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NEW FRACTIONAL INTEGRAL EXTENSIONS FOR INEQUALITIES INVOLVING MONOTONE FUNCTIONS

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Abstract. This paper extends classical results on integral inequalities involving monotone functions to the domain of Riemann-Liouville fractional integrals with positive arbitrary order α. By employing a unified framework, our approach provides a more generalized understanding of the interplay between monotonicity and integrability in the case of fractional integration. We review classical results, introduce Riemann-Liouville integrals, and establish the fractional integral extensions. Our main results are presented, with discussions on their applications, contributing to a broader comprehension of this type of inequalities in mathematical analysis and its applications.

1. Introduction

In the realm of mathematical analysis, inequalities involving monotone functions have been a focal point of investigation due to their significance in various branches of mathematics and applications. Classical results in this domain have laid the groundwork for understanding the behavior of these functions under certain conditions [3, 4, 6, 8, 9, 10, 12]. The extension of these results to the realm of Riemann-Liouville integrals with positive arbitrary order α introduces a novel perspective, unlocking a richer understanding of the interplay between monotonicity and integrability, see for instance [1, 2, 11]. Let us recall some results on the work of [7] that have inspired the present paper. In Theorem 1 of this reference, it was proved that for any positive function $f \in C^1([a,b])$, such that f' does not vanish over $]a,b[$ and for any positive function $w \in C([a,b[)$, one has

$$\int_a^x (x-t)^{n-1} f^{\sigma}(t) \left(\frac{f^2(a)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)dt < \frac{2f^{\sigma+2}(a)}{\sigma(\sigma+2)} \int_a^x (x-t)^{n-1} w(t)dt,$$

provided that $\sigma \in \mathbb{R} \setminus \{0,-2\},\ n = 0,1,2,3,\dots$.

In theorem 2, the authors proved that for any positive function $f \in C^1([a,b])$, such that its first derivative f' does not vanish over $]a,b[$ and for any positive function $w \in C([a,b[)$, the inequality is valid for any integer n and for any $x \in]a,b[$

$$\int_x^b (t-x)^{n-1} f^{\sigma}(t) \left(\frac{f^2(b)}{\sigma} - \frac{f^2(t)}{\sigma+2} \right) w(t)dt < \frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)} \int_x^b (t-x)^{n-1} w(t)dt,$$

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provided that $\sigma \in \mathbb{R} \setminus \{0,-2\}$.

Other integrals results with some important applications were also established in [7]. Motivated by [5, 7] and some of the above cited papers, in this work, we delve into the use of Riemann-Liouville integrals to broaden the scope of classical results on inequalities involving monotone functions. The extension to positive arbitrary order α presents an intriguing challenge and offers an opportunity to explore the nuanced behavior of these functions in a more general context. Our approach aims to provide a unified framework that encompasses and generalizes some of the classical results in [7], opening avenues for applications in diverse analytical scenarios.

This paper is organized as follows: Section 2 introduces the Riemann-Liouville integrals and establishes the framework for extension. Section 3 presents our main results, followed by some examples in Section 4. We conclude with insights into future directions for research in Section 5.

2. Preliminaries

We begin this section by the following definitions [5]:

DEFINITION 1. The left Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function g on $[a,b]$ is defined as

$$J_a^{\alpha}[g(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g(\tau)d\tau, \quad \alpha > 0, \quad a \leqslant t \leqslant b.$$

DEFINITION 2. The right Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function g on $[a,b]$ is defined as

$$J_b^{\alpha}[g(t)] = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} g(\tau)d\tau, \quad \alpha > 0, \quad a \leqslant t \leqslant b.$$

We need the following properties and particular cases

$$J_a^{\alpha} J_a^{\beta}[g(t)] = J_a^{\alpha+\beta}[g(t)], \quad \alpha \geqslant 0, \quad \beta \geqslant 0,$$

$$J_a^{\alpha} J_a^{\beta}[g(t)] = J^{\beta} J_a^{\alpha}[g(t)],$$

$$J_a^{\alpha} 1 = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad J_a^{\alpha}[b] = \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

3. Main results

THEOREM 1. *Let $f \in C^1([a,b])$ be a positive function such that its first derivative f' does not vanish over the same interval. Suppose also that $w \in C([a,b])$ is a positive function. Then, for every $\alpha \geqslant 0$, we have*

$$J_a^{\alpha}\left(f^{\sigma}(x)\left(\frac{f^2(a)}{\sigma} - \frac{f^2(x)}{\sigma+2}\right)w(x)\right) \leqslant \frac{2f^{\sigma+2}(a)}{\sigma(\sigma+2)} J_a^{\alpha} w(x), \ \ x \in [a,b], \tag{1}$$

provided that $\sigma \in \mathbb{R} \setminus \{0,-2\}$

Proof. Let us introduce the quantity

$$F(t) := -f'(t)f^{\sigma-1}(t)\left(f^2(t)-f^2(a)\right), \quad t\in[a,b]$$

Due to the assumptions on f and f', we can state that:

$$F([a,b]) \subset]-\infty,0].$$

Hence, the inequality

$$\int_a^s F(t)dt \leqslant 0; \quad s\in[a,b].$$

is valid. Consequently,

$$\int_a^s \left(-f'(t)f^{\sigma+1}(t)+f^2(a)f'(t)f^{\sigma-1}(t)\right)dt \leqslant 0. \tag{2}$$

On the other hand, one can observe that

$$\begin{aligned}
&\int_a^s \left(-f'(t)f^{\sigma+1}(t)+f^2(a)f'(t)f^{\sigma-1}(t)\right)dt\\
&=\left[-\frac{f^{\sigma+2}(t)}{\sigma+2}+\frac{f^2(a)f^{\sigma}(t)}{\sigma}\right]_{t=a}^{s}\\
&=-\frac{f^{\sigma+2}(s)}{\sigma+2}+\frac{f^2(a)f^{\sigma}(s)}{\sigma}+\frac{f^{\sigma+2}(a)}{\sigma+2}-\frac{f^{\sigma+2}(a)}{\sigma}\\
&=f^{\sigma}(s)\left(\frac{f^2(a)}{\sigma}-\frac{f^2(s)}{\sigma+2}\right)-\frac{2}{\sigma(\sigma+2)}f^{\sigma+2}(a).
\end{aligned}$$

Hence, thanks to (2), it yields that

$$f^{\sigma}(s)\left(\frac{f^2(a)}{\sigma}-\frac{f^2(s)}{\sigma+2}\right)\leqslant\frac{2}{\sigma(\sigma+2)}f^{\sigma+2}(a). \tag{3}$$

If we multiply (3) by $\frac{(x-s)^{(\alpha-1)}}{\Gamma(\alpha)}w(s)$ and then we integrate over $[a,x]$, where $x\in[a,b]$, we get (1). Theorem 1 is thus proved. □

REMARK 1. The particular case where $\alpha = n,\ n \in \mathbb{N}$ and f is a strictly monotone function has been proved in Theorem 1 of [7].

Suppose now the particular case on the negativity of f' to prove the following theorem.

THEOREM 2. *Let* $f \in C^1([a,b])$. *Assume that* $f'([a,b]) \subset]-\infty, 0]$. *Then, for every* $\alpha \geqslant 2$, *we have*

$$J_a^{\alpha+1} f(x) \geqslant J_a^{\alpha}(x-a)f(x); \quad x \in [a,b]. \tag{4}$$

Proof.

$$H(t) = -(t-a)f'(t);\ \ t \in [a,b].$$

Thanks to the hypothesis on f', we have

$$H([a,b]) \subset [0, +\infty[.$$

Therefore, for every $s \in [a,b]$, we can write

$$\int_a^s H(t)dt \geqslant 0. \tag{5}$$

Integrating the right hand side of (6) by parts, we get

$$\begin{aligned}
\int_a^s H(t)dt &= -\int_a^s (t-a)f'(t)dt \\
&= -\left([(t-a)f(t)]_{t=a}^s - \int_a^s f(t)dt \right) \\
&= -\left((s-a)f(s) - \int_a^s f(t)dt \right) \\
&= -(s-a)f(s) + \int_a^s f(t)dt,
\end{aligned}$$

which implies by (5) that

$$\int_a^s f(t)dt \geqslant (s-a)f(s). \tag{6}$$

Multiplying (6) by $\dfrac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}$ and integrating over $[a,x]$, where $x \in [a,b]$, we obtain

$$\int_a^x \left(\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} \int_a^s f(t)dt \right) ds \geqslant \int_a^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}(s-a)f(s)ds.$$

Thanks to Fubini theorem, we have

$$\int_a^x \left(\frac{f(t)}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1} ds \right) dt \geqslant \int_a^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}(s-a)f(s)ds.$$

That is

$$\int_a^x \frac{(x-t)^{\alpha+1-1}}{\Gamma(\alpha+1)} f(t)dt \geqslant \int_a^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}(s-a)f(s)ds.$$

This ends the proof. □

REMARK 2. The particular case where $\alpha = n,\ n = 2,3,\dots,$ and f is a strictly decreasing function has been proved in Theorem 3 of [7].

In the case where f' is positive, we have to prove the following result.

THEOREM 3. *Let $f \in C^1([a,b])$ and its first derivative is positive over the same interval. Then, for every $\alpha \geqslant 2$, and $x \in [a,b]$, it holds that*

$$J_a^{\alpha+1} f(x) \leqslant J_a^{\alpha}(x-a)f(x).$$

Proof. It is sufficient to apply (4) for $-f$ instead of f. □

Now, using the right Riemann-Liouville integral, we present to the reader the following result.

THEOREM 4. *Let $f \in C^1([a,b])$ be a positive function such that f' does not vanish on $[a,b]$. Suppose also that $w \in C([a,b])$ is a positive function. Then, for every $\alpha \geqslant 0$, we have*

$$J_{b-}^{\alpha} f^{\sigma}(x)\left(\frac{f^2(b)}{\sigma} - \frac{f^2(x)}{\sigma+2}\right)w(x) \leqslant \frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)} J_{b-}^{\alpha} w(x).$$

Proof. Let

$$G(t) = -f'(t)f^{\sigma-1}(t)\left(f^2(b) - f^2(t)\right);\ \ t \in [a,b].$$

The assumptions on f and its first derivative allow us to state that:

$$G([a,b]) \subset]-\infty, 0].$$

Then, for every $s \in [a,b]$, we can write

$$\int_s^b G(t)dt \leqslant 0,$$

which is equivalent to

$$\int_s^b \left(-f'(t)f^{\sigma-1}(t)f^2(b) + f'(t)f^{\sigma+1}(t)\right)dt \leqslant 0. \tag{7}$$

On the other hand, we have

$$\int_s^b \left(-f'(t)f^{\sigma-1}(t)f^2(b) + f'(t)f^{\sigma+1}(t)\right)dt$$
$$= f^{\sigma}(s)\left(\frac{f^2(b)}{\sigma} - \frac{f^2(s)}{\sigma+2}\right) - \frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)},$$

which implies by (7) that

$$f^{\sigma}(s)\left(\frac{f^2(b)}{\sigma}-\frac{f^2(s)}{\sigma+2}\right)\leqslant\frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)},\quad s\in[a,b]. \tag{8}$$

Multiplying (8) by $\frac{(s-x)^{(\alpha-1)}}{\Gamma(\alpha)}w(s)$ and integrating over $[x,b]$, where $x\in[a,b]$, we obtain

$$J_{b^-}^{\alpha}f^{\sigma}(x)\left(\frac{f^2(b)}{\sigma}-\frac{f^2(x)}{\sigma+2}\right)w(x)\leqslant\frac{2f^{\sigma+2}(b)}{\sigma(\sigma+2)}J_{b^-}^{\alpha}w(x). \quad \square$$

REMARK 3. The particular case where $\alpha=n,\ n\in\mathbb{N}$, and f is a strictly monotone function has been proved in Theorem 2 of [7].

We present also the following result.

THEOREM 5. *Let $f\in C^1([a,b])$. Assume that f' is negative over the same interval. Then, for every $\alpha\geqslant 2$, and $x\in[a,b]$, we have*

$$J_{b^-}^{\alpha+1}f(x)\leqslant J_{b^-}^{\alpha}(b-x)f(x).$$

Proof. We consider the following function

$$H(t)=-(b-t)f'(t),t\in[a,b].$$

So, with the same arguments as above, the desired inequality follows. $\square$

THEOREM 6. *Let $f\in C^1([a,b])$ and suppose that f' is positive over the same interval. Then, for every $\alpha\geqslant 2$ and $x\in[a,b]$, we have*

$$J_{b^-}^{\alpha+1}f(x)\geqslant J_{b^-}^{\alpha}(b-x)f(x).$$

Proof. We apply Theorem 5 for the function $-f$ instead of f. $\square$

REMARK 4. The particular case where $\alpha=n,\ n=2,3\ldots$, and f is a strictly positive function has been proved in Theorem 6 of [7].

4. Some examples

We shall estimate some fractional inequalities involving trigonometric functions.

EXAMPLE 1. Taking $f(x)=\cos x,\ x\in[0,\pi/2]$, one can see that all the assumptions of Theorem 1 are satisfied. Hence, we have

$$J_0^{\alpha}\left((\cos)^{\sigma}(x)\left(\frac{1}{\sigma}-\frac{(\cos)^2(x)}{\sigma+2}\right)w(x)\right)\leqslant\frac{2}{\sigma(\sigma+2)}J_0^{\alpha}w(x),\quad x\in[0,\pi/2],$$

where $f(0) = 1$. Taking $w(x) = 1$, we obtain

$$J_0^\alpha \left((\cos)^\sigma(x) \left(\frac{1}{\sigma} - \frac{(\cos)^2(x)}{\sigma+2} \right) \right) \leqslant \frac{2}{\sigma(\sigma+2)} \frac{x^\alpha}{\Gamma(\alpha+1)}, \quad x \in [0, \pi/2].$$

The second example is given by the cosinus hyperbolic function. We consider the following example:

EXAMPLE 2. We take $f(x) = \operatorname{ch} x$, $x \in [0,1]$, one can see that all the assumptions of Theorem 1 are satisfied. If we consider the case $w(x) = 1$, then we can write

$$J_0^\alpha \left((\operatorname{ch})^\sigma(x) \left(\frac{1}{\sigma} - \frac{(\operatorname{ch})^2(x)}{\sigma+2} \right) \right) \leqslant \frac{2}{\sigma(\sigma+2)} \frac{x^\alpha}{\Gamma(\alpha+1)}, \quad x \in [0,1].$$

5. Conclusion

In summary, the generalization of integral inequalities presented in this study represents an advancement, extending beyond the limitations of previous works that focused on cases dependent on a natural parameter n. By generalizing these inequalities to positive arbitrary orders, utilizing the Riemann-Liouville fractional integral, this research introduces a more inclusive framework. The outcomes not only enrich the existing mathematical literature but also provide a foundation for exploring new dimensions in fractional calculus and integral inequalities. This work opens avenues for further investigations and applications, especially in fractional differential equations.

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