l m-k}(x).$$

Hence, using estimate (17), we have

$$|D_t^\alpha u(t,x)| \leqslant C\left(\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \left[\left|\psi_{2^l m-k}\right| + \left|\varphi_{2^l m-k}\right|\right] v_{2^l m-k}(x)\right),$$

$$|L_x u(t,x)| \leqslant C\left(\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1} \left[\lambda_{2^l m-k}\left|\varphi_{2^l m-k}\right| + \lambda_{2^l m-k}\left|\psi_{2^l m-k}\right|\right] \left|v_{2^l m-k}(x)\right|\right).$$

Under the conditions of the theorem, the last two series converge uniformly in a closed domain $\bar{\Omega}$. Then the series representing the functions $D_t^\alpha u(t,x)$ and $L_x u(t,x)$ converge uniformly in a closed domain $\bar{Q}$ and their sums, i.e. functions $D_t^\alpha u(t,x)$ and $L_x u(t,x)$ $f(x)$ belong to the class $C\left(\bar{Q}\right)$. The theorem is proved. □

5. Inverse problem for the case $g(t) \neq 1$

Let now the function $g(t)$ be not identical to 1 and the function $u(t,x)$ be a solution to Problem (1)–(4). As in the case $g(t) = 1$, consider the function

$$u_{2^l m-k}(t) = \int_\Omega u(t,x) v_{2^l m-k}(x) dx.$$

In this case, with respect to the function $u_{2^l m-k}(t)$, we obtain the equation

$$D_t^\alpha u_{2^l m-k}(t) = -\lambda_{2^l m-k} u_{2^l m-k}(t) + g(t) f_{2^l m-k}. \tag{28}$$

The general solution to equation (28) is written as

$$u_{2^l m-k}(t) = C_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t^\alpha\right) + f_{2^l m-k} g_{2^l m-k}(t),$$

where

$$g_{2^l m-k}(t)=\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}(t-\tau)^{\alpha}\right)g(\tau)d\tau. \tag{29}$$

Using the conditions of Problem (1)–(4), we obtain

$$\varphi_{2^l m-k}=u_{2^l m-k}(0)=C_{2^l m-k},$$

$$\psi_{2^l m-k}=u_{2^l m-k}(t_0)=\varphi_{2^l m-k}E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^{\alpha}\right)+f_{2^l m-k}g_{2^l m-k}(t_0).$$

Hence,

$$f_{2^l m-k}g_{2^l m-k}(t_0)=\psi_{2^l m-k}-\varphi_{2^l m-k}E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^{\alpha}\right). \tag{30}$$

Further, two cases are possible: $g_{2^l m-k}(t_0)\neq 0$ and $g_{2^l m-k}(t_0)=0$.

If for some values, the condition $g_{2^l m-k}(t_0)=0$ is satisfied, then the uniqueness may be violated. As it is noted in [8], this is due to the variable sign of the function $g(t)$. Following the results of [8], consider the following example.

EXAMPLE 3. Consider equation (1) with the function $F(t,x)=f(x)g(t)$ and homogeneous conditions (2)–(4).

Take any eigenfunction $v(x)$ of the nonlocal Laplace operator obeying homogeneous Dirichlet boundary conditions, i.e. $L_x v(x)+\lambda v(x)=0$ with $v(x)|_{\partial\Omega}=0$ and set $t_0=1$, $g(t)=t^{\alpha}\left(1-t^{b}\right)$, $b>0$. In this case, $u(t,x)=g(t)v(x)$ satisfies problem (2)–(4) with $\varphi(x)=\psi(x)=0$, $f(x)=v(x)$ and $g(t)=t^{\alpha}\left(1-t^{\beta}\right)$. Then, besides the trivial solution $\{u(t,x),f(x)\}=\{0,0\}$ to problem (2)–(4), we also have the following non-trivial solution $u(t,x)=g(t)v(x)$, $f(x)=v(x)$.

For further investigation, as suggested in [8], we divide the set $N_0=N\cup\{0\}$ into two groups: $N_0=K_{\alpha}\cup K_{0,\alpha}$, where the number $m\in N$, $k\in\left\{0,1,\dots,2^l-1\right\}$ is assigned to $K_{0,\alpha}$, if $g_{2^l m-k}(t_0)=0$, and if $g_{2^l m-k}(t_0)\neq 0$, then this number is assigned to K_{α}. Note that for some t_0 the set $K_{0,\alpha}$ can be empty, then $K_{\alpha}=N_0$. For example, if $g(t)$ does not change the sign, then $K_{0,\alpha}=N_0$ for all t_0.

Let us establish two-sided estimates for $g_{2^l m-k}(t_0)$. First, we suppose that $g(t)$ does not change sign for the diffusion equation.

LEMMA 5. *Let $g(t)\in C[0,T]$ and $g(t)\neq 0$, $t\in[0,T]$. Then there are constants $C_1,C_2>0$, depending on t_0, such that for all $m\geqslant 1$, $k=0,1,\dots,2^l-1$:*

$$\frac{C_1}{\mu_{2^l m-k}}\leqslant\left|g_{2^l m-k}(t_0)\right|\leqslant\frac{C_2}{\mu_{2^l m-k}}. \tag{31}$$

Proof. According to the Weierstrass theorem, we have $\left|g_{2^l m-k}(t)\right|\geqslant g_0=const>$

0. From equality (29), on the basis of the mean value theorem, we have

$$\begin{aligned}\left|g_{2^l m-k}(t_0)\right| &= \left|\int_0^{t_0} \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}\tau^\alpha\right) g(t_0-\tau)d\tau\right| \\ &= \left|g(\xi_k)\int_0^{t_0} \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}\tau^\alpha\right) d\tau\right| \\ &= |g(\xi_k)|\, t_0^\alpha E_{\alpha,\alpha+1}\left(-\lambda_{2^l m-k}t_0^\alpha\right) \\ &= \frac{|g(\xi_k)|}{\lambda_{2^l m-k}}\left[1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)\right], \quad \xi_k\in[0,t_0].\end{aligned}$$

From here, using the boundedness of the function $1-E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)$, inequalities (15) and the estimate $\left|g_{2^l m-k}(t)\right| \geqslant g_0$, we obtain (31). The lemma is proved. □

Further, if the condition $g_{2^l m-k}(t_0)\neq 0$, is satisfied for all $m\geqslant 1$, $k=0,1,\dots,2^l-1$, then for the coefficients $f_{2^l m-k}$ and $u_{2^l m-k}(t)$ we get

$$f_{2^l m-k} = \frac{1}{g_{2^l m-k}(t_0)}\left[\psi_{2^l m-k} - \varphi_{2^l m-k}E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0\right)\right] \tag{32}$$

and

$$u_{2^l m-k}(t) = \left(E_{\alpha,1}\left(-\lambda_{2^l m-k}t^\alpha\right) - \frac{g_{2^l m-k}(t)}{g_{2^l m-k}(t_0)}E_{\alpha,1}\left(-\lambda_{2^l m-k}t_0^\alpha\right)\right)\varphi_{2^l m-k} + \frac{g_{2^l m-k}(t)}{g_{2^l m-k}(t_0)}\psi_{2^l m-k}. \tag{33}$$

Using the estimate (31) for the coefficients $f_{2^l m-k}$ and $u_{2^l m-k}(t)$ as well as representations (32) and (33), we get the following inequalities:

$$\left|f_{2^l m-k}\right| \leqslant C\mu_{2^l m-k}\left[|\varphi_{2^l m-k}|+|\psi_{2^l m-k}|\right], \tag{34}$$

$$u_{2^l m-k}(t) = C\left[|\varphi_{2^l m-k}|+|\psi_{2^l m-k}|\right], \tag{35}$$

$$\left|D_t^\alpha u_{2^l m-k}(t)\right| \leqslant C\mu_{2^l m-k}\left[|\varphi_{2^l m-k}|+|\psi_{2^l m-k}|\right]. \tag{36}$$

Let us study the main assertion from Problem (1)–(4) for the case $g(t)\neq 0$. First, consider the case when the function does not change its sign. The following assertion is valid.

THEOREM 3. *Let $g(t)\in C[0,T]$ and $g(t)\neq 0$, functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 3 for $p=[n/2]+2$. Then, the solution to Problem* (1)–(4) *exists, is unique, and is defined by the series*

$$f(x) = \sum_{p=1}^{\infty}\sum_{k=0}^{2^l-1}\frac{1}{g_{2^l p-k}(t_0)}\left[\psi_{2^l p-k} - \varphi_{2^l p-k}E_{\alpha,1}\left(-\lambda_{2^l p-k}t_0\right)\right]v_{2^l p-k}(x), \tag{37}$$

$$u(t,x)=\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left[\left(E_{\alpha,1}\left(-\lambda_{2^lm-k}t^{\alpha}\right)-\frac{g_{2^lm-k}(t)}{g_{2^lm-k}(t_0)}E_{\alpha,1}\left(-\lambda_{2^lm-k}t_0^{\alpha}\right)\right)\varphi_{2^lm-k}\right.$$
$$\left.+\frac{g_{2^lm-k}(t)}{g_{2^lm-k}(t_0)}\psi_{2^lm-k}\right]v_{2^lm-k}(x). \quad (38)$$

Proof. By construction, the functions $f(x)$ and $u(t,x)$ from equalities (37) and (38) formally satisfy all the conditions of Problem (1)–(4). Estimate (34) implies uniform convergence of the series (37) in a closed domain $\bar{\Omega}$ and, therefore, $f(x)\in C\left(\bar{\Omega}\right)$. Similarly, the estimate (35) implies uniform convergence of the series (38) in a closed domain $\bar{Q}$ and, therefore, the sum of this series, i.e. the function $u(t,x)$ will be continuous in $\bar{Q}$. If we apply the operator ∂^{α} to the series (38), then from estimate (36) for $\partial^{\alpha}u(t,x)$ we obtain

$$|\partial^{\alpha}u(t,x)|\leqslant C\sum_{m=1}^{\infty}\sum_{k=0}^{2^l-1}\left(\mu_{2^lm-k}|\varphi_{2^lm-k}|+\mu_{2^lm-k}|\psi_{2^lm-k}|\right)|v_{2^lm-k}(x)|.$$

As the series on the right side of the last inequality converges uniformly in the closed domain $\bar{\Omega}$, the series representing the function $\partial^{\alpha}u(t,x)$ also converges uniformly in a closed domain $\bar{Q}$, and therefore $\partial^{\alpha}u(t,x)\in C\left(\bar{Q}\right)$. The inclusion $L_nu(t,x)$ $\in C\left(\bar{Q}\right)$ is proved similarly.

To prove the uniqueness of the solution, we assume the contrary. Let there exist two different solutions $\{u_1(t,x),f_1(x)\}$ and $\{u_2(t,x),f_2(x)\}$ satisfying inverse problem (1)–(4). We must show that $u(t,x)=u_1(t,x)-u_2(t,x)\equiv 0$, $f(t)=f_1(x)-f_2(x)\equiv 0$. For $\{u(t,x),f(x)\}$ we have the following problem:

$$\begin{cases}\partial^{\alpha}u(t,x)-L_xu(t,x)=f(x)g(t), & (t,x)\in Q,\\ u(t,x)=0, & x\in\partial\Omega,\\ u(0,x)=0, & 0\leqslant t\leqslant T,\ x\in\Omega\\ u(t_0,x)=0, & x\in\Omega,\ t_0\in(0,T]\end{cases}\quad (39)$$

Let $\{u(t,x),f(x)\}$ be a solution to problem (39). Determine the coefficients $u_{2^lm-k}(t)=\left\langle u,v_{2^lm-k}\right\rangle$, $f_{2^lm-k}=\left\langle f,v_{2^lm-k}\right\rangle$. Then, using the conditions $\partial_t^{\alpha}u(t,x)$, $L_xu(t,x)\in C\left(\bar{Q}\right)$ and taking into account that $\partial^{\alpha}u(t,x)-L_xu(t,x)=f(x)g(t)$ we get

$$\partial^{\alpha}u_{2^lm-k}(t)=\left\langle\partial^{\alpha}u,v_{2^lm-k}\right\rangle=\left\langle L_xu,v_{2^lm-k}\right\rangle+g(t)\left\langle f,v_{2^lm-k}\right\rangle$$
$$=-\left\langle\lambda_{2^lm-k},v_{2^lm-k}\right\rangle+f_{2^lm-k}g(t)=-\lambda_{2^lm-k}u_{2^lm-k}+f_{2^lm-k}g(t).$$

Thus, for the function $u_{2^lm-k}(t)$ we obtain the following Cauchy problem

$$\partial^{\alpha}u_{2^lm-k}(t)=-\lambda_{2^lm-k}u_{2^lm-k}(t)+g(t)f_{2^lm-k},\ \ t>0,\ \ u_{2^lm-k}(0)=0.$$

If $f_{2^l m-k}$ is known, then the unique solution to the Cauchy problem has the form

$$u_{2^l m-k}(t) = f_{2^l m-k}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{2^l m-k}(t-\tau)^{\alpha}\right)g(\tau)d\tau = f_{2^l m-k}g_{2^l m-k}(t).$$

Applying the additional condition, we get

$$u_{2^l m-k}(t_0) = f_{2^l m-k}g_{2^l m-k}(t_0) = 0.$$

It is assumed that $g_{2^l m-k}(t_0) \neq 0$ for all $m \in N$, $k \in \{0,1,\dots,2^l-1\}$. Then, $f_{2^l m-k} \equiv 0$ and $u_{2^l m-k}(t) \equiv 0$. By virtue of completeness of the system of eigenfunctions $v_{2^l m-k}(x)$ in $L_2(\Omega)$ we obtain that $f(x) \equiv 0$ and $u(t,x) \equiv 0$. The theorem is proved. $\square$

Let us consider the case when $g(t)$ changes its sign. In this case, the function $g_{2^l m-k}(t_0)$ can go to zero, and as a result, the set $K_{0,\alpha}$ may turn out to be non-empty. Therefore, we should consider separately the case of diffusion $(\alpha = 1)$ and subdiffusion $(0 < \alpha < 1)$ equations.

The following assertions are proved in the same way as in [8].

LEMMA 6. *Let $\alpha = 1$, $g(t) \in C^1[0,T]$ and $g(t_0) \neq 0$. Then there exist numbers $m_0 \in N$ and $k_0 \in \{0,1,\dots,2^l-1\}$ such that, starting from the number $m \geqslant m_0$, the following estimates*

$$\frac{C_1}{\mu_{2^l m_0-k_0}} \leqslant \left|g_{2^l m_0-k_0}(t_0)\right| \leqslant \frac{C_2}{\mu_{2^l m_0-k_0}}, \tag{40}$$

where constants C_1 and $C_2 > 0$ depend on m_0, k_0 and t_0, are satisfied.

LEMMA 7. *Let $0 < \alpha < 1$, $g(t) \in C^1[0,T]$ and $g(0) \leqslant 0$. Then there exist numbers $m_0 > 0$, $k_0 \in 0,1,\dots,2^{l-1}$ and n_0 such that, for all $t_0 \leqslant n_0$ and $m \geqslant m_0$, the following estimates*

$$\frac{C_1}{\mu_{2^l m_0-k_0}} \leqslant \left|g_{2^l m_0-k_0}(t_0)\right| \leqslant \frac{C_2}{\mu_{2^l m_0-k_0}}, \tag{41}$$

where constants C_1 and $C_2 > 0$ depend on n_0 and m_0, are satisfied.

COROLLARY 5. *If conditions of Lemma 6 are satisfied, then estimate (40) holds for all $m,k \in K_1$.*

COROLLARY 6. *If conditions of Lemma 6 are satisfied, then the set $K_{0,1}$ has a finite number of elements.*

COROLLARY 7. *If conditions of Lemma 7 are satisfied, then estimate (41) holds for all $t_0 \leqslant n_0$ and $m,k \in K_{0,\alpha}$.*

COROLLARY 8. *If conditions of Lemma 7 are satisfied and t_0 is sufficiently small, then the set $K_{0,\alpha}$ has a finite number of elements.*

THEOREM 4. *Let $g(t) \in C^1[0,T]$, functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 3 for $p = [n/2]+2$. Further, let us assume that for $\alpha = 1$ the conditions of Lemma 6 are satisfied, and for $0 < \alpha < 1$ the conditions of Lemma 7 are satisfied and t_0 is sufficiently small.*

1) If the set $K_{0,\alpha}$ is empty, i.e. $g_{2^l m-k}(t_0) \neq 0$ for all $m \in N$, $k \in \{0,1,\ldots,2^l-1\}$, then there is a unique solution to the inverse problem (1)- (4) which is represented in the form of series (37) and (38);

2) If the set $K_{0,\alpha}$ is not empty, then for the existence of a solution to the inverse problem it is necessary and sufficient that the following conditions be satisfied:

$$\psi_{2^l m-k} = \varphi_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0^{\alpha}\right), \quad m,k \in K_1 \tag{42}$$

If a solution to problem (1)–(4) exists, then it is not unique:

$$f(x) = \sum_{m,k \in K_\alpha} \frac{1}{g_{2^l m-k}(t_0)} \left[\psi_{2^l m-k} - \varphi_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t_0\right)\right] v_{2^l m-k}(x)$$

$$+ \sum_{m,k \in K_\alpha} \sum f_{2^l m-k} v_{2^l m-k}(x),$$

$$u(t,x) = \sum_{m=1}^{\infty} \sum_{k=0}^{2^l-1} \left[\varphi_{2^l m-k} E_{\alpha,1}\left(-\lambda_{2^l m-k} t^{\alpha}\right) + f_{2^l m-k}\right] v_{2^l m-k}(x),$$

where $f_{2^l m-k}, m,k \in K_{0,\alpha}$ are arbitrary real numbers.

Proof. In the first case, the theorem is proved in exactly the same way as in the case of Theorem 2. Let us consider the second case. If $m,k \in K_\alpha$, then from equality (30) we get (32) and (33). If $m,k \in K_{0,\alpha}$, i.e. $g_{2^l m-k}(t_0) = 0$, then a solution to equation (30) with respect to $f_{2^l m-k}$ exists if and only if conditions (42) are satisfied. In this case, the solution to equation (30) can be arbitrary numbers $f_{2^l m-k}$. As shown above (see Corollaries 6 and 8), under the conditions of the theorem, the set $K_{0,\alpha}$, $\alpha \in (0,1]$ contains a finite number of elements. The theorem is proved. □

6. Conclusion

In this paper, the solvability of some inverse problems for a nonlocal analogue of a parabolic equation is studied. The nonlocal operator is introduced using involutive mappings.

Unlike the previous works of the authors, in this work the problems are studied in the n-dimensional case. Solutions to the main problems are constructed in the form of series using the completeness of the system of eigenfunctions of the nonlocal Laplace operator. The results of this work can be generalized to the case of high-order equations.

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Batirkhan Turmetov
Department of Mathematics
Khoja Akhmet Yassawi International Kazakh-Turkish University
pr. B. Sattarkhanova, 29, Turkistan, 160200, Kazakhstan
e-mail: batirkhan.turmetov@ayu.edu.kz

Maira Koshanova
Department of Mathematics
Khoja Akhmet Yassawi International Kazakh-Turkish University
pr. B. Sattarkhanova, 29, Turkistan, 160200, Kazakhstan
e-mail: maira.koshanova@ayu.edu.kz

Moldir Muratbekova
Department of Mathematics
Khoja Akhmet Yassawi International Kazakh-Turkish University
pr. B. Sattarkhanova, 29, Turkistan, 160200, Kazakhstan
e-mail: moldir.muratbekova@ayu.edu.kz

Isabek Orazov
Department of Mathematics
Khoja Akhmet Yassawi International Kazakh-Turkish University
pr. B. Sattarkhanova, 29, Turkistan, 160200, Kazakhstan
e-mail: isabek.orazov@ayu.edu.kz