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STURM–LIOUVILLE AND BOUNDARY VALUE PROBLEMS IN NABLA DISCRETE FRACTIONAL CALCULUS

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Abstract. We consider a novel discrete nabla fractional self-adjoint operator L in the Caputo sense. We demonstrate basic properties of boundary-value problems under L, including explicit solutions. Furthermore, we consider Sturm-Liouville problems under L and prove eigenvalue results analogous to continuous classical Sturm-Liouville systems.

1. Introduction

Time and again, Sturm-Liouville theory emerges as an indispensable tool of applied mathematics among other fields of science. Originating in the mid-1800s, Sturm-Liouville theory quickly found its way into classical mathematics, for example, in the analysis of certain partial differential equations. In modern times, Sturm-Liouville theory has been used to describe quantum mechanical states [10], [16] and has influenced the study of self-adjoint operators and spectral theory for linear differential equations [8].

Fractional calculus generalizes (and unites) the integral and differential operators to non-integer order [15]; from its inception until the 21st century, it has found sporadic application. In the last two decades, however, modern technical advances have allowed for the substantial application of fractional calculus in scientific modeling and engineering, in fields such as physics and bioengineering [13], [19], [22].

Recently, fractional calculus concepts have found their way into the calculus of differences. See [17] for a self-contained introduction to difference calculus over several operators, including their fractional extensions. In this work, we study the *nabla (backwards) difference operator*, and in particular, the *nabla Caputo fractional difference*. Nabla fractional calculus has been developed largely in the last decade by authors such as, but not limited to, Anastassiou, Atici and Eloe, and Hein et al. [9], [11], [18]. The nabla Caputo fractional difference in particular has been of interest to [6], [20], [23].

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Initiative work in fractional Sturm-Liouville problems involving a continuous fractional order derivative is done in [7]. Sturm-Liouville Equations in the context of fractional and discrete fractional operators are studied in the recent works [4] and [5]. Our self-adjoint operator of study is inspired by the analogue in the classical ordinary differential equations setting given by $Lx(t) = (p(t)x'(t))' + q(t)x(t)$, where p,q are continuous on some interval I and $p(t) > 0$ on I [21].

Section 2 details preliminary definitions and theorems relating to discrete fractional calculus, culminating with a recap of useful theorems regarding the Caputo and Riemann-Liouville differences and their interplay. In Section 3, we define the *nabla self-adjoint operator* L:

$$(Lx)(t) = {}^{C}\nabla_a^{\alpha}[p(t)_b^C\nabla^{\alpha}x(t)] + q(t)x(t). \tag{1}$$

We then derive a novel integration by parts formula for the Caputo operator and prove some basic properties of L, including that it is self-adjoint. In Section 4, we derive the solution to a class of simple boundary-value problems under L. Finally, in Section 5, we develop nabla discrete fractional Sturm-Liouville theory under L and prove behavioral characteristics of the Sturm-Liouville eigenvalue problem,

$$(Lx)(t) = \lambda\omega(t)x(t). \tag{2}$$

2. Preliminaries

DEFINITION 1. ([1], [17]) Let $a,b \in \mathbb{R}$ such that $b-a \in \mathbb{N}$. Then, $\mathbb{N}_a$ is given by $\{a, a+1, a+2, \ldots\}$, and ${}_b\mathbb{N}$ is given by $\{\ldots, b-2, b-1, b\}$. Furthermore, $\mathbb{N}_a^b$ is given by $\{a, a+1, \ldots, b-1, b\}$.

DEFINITION 2. ([17]) We define the *generalized rising factorial* to be

$$t^{\overline{\alpha}} := \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \tag{3}$$

for $\alpha \in \mathbb{R}$ and $t \in \mathbb{C} - \{\ldots, -2, -1, 0\}$. As a convention, we define $0^{\overline{\alpha}} := 0$.

DEFINITION 3. ([1], [17]) Let $f : \mathbb{N}_a \to \mathbb{C}$ and $g : {}_b\mathbb{N} \to \mathbb{C}$ where $b-a \in \mathbb{N}$. We define the *nabla difference of f* by

$$(\nabla f)(t) := f(t) - f(t-1), \tag{4}$$

for $t \in \mathbb{N}_{a+1}$. Furthermore, we define the *delta difference of g* by

$$(\Delta g)(t) := g(t+1) - g(t), \tag{5}$$

for $t \in {}_{b-1}\mathbb{N}$. We will use the notation that $\nabla f(t) := (\nabla f)(t)$ and $\Delta g(t) := (\Delta g)(t)$, and we will extend these conventions to similar operators without further note.

We recursively extend these operators to arbitrary integer order n differences as follows:

$$(\nabla^n f)(t) := \nabla(\nabla^{n-1} f)(t), \tag{6}$$

for $t \in \mathbb{N}_{a+n}$, and

$$(\Delta^n g)(t) := \Delta(\Delta^{n-1} g)(t), \tag{7}$$

for $t \in {}_{b-n}\mathbb{N}$. We also define the following useful operators:

$$(\nabla_\theta^n f)(t) := (-1)^n (\nabla^n f)(t), \tag{8}$$

for $t \in \mathbb{N}_{a+n}$, and

$$({}_\theta\Delta^n g)(t) := (-1)^n (\Delta^n g)(t), \tag{9}$$

for $t \in {}_{b-n}\mathbb{N}$.

Finally, by convention, we take that any of the above operators for $n = 0$ is the identity operator.

DEFINITION 4. ([1], [17]) Let $f : {}_{b-1}\mathbb{N} \to \mathbb{C}$, and let $\nu > 0$. The ν^{th}*-order right fractional sum of* f ending at b is

$$ {}_b\nabla^{-\nu} f(t) := \sum_{s=t}^{b-1} \frac{(s+1-t)^{\overline{\nu-1}}}{\Gamma(\nu)} f(s), \tag{10}$$

for $t \in {}_{b-1}\mathbb{N}$,

Let $f : \mathbb{N}_{a+1} \to \mathbb{C}$, and let $\nu > 0$. The ν^{th}*-order left fractional sum of* f *based at* a is

$$\nabla_a^{-\nu} f(t) := \sum_{s=a+1}^{t} \frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)} f(s), \tag{11}$$

for $t \in \mathbb{N}_{a+1}$.

DEFINITION 5. (Caputo Differences [1], [17]) Let $\nu > 0$, $N-1 < \nu \leqslant N$ for some $N \in \mathbb{N}_1$, and let $f : \mathbb{N}_{a-N+1} \to \mathbb{C}$. The ν^{th}*-order left Caputo fractional difference of* f is given by

$${}^C\nabla_a^\nu f(t) := \nabla_a^{-(N-\nu)} \nabla^N f(t), \tag{12}$$

for $t \in \mathbb{N}_{a+1}$. Let $\nu > 0$, $N-1 < \nu \leqslant N$ for some $N \in \mathbb{N}_1$, and let $f : {}_{b+N-1}\mathbb{N} \to \mathbb{C}$. The ν^{th}*-order right Caputo fractional difference of* f is given by

$${}_b^C\nabla^\nu f(t) := {}_b\nabla^{-(N-\nu)} {}_\theta\Delta^N f(t), \tag{13}$$

for $t \in {}_{b-1}\mathbb{N}$.

REMARK 1. As an alternative equivalent way to view the Caputo differences above, one may choose to shift the domain of the differences instead of f's domain; i.e., in (12), the difference can instead be based to start at $a+N-1$ (so the difference is defined on $\mathbb{N}_{a+N}$), and similarly the difference operator in (13) to end at $b-N+1$ (so the difference is defined on ${}_{b-N}\mathbb{N}$). This would mean the function f would be defined instead on $\mathbb{N}_a$ in the first case and ${}_b\mathbb{N}$ in the second case. For example, see Definition 5.1 in [1]. These two views coincide when $0 < \nu \leqslant 1$.

In this work, we are primarily concerned with the Caputo fractional difference; however, the similarly defined Riemann-Liouville fractional differences will also play a role in our proofs, so we define them here.

DEFINITION 6. (Riemann-Liouville Differences [1], [17]) Let $\nu > 0, N-1 < \nu \leqslant N$ for some $N \in \mathbb{N}_1$, and let $f : \mathbb{N}_{a-N+1} \to \mathbb{C}$. The ν^{th}*-order left Riemann-Liouville fractional difference of* f is given by

$$\nabla_a^{\nu} f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t), \tag{14}$$

for $t \in \mathbb{N}_{a+1}$. Let $\nu > 0$, $N-1 < \nu \leqslant N$ for some $N \in \mathbb{N}_1$, and let $f : {}_{b+N-1}\mathbb{N} \to \mathbb{C}$. The ν^{th}*-order right Riemann-Liouville fractional difference of* f is given by

$${}_b\nabla^{\nu} f(t) := {}_{\theta}\Delta_b^N \nabla^{-(N-\nu)} f(t), \tag{15}$$

for $t \in {}_{b-1}\mathbb{N}$.

We now give relevant theorems concerning the fractional difference operators.

THEOREM 1. ([1], [2], [3]) *For any* $0 < \alpha < 1$ *and* f *defined on suitable domains* $\mathbb{N}_a$ *and* ${}_b\mathbb{N}$, *we have the conversion identity*

$${}^C\nabla_a^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \frac{(t-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a), \quad \textit{for } t \in \mathbb{N}_{a+1}, \tag{16}$$

$${}_b^C\nabla^{\alpha} f(t) = {}_b\nabla^{\alpha} f(t) - \frac{(b-t)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b), \quad \textit{for } t \in \mathbb{N}^{b-1}. \tag{17}$$

THEOREM 2. ([1], [2]) *Let* $\alpha > 0$ *be noninteger and* $a, b \in \mathbb{R}$ *such that* $b-a \in \mathbb{N}$. *If* f *and* g *are defined appropriately on* $\mathbb{N}_a^b$, *then*

$$\sum_{s=a+1}^{b-1} f(s) \nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} g(s) {}_b\nabla^{\alpha} f(s). \tag{18}$$

THEOREM 3. ([1], [2]) *For* $\alpha > 0$, *and* f *defined on a suitable domain* ${}_b\mathbb{N}$, *one has*

$${}_b\nabla^{\alpha} {}_b\nabla^{-\alpha} f(t) = f(t), \tag{19}$$

$${}_b\nabla^{-\alpha} {}_b\nabla^{\alpha} f(t) = f(t), \quad \textit{when } \alpha \notin \mathbb{N}, \tag{20}$$

$${}_b\nabla^{-\alpha} {}_b\nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k}}}{k!} {}_a\Delta^k f(b), \quad \textit{when } \alpha = n \in \mathbb{N}. \tag{21}$$

3. The nabla self-adjoint operator L

DEFINITION 7. Let a,b be real numbers such that $b-a\in\mathbb{N}$. Then, we define V as the Hermitian inner product space of complex, $(b-a+1)$-tuples x indexed as follows:

$$\big(x(a),x(a+1),\dots,x(b-1),x(b)\big), \tag{22}$$

equipped with the inner product,

$$\langle x,y\rangle := \sum_{t=a}^{b} x(t)\overline{y(t)}, \tag{23}$$

where $x,y\in V$ and the weighted inner product

$$\langle x,y\rangle_\omega := \sum_{t=a}^{b} x(t)\overline{y(t)}\omega(t), \tag{24}$$

where ω is a real-valued weight function, and $\omega(t)\geqslant 0$.

REMARK 2. Note that since ω is real-valued, $\langle x,y\rangle_\omega = \langle x,\omega y\rangle = \langle \omega x,y\rangle$.

Now we define a nabla self-adjoint operator L, which is henceforth the main operator of study.

DEFINITION 8. For any function $x:\mathbb{N}_a^b\to\mathbb{R}$, let L be the operator

$$(Lx)(t) = {}^{C}\nabla_a^{\alpha}[p(t)\,{}_b^{C}\nabla^{\alpha}x(t)] + q(t)x(t) \tag{25}$$

for $t\in\mathbb{N}_{a+1}^{b-1}$, $0<\alpha<1$, and p,q real-valued functions with $p(t)\neq 0$ for all $t\in\mathbb{N}_{a+1}^{b-1}$.

REMARK 3. L is a linear operator.

We now define the set U, a subset of V, which enforces boundary conditions on the elements.

DEFINITION 9. Let a,b be real numbers such that $b-a\in\mathbb{N}$. Then,

$$U := \{x\in V : {}_b^{C}\nabla^{\alpha}x(a) = 0 \text{ and } x(b)=0\}.$$

REMARK 4. Using the linearity of L and the linearity of the boundary conditions (from the linearity of the Caputo difference operator), it can be shown that U is a subspace of V.

3.1. Basic properties of L

First, we show that complex conjugates distribute over Caputo differences.

LEMMA 1. *Let x^* denote the complex conjugate of x. Furthermore, let $\alpha > 0$ be given such that $N-1 < \alpha \leqslant N$, for some $N \in \mathbb{N}_1$, and let $f : \mathbb{N}_{a-N+1} \to \mathbb{C}$. Then*

$$\left({}^C\nabla_a^\alpha f(t)\right)^* = {}^C\nabla_a^\alpha\left(f(t)^*\right), \tag{26}$$

for $t \in \mathbb{N}_{a+1}$.

Proof. Let $\alpha > 0$ be given such that $N-1 < \alpha \leqslant N$, for some $N \in \mathbb{N}_1$, and let $f : \mathbb{N}_{a-N+1} \to \mathbb{C}$. We then obtain from definitions and complex arithmetic,

$$\begin{aligned}
\left({}^C\nabla_a^\alpha f(t)\right)^* &= \left(\nabla_a^{-(N-\alpha)}\nabla^N f(t)\right)^* \\
&= \left(\sum_{s=a+1}^{t} \frac{(t-s+1)^{\overline{N-\alpha-1}}}{\Gamma(N-\alpha)}\left(\nabla^N f(s)\right)\right)^* \\
&= \left(\sum_{s=a+1}^{t} \frac{(t-s+1)^{\overline{N-\alpha-1}}}{\Gamma(N-\alpha)}\left(\nabla^N f(s)\right)^*\right), \qquad \text{(i)}
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$. Thus it remains to show that $\left(\nabla^N f(t)\right)^* = \left(\nabla^N f(t)^*\right)$. We show this via induction; observe that when $N = 1$ we have

$$\begin{aligned}
\left(\nabla f(t)\right)^* &= \left(f(t) - f(t-1)\right)^* \\
&= \left(f(t)^* - f(t-1)^*\right) \\
&= \left(\nabla f(t)^*\right),
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$. Now assume, for the inductive hypothesis, that for some $N > 1$

$$\left(\nabla^{N-1} f(t)\right)^* = \left(\nabla^{N-1} f(t)^*\right),$$

for all $t \in \mathbb{N}_{a+1}$. Then, we have

$$\begin{aligned}
\left(\nabla^N f(t)\right)^* &= \left(\nabla(\nabla^{N-1} f(t))\right)^* \\
&= \left(\nabla^{N-1} f(t) - \nabla^{N-1} f(t-1)\right)^* \\
&= \left(\nabla^{N-1} f(t)\right)^* - \left(\nabla^{N-1} f(t-1)\right)^* \\
&= \left(\nabla^{N-1} f(t)^*\right) - \left(\nabla^{N-1} f(t-1)^*\right) \qquad \text{(Ind. Hyp.)} \\
&= \left(\nabla(\nabla^{N-1} f(t)^*)\right) \\
&= \left(\nabla^N f(t)^*\right).
\end{aligned}$$

By plugging this result into (i) from above and reducing the expansion, we obtain the desired identity and the proof is complete. □

THEOREM 4. (Caputo Summation by Parts) *Let* $0 < \alpha < 1$ *and* $a, b \in \mathbb{R}$ *such that* $b-a \in \mathbb{N}_1$. *If* f *is defined on* ${}_b\mathbb{N}$ *and* g *is defined on* $\mathbb{N}_a$, *then*

$$\sum_{s=a+1}^{b-1} f(s)^C\nabla_a^\alpha g(s) = \sum_{s=a+1}^{b-1} \left(g(s)_b^C\nabla^\alpha f(s)\right) + C, \tag{27}$$

where

$$C := \frac{1}{\Gamma(1-\alpha)} \sum_{s=a+1}^{b-1} \left[g(s)(b-s)^{\overline{-\alpha}} f(b) - f(s)(s-a)^{\overline{-\alpha}} g(a)\right].$$

Proof. Let $0 < \alpha < 1$ and $a, b \in \mathbb{R}$ be given such that $b-a \in \mathbb{N}$. Also let f be defined on ${}_b\mathbb{N}$ and g be defined on $\mathbb{N}_a$. We have

$$\begin{aligned}\sum_{s=a+1}^{b-1} f(s)\nabla_a^\alpha g(s) &= \sum_{s=a+1}^{b-1} f(s)\left({}^C\nabla_a^\alpha g(s) + \frac{(s-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} g(a)\right) && \text{(Theorem 1)}\\ &= \sum_{s=a+1}^{b-1} f(s)^C\nabla_a^\alpha g(s) + \sum_{s=a+1}^{b-1} f(s)\frac{(s-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} g(a). && (*)\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{s=a+1}^{b-1} g(s)_b\nabla^\alpha f(s) &= \sum_{s=a+1}^{b-1} g(s)\left({}_b^C\nabla^\alpha f(s) + \frac{(b-s)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b)\right) && \text{(Theorem 1)}\\ &= \sum_{s=a+1}^{b-1} g(s)_b^C\nabla^\alpha f(s) + \sum_{s=a+1}^{b-1} g(s)\frac{(b-s)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(b). && (**)\end{aligned}$$

By Theorem 2, we have $\sum_{s=a+1}^{b-1} f(s)\nabla_a^\alpha g(s) = \sum_{s=a+1}^{b-1} g(s)_b\nabla^\alpha f(s)$. Therefore we may equate $(*)$ and $(**)$ and obtain the desired identity. □

COROLLARY 1. *When* $f(b) = g(a) = 0$, *the Caputo summation by parts formula yields the identity*

$$\sum_{s=a+1}^{b-1} f(s)^C\nabla_a^\alpha g(s) = \sum_{s=a+1}^{b-1} g(s)_b^C\nabla^\alpha f(s). \tag{28}$$

REMARK 5. It can be shown using definitions of the fractional differences and Theorem 1 that the Caputo Summation by Parts formula given above is equivalent to the one given in Theorem 6.7 of [1], which is formulated in terms of the Riemann-Liouville difference.

THEOREM 5. *If* $f, g \in U$, *then*

$$\langle Lf, g\rangle = \langle f, Lg\rangle; \tag{29}$$

i.e., L is a self-adjoint operator.

Proof. Let $f,g \in U$. We have

$$\begin{aligned}
\langle f, Lg\rangle &= \sum_{t=a+1}^{b-1} f(t)\left({}^{C}\nabla_a^{\alpha}[p(t){}_b^C\nabla^{\alpha} g(t)] + q(t)g(t)\right)^{*} \\
&= \sum_{t=a+1}^{b-1} f(t){}^{C}\nabla_a^{\alpha}\left[p(t){}_b^C\nabla^{\alpha}\left(g(t)^{*}\right)\right] + \sum_{t=a+1}^{b-1} q(t)f(t)\left(g(t)^{*}\right) \\
&= \sum_{t=a+1}^{b-1} \left[p(t){}_b^C\nabla^{\alpha}\left(g(t)^{*}\right)\right]{}_b^C\nabla^{\alpha} f(t) + C_1 + \sum_{t=a+1}^{b-1} q(t)f(t)\left(g(t)^{*}\right) \\
&= \sum_{t=a+1}^{b-1} \left[p(t){}_b^C\nabla^{\alpha} f(t)\right]{}_b^C\nabla^{\alpha}\left(g(t)^{*}\right) + C_1 + \sum_{t=a+1}^{b-1} q(t)f(t)\left(g(t)^{*}\right) \\
&= \sum_{t=a+1}^{b-1} \left(g(t)^{*}\right){}^{C}\nabla_a^{\alpha}\left[p(t){}_b^C\nabla^{\alpha} f(t)\right] + C_1 - C_2 + \sum_{t=a+1}^{b-1} q(t)f(t)\left(g(t)^{*}\right) \\
&= (C_1 - C_2) + \sum_{t=a+1}^{b-1} \left(g(t)^{*C}\nabla_a^{\alpha}\left[p(t){}_b^C\nabla^{\alpha} f(t)\right] + q(t)f(t)\left(g(t)^{*}\right)\right) \\
&= (C_1 - C_2) + \sum_{t=a+1}^{b-1} \left({}^{C}\nabla_a^{\alpha}\left[p(t){}_b^C\nabla^{\alpha} f(t)\right] + q(t)f(t)\right)\left(g(t)^{*}\right) \\
&= (C_1 - C_2) + \sum_{t=a+1}^{b-1} \left({}^{C}\nabla_a^{\alpha}\left[p(t){}_b^C\nabla^{\alpha} f(t)\right] + q(t)f(t)\right)\left(g(t)^{*}\right) \\
&= (C_1 - C_2) + \langle Lf, g\rangle.
\end{aligned}$$

If $C_1 - C_2 = 0$, then the proof is complete.

By Theorem 4, we have that in the first application of Caputo summation by parts, $f(t)$ plays the role of $f(s)$, and $p(t){}_b^C\nabla^{\alpha}\left(g(t)^{*}\right)$ plays the role of $g(s)$. In C_1 given by the formula in Theorem 4, the first term is multiplied by $f(b) = 0$ and the second by

$$\begin{aligned}
g(a) &= p(a){}_b^C\nabla^{\alpha}\left(g(a)^{*}\right) \\
&= p(a)\left({}_b^C\nabla^{\alpha} g(a)\right)^{*} \\
&= 0.
\end{aligned}$$

Thus, $C_1 = 0$, as $f,g \in U$. We can use a similar analysis for C_2, however we are now working backwards in the Caputo summation by parts formula (rearrange to see how f and g are "switched"): $\left[p(t){}_b^C\nabla^{\alpha} f(t)\right]$ plays the role of $g(s)$, and $\left(g(t)^{*}\right)$ plays the role of $f(s)$. We have $\left(g(b)^{*}\right) = 0^{*} = 0$, and $\left[p(a){}_b^C\nabla^{\alpha} f(a)\right] = 0$. Thus $C_2 = 0$, and the proof is complete. □

4. Boundary value problems under L

4.1. General solution to non-autonomous problems

Consider, for $0<\alpha<1$, the boundary-value problem,

$$\begin{cases} {}^{C}\nabla_a^{\alpha}[p(t){}_b^C\nabla^{\alpha}x(t)] = h(t), & \text{for } t\in\mathbb{N}_{a+1}^{b-1},\\ {}_b^C\nabla^{\alpha}x(a)=0,\\ x(b)=0. \end{cases}$$

We will find a $x(t)$. First, call $y(t)=p(t){}_b^C\nabla^{\alpha}x(t)$. We define the following IVP as a sub-problem:

$$\begin{cases} {}^{C}\nabla_a^{\alpha}y(t) = h(t), & \text{for } t\in\mathbb{N}_{a+1}^{b-1},\\ {}_b^C\nabla^{\alpha}y(a)=0. \end{cases}$$

Since $y(t)=p(t){}_b^C\nabla^{\alpha}x(t)$, we have $y(a)=p(a)({}_b^C\nabla^{\alpha}x(a))=0$ by the boundary value condition in the original problem. So the sub-problem is equivalent to

$$\begin{cases} {}^{C}\nabla_a^{\alpha}y(t) = h(t), & \text{for } t\in\mathbb{N}_{a+1}^{b-1},\\ y(a)=0. \end{cases}$$

The solution to this IVP is $y(t)=\nabla_a^{-\alpha}h(t)$ by Theorem 3.120 in [17]. Since we defined $y(t)=p(t){}_b^C\nabla^{\alpha}x(t)$, we have $y(t)=p(t){}_b^C\nabla^{\alpha}x(t)=\nabla_a^{-\alpha}h(t)$, or ${}_b^C\nabla^{\alpha}x(t)=\frac{1}{p(t)}\nabla_a^{-\alpha}h(t)$.

Now we introduce the other boundary condition $x(b)=0$ and solve

$$\begin{cases} {}_b^C\nabla^{\alpha}x(t) = \frac{1}{p(t)}\nabla_a^{-\alpha}h(t), & \text{for } t\in\mathbb{N}_{a+1}^{b-1}\\ x(b)=0. \end{cases}$$

By Theorem 1, since $x(b)=0$, this is equivalent to

$$\begin{cases} {}_b\nabla^{\alpha}x(t) = \frac{1}{p(t)}\nabla_a^{-\alpha}h(t), & \text{for } t\in\mathbb{N}_{a+1}^{b-1}\\ x(b)=0, \end{cases}$$

which, by Theorem 3, has a solution of the form

$$x(t)={}_b\nabla^{-\alpha}\left[\frac{1}{p(t)}\nabla_a^{-\alpha}h(t)\right].$$

EXAMPLE 1. Solve the following boundary-value problem.

$$\begin{cases} {}^{C}\nabla_0^{0.5}\left({}_{10}^{C}\nabla^{0.5}x(t)\right) = 1, & \text{for } t\in\mathbb{N}_1^9,\\ {}_{10}^C\nabla^{\alpha}x(0)=0,\\ x(10)=0. \end{cases}$$

SOLUTION. According to the general solution for non-autonomous problems above, the general solution to this boundary value problem is

$$x(t) = \frac{1}{\pi} \sum_{s=t}^{9} \frac{\Gamma(s-t+0.5)(2s-1)\Gamma(s-0.5)}{\Gamma(s-t+1)\Gamma(s)}.$$

5. Sturm-Liouville problems under L

In this section we extend the work of Bas and Ozarslan [14] to develop the theory of discrete fractional Sturm-Liouville theory in the Caputo case. Recall that we have constructed the set U containing sequences $x(t)$ which satisfy our boundary conditions, so that L is self-adjoint on elements of U. We now define the nabla discrete fractional Sturm-Liouville (NDFSL) eigenvalue problem as follows:

DEFINITION 10. (NDFSL Eigenvalue Problem) Suppose $x \in U$ is an unknown function defined on $\mathbb{N}_a^b$, for $a,b \in \mathbb{R}$ such that $b-a \in \mathbb{N}$, and ω is a real-valued weight function with $\omega(t) \geqslant 0$. Then, the NDFSL eigenvalue problem is given by the equation

$$(Lx)(t) = \lambda\omega(t)x(t). \tag{30}$$

THEOREM 6. *All eigenvalues to the nabla discrete fractional Sturm-Liouville equation are real.*

Proof. Let $f \in U$ be a nontrivial solution to (30). The result follows due to the self-adjointness of L together with conjugate symmetry of our inner product:

$$\begin{aligned} &\langle Lf, f\rangle = \langle f, Lf\rangle \\ &\implies \langle \lambda\omega f, f\rangle = \langle f, \lambda\omega f\rangle \\ &\implies \lambda\langle \omega f, f\rangle = \overline{\lambda}\langle f, \omega f\rangle \\ &\implies (\lambda-\overline{\lambda})\langle f, f\rangle_\omega = 0. \end{aligned}$$

Since $\langle f, f\rangle_\omega$ is nonzero due to positive definiteness, $\lambda = \overline{\lambda}$, and the result follows. □

THEOREM 7. *Eigenfunctions corresponding to distinct eigenvalues of the NDFSL equation are orthogonal with respect to the weighted inner product.*

Proof. Since L is self-adjoint, we have for any $f,g \in U$ which are solutions to (30),

$$\langle Lf, g\rangle = \langle f, Lg\rangle \implies \langle \lambda_f\omega f, g\rangle = \langle f, \lambda_g\omega g\rangle.$$

By the properties of our inner product, we have

$$\lambda_f\langle \omega f, g\rangle = \overline{\lambda_g}\langle f, \omega g\rangle,$$

but Theorem 6 guarantees all eigenvalues are real and $\langle \omega f, g\rangle = \langle f, \omega g\rangle = \langle f, g\rangle_\omega$, so we have

$$(\lambda_f - \lambda_g)\langle f, g\rangle_\omega = 0.$$

Since eigenvalues are distinct, we necessarily have $\langle f, g\rangle_\omega = 0$. □

If we define the subspace W of U as follows, then we get that the operator L maps W to W, which will allow us to make an existence claim regarding the eigenvalue problem, so that the previous theorems given in this section are not vacuous.

DEFINITION 11. Let U be as defined previously in Definition 9, and a, b be real numbers such that $b-a \in \mathbb{N}_3$. Then we define

$$W := \{x \in U : {}_b^C\nabla^\alpha (Lx)(a) = 0 \text{ and } (Lx)(b) = 0\}.$$

Then, as a consequence of W being a nontrivial subspace of $V \cong \mathbb{C}^{b-a+1}$ we obtain an existence result about the NDFSL eigenvalue problem. The fact that W is nontrivial can be verified by observing that the boundary conditions ${}_b^C\nabla^\alpha (Lx)(a) = 0$, $(Lx)(b) = 0$, ${}_b^C\nabla^\alpha x(a) = 0$, and $x(b) = 0$ amount to four linear equations in $b-a+1$ unknowns.

LEMMA 2. *Eigenvalues and corresponding eigenfunctions always exist to the NDFSL problem.*

Proof. It is straightforward that L is a linear operator. Given this and the fact that W is a nontrivial finite-dimensional complex vector space, it follows that eigenvalues and eigenfunctions necessarily exist ([12]). □

6. Conclusion and future work

We have established analogues to some classical results in self-adjoint ordinary differential equations for our self-adjoint operator involving nabla Caputo fractional differences. We proved that eigenvalues to the NDFSL are real and the eigenfunctions are orthogonal. We also established the existence of solutions to the NDFSL problem. Similiar results in a continuous fractional context are developed, for example, in [7]. Our results lay the groundwork for future study and results analogous to those in ordinary differential equations and can be explored in various directions. For example, further analysis of eigenvalues and eigenfunctions and their properties may be studied, perhaps by considering specific conditions on $p(t)$ or $q(t)$. For example, in [14], the case of $p(t) \equiv 1$ is explored in further detail with the form of an explicit solution. Results for operators of higher order versions of our operator ($\alpha > 1$), may also be explored, such as those developed in [6].

REFERENCES

[1] T. ABDELJAWAD, *On Delta and Nabla Caputo Fractional Differences and Dual Identities*, Discrete Dyn. Nature Soc. **2013**, (2013), 1–12.

[2] T. ABDELJAWAD AND F. M. ATICI, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal. **2012**, (2012).

[3] T. ABDELJAWAD, *Dual identities in fractional difference calculus within Riemann*, Adv. Difference Equ. **2013**, 36, (2013).

[4] T. ABDELJAWAD, R. MERT, AND A. PETERSON, *Sturm Liouville Equations in the frame of fractional operators with exponential kernels and their discrete versions*, Quaestiones Mathematicae **42**, 9, (2019), 1271–1289.

[5] T. ABDELJAWAD AND M. AL-REFAI, *Fundamental Results of Conformable Sturm-Liouville Eigenvalue Problems*, Wiley/Hindawi **2017**, (2017).

[6] K. AHRENDT AND C. KISSLER *Green's Function for Higher-Order Boundary Value Problems Involving a Nabla Caputo Fractional Operator*, J. Differ. Equ. **25**, 6, (2019), 788-800.

[7] Q. M. AL-MDALLAL *On the numerical solution of fractional Sturm-Liouville problems*, International Journal of Computer Mathematics **87**, 12, (2010), 2837–2845.

[8] W. O. AMREIN, A. M. HINZ, AND D. B. PEARSON, *Sturm-Liouville Theory: Past and Present*, Birkhäuser, 2005.

[9] G. A. ANASTASSIOU, *Nabla Discrete Fractional Calculus and Nabla Inequalities*, Math. Comput. Model. **51**, 5–6 (2010), 562–571.

[10] G. B. ARFKEN AND H. J. WEBER, *Mathematical Methods for Physicists*, Elsevier, 2005.

[11] F. M. ATICI AND P. ELOE *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc. **137**, (2009), 981–989.

[12] S. AXLER, *Linear Algebra Done Right*, Springer, 2015.

[13] R. L. BAGLEY AND P. J. TORVIK, *A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity*, J. Rheol. **27**, 1, (1983).

[14] E. BAS AND R. OZARSLAN, *Theory of discrete fractional Sturm-Liouville equations and visual results*, AIMS Math. **4**, (2019), 593–612.

[15] A. CARPINTERI et al., *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, 1997.

[16] E. GERCK et al., *Solution of the Schrödinger equation for bound states in closed form*, Phys. Rev. A. **26**, (1982), 662–664.

[17] C. GOODRICH AND A. C. PETERSON, *Discrete Fractional Calculus*, Springer, (2015).

[18] J. HEIN et al., *Laplace Transforms for the Nabla Difference Operator*, PanAmer. Math. J. **23**, (2011), 79–96.

[19] R. HILFER, *Applications of fractional calculus in physics*, World Scientific, 2000.

[20] A. IKRAM *Green's Functions and Lyapunov Inequalities for Nabla Caputo Boundary Value Problems*, University of Nebraska-Lincoln, (2018).

[21] W. G. KELLEY AND A. C. PETERSON, *The Theory of Differential Equations*, Springer, (2010).

[22] R. L. MAGIN, *Fractional Calculus in Bioengineering*, Begell House, 2006.

[23] J. ST. GOAR, *A Caputo Boundary Value Problem in Nabla Fractional Calculus*, University of Nebraska-Lincoln, (2016).

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