Fractional
Differential
Calculus
Volume 14, Number 2 (2024), 255–271 doi:10.7153/fdc-2024-14-12

EXISTANCE OF NON–NEGATIVE AND NONDECREASING SOLUTION OF ERDÉLYI–KOBER FRACTIONAL INTEGRAL EQUATION WITH THE HELP OF SIMULATION TYPE CONDENSING OPERATOR

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(*Communicated by S. K. Ntouyas*)

Abstract. The purpose of this paper is to prove the existence of a solution for Erdélyi-Kober fractional integral equations using the generalised form of Darbo's fixed point theorem and a simulation form condensing operator in Banach space Ω. The primary purpose is to show that the Erdélyi-Kober fractional integral equation has solutions in $\mathbb{C}[0,1]$. Finally, to illustrate that our abstract conclusions are simple to verify, we give an example.

1. Introduction

In the theory of neutron transport, radiation, gas kinetic theory and traffic theory, the integral equations play significant roles in applied problems, numerous problems also occurrences of the real world (see [4,5,9,20,24,25]). Darbo's stated the fixed point theorem using the definition of a measure of noncompactness in 1955 [7]. The fixed point theorem of Darbo is a significant extension of Schauder's fixed point theorem and serves an important role in the study of an integral equation form (see [6,8,10,11,28]).

Kuratowski's measure of noncompactness introduced in 1930 [1], has been extensively studied in various mathematical contexts. Different research papers have explored the application of this measure in other areas. Application of the measure of noncompactness proposed the concept of condensing operators, focusing on best proximity points and fractional differential equations. The Kuratowski measure of noncompactness in studying boundary-value problems for fractional differential equations with variable order and delays showcases the measure's applicability in stability criteria.

J. Banaś [3] introduced the measure of noncompactness in 1980, and its study in Banach spaces has since become a key research area. These measures play a crucial role in determining the existence of solutions to integral boundary value problems, studying the representation of a measure of noncompactness and their applications in Banach spaces, exploring interpolation of a measure of noncompactness of polynomials on Banach spaces, examining solvability conditions for fractional integral equations in Banach spaces using the theory of a measure of noncompactness, and applying the

Mathematics subject classification (2020): 47H08, 47H09, 47H10, 34A12, 26A33.
Keywords and phrases: Simulation function, measure of noncompactness, fixed point theorem, Erdélyi-Kober fractional integral.
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measure of noncompactness in characterizing classes of compact operators, solving integral equations, and establishing the existence of optimal solutions for systems of integro-differentials in Banach spaces (see [2,19,23,26,30,31,33]). These research efforts collectively contribute to a deeper understanding of measures of noncompactness and their diverse applications in various mathematical contexts.

The noncompactness measure can be extended and show the effects of existence on integral equations of various forms, integro-differential equations, differential equations, and their infinite structures (see [12,14,29]). Recently, the solvability of fractional integral equations has been established (see [15–18,21,22]).

In this work, we have presented Darbo's fixed point theorem to solve Erdélyi-Kober fractional integral equations in Banach space with the different functions used. Herein, we also illustrated a suitable example for verifying the proposed theorem.

Let Ω be a real Banach space with the norm $\|\,.\,\|$. Suppose $\mathbb{B}(a,b)$ in Ω radius b is also a closed ball,centered on a. If $\mathbb{Y}$ is a nonempty subset of Ω. Then, $\bar{\mathbb{Y}}$ and Conv$\,\mathbb{Y}$ notation are the closure and convex closure of $\mathbb{Y}$. Moreover, suppose $\mathscr{M}_\Omega$ is denote the family of all bounded and nonempty subsets of Ω and $\mathscr{N}_\Omega$ its subfamily consisting of all relatively compact sets. Also, the real numbers set is denote by $\mathbb{R}$ and $\mathbb{R}_+ = [0,\infty)$.

The following definition of a noncompactness measure is presented in [3].

DEFINITION 1.1. A function $\Sigma : \mathscr{M}_\Omega \to \mathbb{R}_+$ is called a measure of noncompactness(MNC) in Ω if it satisfies the following conditions:

(i) The family ker $\Sigma = \{\mathbb{Y} \in \mathscr{M}_\Omega : \Sigma(\mathbb{Y}) = 0\}$ is nonempty and ker $\Sigma \subset \mathscr{N}_\Omega$.

(ii) $\mathbb{Y} \subseteq \mathbb{Z} \implies \Sigma(\mathbb{Y}) \leqslant \Sigma(\mathbb{Z})$.

(iii) $\Sigma\left(\bar{\mathbb{Y}}\right) = \Sigma(\mathbb{Y})$.

(iv) $\Sigma(Conv\mathbb{Y}) = \Sigma(\mathbb{Y})$.

(v) $\Sigma(\rho\mathbb{Y} + (1-\rho)\mathbb{Z}) \leqslant \rho\Sigma(\mathbb{Y}) + (1-\rho)\Sigma(\mathbb{Z})$ for $\rho \in [0,1]$.

(vi) If $\mathbb{Y}_n \in \mathscr{M}_\Omega$, $\mathbb{Y}_n = \bar{\mathbb{Y}}_n$, $\mathbb{Y}_{n+1} \subset \mathbb{Y}_n$ for $n = 1,2,3,\ldots$ and $\lim_{n\to\infty} \Sigma(\mathbb{Y}_n) = 0$ then $\mathbb{Y}_\infty = \bigcap_{n=1}^{\infty} \mathbb{Y}_n \neq \phi$.

The $ker\Sigma$ family is *kernel of measure* Σ. Note that the intersection set $\mathbb{Y}_\infty$ from (vi) is a member of the family $ker\Sigma$. In fact, since $\Sigma(\mathbb{Y}_\infty) \leqslant \Sigma(\mathbb{Y}_n)$ for any n, we conclude that $\Sigma(\mathbb{Y}_\infty) = 0$. This is giving $\mathbb{Y}_\infty \in ker\Sigma$.

We recall the following well-known theorems:

THEOREM 1.2. (Schauder [1]) *Let $\mathbb{S}$ be a non-empty, closed and convex subset of a Banach space Ω. Then every compact, continuous map $W : \mathbb{S} \to \mathbb{S}$ has at least one fixed point.*

THEOREM 1.3. (Darbo [7]) *Let D be a non-empty, bounded, closed and convex(NBCC) subset of a Banach space Ω. Let $W : D \to D$ be a continuous mapping. Assume that there is a constant $p \in [0,1)$ such that*

$$\Sigma(W\mathbb{Y}) \leqslant p\Sigma(\mathbb{Y}), \quad \mathbb{Y} \subseteq D.$$

Then W has a fixed point.

In formulating our fixed point theorem, we require certain analogous principles from the following set of methods.

DEFINITION 1.4. ([32]) Let C denote the class of all simulation function of the form $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

1. $\zeta(0,0) = 0$;
2. $\zeta(s,t) < t - s$, for all $s,t > 0$;
3. if $\{s_n\},\{t_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$ and $s_n < t_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \sup \zeta(s_n,t_n) < 0$.

EXAMPLE 1.5. Let $\zeta_j : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},\ j = 1,2,3,4$ be defined by

1. $\zeta_1(s,t) = \eta(t) - \gamma(s)\ \forall\ s,t \in \mathbb{R}_+$, where $\gamma,\eta : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\gamma(s) = \eta(s) = 0 \Leftrightarrow s = 0$ and $\eta(s) < s \leqslant \gamma(s)$ for all $s > 0$.

   If we consider, $\eta(s) = s$ and $\gamma(s) = px\ \forall\ s \geqslant 0$, where $p \in [0,1)$, then we get the particular case of simulation function $\overline{\zeta}_1(s,t) = ps - t\ \forall\ s,t \in \mathbb{R}_+$.
2. $\zeta_2(s,t) = t - \eta(t) - s\ \forall\ s,t \in \mathbb{R}_+$, where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is lower semi-continuous mapping such that $\eta^{-1}(0) = 0$
3. $\zeta_3(s,t) = \eta(t) - s\ \forall\ s,t \in \mathbb{R}_+$, where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a upper semi-continuous function with $\eta(s) < s\ \forall\ s > 0$ also $\eta(0) = 0$.
4. If $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\int_0^\varepsilon g(v)dv$ exists and $\int_0^\varepsilon g(v)dv > \varepsilon$ for each $\varepsilon > 0$, with we set $\zeta_4(s,t) = t - \int_0^s g(v)dv\ \forall\ s,t \geqslant 0$.

DEFINITION 1.6. ([13]) Let C be the class of all functions $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

1. $\max\{m_1,m_2\} \leqslant M(m_1,m_2)$ for $m_1,m_2 \geqslant 0$.
2. M is a continuous
3. $M(m_1+m_2,n_1+n_2) \leqslant M(m_1,n_1) + M(m_2,n_2)$.

Example: $M(m_1+m_2) = m_1+m_2$.

DEFINITION 1.7. Let D be a non-empty subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. We say that an operator $W : D \to D$ is generalized *ζ-condensing* operator if there exists $\zeta \in C$ such that

$$\zeta(M(\Sigma(W\mathbb{Y}), H(\Sigma(W\mathbb{Y}))), M(\Sigma(\mathbb{Y}), H(\Sigma(\mathbb{Y})))) \geqslant 0 \tag{1.1}$$

for any bounded subset $\mathbb{Y}$ of D, where $M \in C$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing continuous function.

In this article, we establish a generalization of Darbo's fixed point theorem and then apply it to check the solvability of Erdélyi-Kober fractional integral equation.

2. Main result

THEOREM 2.1. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC. If $W : D \to D$ is a continuous with generalized ζ-condensing operator and $M \in C$. Then W has at least one fixed point.*

Proof. Let $D_1 = D$ and construct a sequence $\{D_n\}$ such that $D_{n+1} = Conv(WD_n)$ for $n \in N$. Also $WD_1 = WD \subseteq D = W_1$, $D_2 = Conv(WD_1) \subseteq D = D_1$, therefore by continuing this process, we have

$$D_1 \supseteq D_2 \supseteq \ldots \supseteq D_n \supseteq D_{n+1} \supseteq \ldots$$

If there exists a positive integer $N_0 \in \mathbb{N}$ such that $M(\Sigma(D_{N_0}), H(\Sigma(D_{N_0}))) = 0$. Then $\Sigma(D_{N_0}) = 0$. So D_{N_0} is relatively compact and since $W(D_{N_0}) \subseteq Conv(WD_{N_0}) = D_{N_0+1} \subseteq D_{N_0}$. Then by Theorem 2.1, we conclude that W has a fixed point. So, we assume that $M(\Sigma(D_n), H(\Sigma(D_n))) > 0 \ \forall \ n \in \mathbb{N}$. Suppose that

$$M(\Sigma(D_n), H(\Sigma(D_n))) < M(\Sigma(D_{n+1}), H(\Sigma(D_{n+1}))) \tag{2.1}$$

for some $n_0 \in \mathbb{N}$. By using equation (1.1) and definition (1.7) with $\mathbb{Y} = D_n$, we have

$$\begin{aligned} 0 &\leqslant \zeta[M(\Sigma(WD_{n_0}), H(\Sigma(WD_{n_0}))), M(\Sigma(D_{n_0}), H(\Sigma(D_{n_0})))] \\ &= \zeta[M(\Sigma(convWD_{n_0}), H(\Sigma(convWD_{n_0}))), M(\Sigma(D_{n_0}), H(\Sigma(D_{n_0})))] \\ &= \zeta[M(\Sigma(D_{n_0+1}), H(\Sigma(D_{n_0+1}))), M(\Sigma(D_{n_0}), H(\Sigma(D_{n_0})))] \\ &< M(\Sigma(D_{n_0}), H(\Sigma(D_{n_0}))) - M(\Sigma(D_{n_0+1}), H(\Sigma(D_{n_0+1}))) \\ &< 0, \end{aligned}$$

which is contradiction. So, this implies,

$$M(\Sigma(D_{n+1}), H(\Sigma(D_{n+1}))) \leqslant M(\Sigma(D_n), H(\Sigma(D_n))) \text{ for all } n \in \mathbb{N}.$$

Therefore, $M(\Sigma(D_n), H(\Sigma(D_n)))$ is nonnegative and non-increasing, we infer that

$$\lim_{n\to\infty} M(\Sigma(D_n), H(\Sigma(D_n))) = a. \tag{2.2}$$

We need to show that $a = 0$.

Suppose to the contrary that $a > 0$. Then by using equation (1.1) with $\mathbb{Y} = D_n$, we have

$$\begin{aligned} 0 &\leqslant \zeta[M(\Sigma(WD_n), H(\Sigma(WD_n))), M(\Sigma(D_n), H(\Sigma(D_n)))] \\ &= \zeta[M(\Sigma(convWD_n), H(\Sigma(convWD_n))), M(\Sigma(D_n), H(\Sigma(D_n)))] \\ &= \zeta[M(\Sigma(D_{n+1}), H(\Sigma(D_{n+1}))), M(\Sigma(D_n), H(\Sigma(D_n)))]. \end{aligned}$$

Using the above inequality and the definition (1.7) with $t_n = M(\Sigma(D_{n+1}), H(\Sigma(D_{n+1})))$ and $s_n = M(\Sigma(D_n), H(\Sigma(D_n)))$, we have

$$0 \leqslant \lim_{n\to\infty} \sup \zeta[M(\Sigma(D_{n+1}), H(\Sigma(D_{n+1}))), M(\Sigma(D_n), H(\Sigma(D_n)))] < 0$$

which is contradiction.

So, $a = 0$ and from (2.2), we get

$$\lim_{n\to\infty} M(\Sigma(D_n), H(\Sigma(D_n))) = 0. \tag{2.3}$$

Also, using definition (1.6), we get

$$\max\{\Sigma(D_n), H(\Sigma(D_n))\} \leqslant M(\Sigma(D_n), H(\Sigma(D_n))).$$

As $n \to \infty$, we get

$$0 \leqslant \max\{\lim_{n\to\infty} \Sigma(D_n), \lim_{n\to\infty} H(\Sigma(D_n))\} \leqslant 0.$$

Since $H \geqslant 0$, we obtain

$$\lim_{n\to\infty} \Sigma(D_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} H(\Sigma(D_n)) = 0.$$

Since, $D_n \supseteq D_{n+1}$ in the view of definition (1.1), we conclude that $D_\infty = \bigcap_{n=1}^{\infty} D_n$ is non-empty, convex closed set, invariant under the mapping W and belongs to $ker\Sigma$. Thus, Theorem 1.2 conclude that W has a fixed point in D. This is completing the proof. □

An essential significance of Theorem 2.1 is as follows:

THEOREM 2.2. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. If $W : D \to D$ is a continuous and satisfy*

$$M(\Sigma(W\mathbb{Y}), H(\Sigma(W\mathbb{Y}))) \leqslant pM(\Sigma(\mathbb{Y}), H(\Sigma(\mathbb{Y}))); \qquad p \in [0,1)$$

for any bounded subset $\mathbb{Y}$ of D, where $M \in C$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Then W has atleast one fixed point.

Proof. Taking $\zeta(s,t) = pt - s$; $s,t \geqslant 0$ in Theorem 2.1. □

The statement in the next corollary is a result of Theorem 2.1.

COROLLARY 2.3. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that a constant $p \in [0,1)$ with the property*

$$\Sigma(W\mathbb{Y}) \leqslant p\Sigma(\mathbb{Y})$$

for any bounded subset $\mathbb{Y}$ of D. Then W has atleast one fixed point in the set D.

Proof. Taking $M(m_1,m_2) = m_1 + m_2$ and $H \equiv 0$ in Theorem 2.2. So, we get the required result. □

THEOREM 2.4. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that there exist altering distance functions $\eta,\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ verifying $\eta(s) < s \leqslant \gamma(s)$ for all $s > 0$ and $\gamma(s) = \eta(s) = 0 \Leftrightarrow s = 0$ satisfying*

$$\gamma(M(\Sigma(W\mathbb{Y}),H(\Sigma(W\mathbb{Y})))) \leqslant \eta(M(\Sigma(\mathbb{Y}),H(\Sigma(\mathbb{Y}))))$$

for any bounded subset $\mathbb{Y}$ of D, where $M \in C$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Then W has atleast one fixed point.

Proof. Taking $\zeta(s,t) = \eta(t) - \gamma(s);\ s,t \geqslant 0$ in Theorem 2.1. □

COROLLARY 2.5. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that there exist maps $\eta,\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ verifying $\eta(s) < s \leqslant \gamma(s)\ \forall\ s > 0$ and $\eta(s) = \gamma(s) = 0 \Leftrightarrow s = 0$ satisfying*

$$\gamma(\Sigma(W\mathbb{Y})) \leqslant \eta(\Sigma(\mathbb{Y}))$$

for any bounded subset $\mathbb{Y}$ of D. Then W has atleast one fixed point.

Proof. Taking $M(m_1,m_2) = m_1 + m_2$ and $H \equiv 0$ in Theorem 2.4. So, we obtain the required result. □

COROLLARY 2.6. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that a constant $p \in [0,1)$ with the property*

$$\Sigma(W\mathbb{Y}) \leqslant p\Sigma(\mathbb{Y})$$

for any bounded subset $\mathbb{Y}$ of D. Then W has atleast one fixed point.

Proof. Taking $\eta(t) = pt,\ \gamma(t) = t,\ p \in [0,1)$ in Corollary 2.5, we get required result. □

COROLLARY 2.7. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that there exist a lower semi-continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ verifying $\eta^{-1}(0) = 0$ with*

$$\Sigma(W\mathbb{Y}) \leqslant \Sigma(\mathbb{Y}) - \eta(\Sigma(\mathbb{Y}))$$

for any bounded subset $\mathbb{Y}$ of D. Then W has atleast one fixed point.

Proof. Taking $\zeta(s,t) = t - \eta(t) - s$ for all $s,t \in [0,\infty)$, $M(m_1,m_2) = m_1 + m_2$ and $H \equiv 0$ in Theorem 2.1. So, we get the required result. □

THEOREM 2.8. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that there exist a upper semi continuous mapping $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ verifying $\eta(0) = 0$ and $\eta(s) < s$ $\forall\ x > 0$ satisfying*

$$M(\Sigma(W\mathbb{Y}), H(\Sigma(W\mathbb{Y})) \leqslant \eta(M(\Sigma(\mathbb{Y}, H(\Sigma(\mathbb{Y})))$$

for any bounded subset $\mathbb{Y}$ of D, where $M \in C$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Then W has atleast one fixed point.

Proof. Taking $\zeta(s,t) = \eta(t) - s$; $s,t \geqslant 0$ in Theorem 2.1. □

COROLLARY 2.9. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping such that there exist a upper semi continuous mapping $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ verifying*

$$\Sigma(W\mathbb{Y}) \leqslant \eta\Sigma(\mathbb{Y})$$

for any bounded subset $\mathbb{Y}$ of D. Then W has atleast one fixed point.

Proof. Taking $M(m_1,m_2) = m_1 + m_2$ and $H \equiv 0$ in Theorem 2.8. So, we get the required result. □

THEOREM 2.10. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping satisfying*

$$\int_0^{M(\Sigma(W\mathbb{Y}),H(\Sigma(W\mathbb{Y})))} g(v)dv \leqslant M(\Sigma(\mathbb{Y}), H(\Sigma(\mathbb{Y})))$$

for any bounded subset $\mathbb{Y}$ of D, where $M \in C$ and $g,H : \mathbb{R}_+ \to \mathbb{R}_+$ are a functions such that $\int_0^{\varepsilon} g(v)dv$ exists and $\int_0^{\varepsilon} g(v)dv > \varepsilon$ for each $\varepsilon > 0$. Then W has atleast one fixed point.

Proof. Taking $\zeta(s,t) = t - \int_0^s g(v)dv$; $s,t \geqslant 0$ in Theorem 2.1. □

COROLLARY 2.11. *Let D be a NBCC subset of a Banach space Ω and Σ is an arbitrary MNC on Ω. Also, let $W : D \to D$ be a continuous mapping satisfying the following condition :*

$$\int_0^{\Sigma(W\mathbb{Y})} g(v)dv \leqslant \Sigma(\mathbb{Y})$$

for any bounded subset $\mathbb{Y}$ of D, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\int_0^{\varepsilon} g(v)dv$ exists and $\int_0^{\varepsilon} g(v)dv > \varepsilon$ for each $\varepsilon > 0$. Then W has atleast one fixed point.

Proof. Taking $M(m_1,m_2) = m_1 + m_2$ and $H \equiv 0$ in Theorem 2.10. So, we get the required result. □

3. Measure of noncompactness on $\mathbb{C}(I)$

From [27], we recall that the fractional integral as Erdélyi-Kober of a continuous mapping f is defined as

$$I_{\beta}^{\alpha} f(t) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \frac{\theta^{\beta-1} f(\theta)}{(t^{\beta}-\theta^{\beta})^{1-\alpha}} d\theta, \qquad \beta > 0, \quad 0 < \alpha < 1.$$

Consider, the Erdélyi-Kober fractional integral equation is given below:

$$x(\theta) = F_1(\theta, x(\theta)) + \frac{\beta F_2(\theta, x(\theta))}{\Gamma(\alpha)} \int_0^{\theta} \frac{s^{\beta-1} k(\theta, s, x(s))}{(\theta^{\beta}-s^{\beta})^{1-\alpha}} ds, \qquad \theta \in I = [0,1], \quad (3.1)$$

where $\beta > 0$ and $\alpha \in (0,1)$.

The solvability of equation (3.1) in the Banach space $\mathbb{C}(I)$ where $I = [0,1]$ consisting of all real continuous maps based on I with norm $||x|| = \max\{|x(t)| : t \in I\}$, will be studied in this section.

Let us fix the non-empty and bounded subset X of $\mathbb{C}(I)$. For $x \in X$ and $\varepsilon > 0$, We define by $\omega(x,\varepsilon)$ the modulus of continuity of a mapping x, that is (i.e.)

$$\omega(x,\varepsilon) = \sup\{|x(\theta) - x(t)| : t, \theta \in I, |\theta - t| \leqslant \varepsilon\}.$$

Next, the succeeding quantities are known

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon).$$

It is possible to show [3] that $\omega_0(X)$ is an MNC in $\mathbb{C}(I)$.

4. Application

In this section, we shall consider in our study eq. (3.1) the following assumptions:

(i) $F_i : I \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are nondecreasing and continuous function. Then, there exist constants $C_i \geqslant 0$; $i = 1,2$ with

$$|F_i(\theta,x) - F_i(\theta,y)| \leqslant C_i |x-y| \qquad \text{for all} \quad \theta \in I \quad \text{and} \quad x,y \in \mathbb{R}.$$

(ii) A continuous function $k : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $k(\theta,s,x)$ is non-decreasing mapping with respect to each variable θ, s and x separately.

(iii) There exist a nondecreasing mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $|k(\theta,s,x)| \leqslant \phi(|x|)$; for all θ, $s \in I$ and all $x \in \mathbb{R}$.

(iv) There exists a fix constant $b > 0$ that satisfies the inequality that follows

$$(bc_1 + F_1^*)\Gamma(\alpha+1) + (bc_2 + F_2^*)\phi(b) \leqslant b\Gamma(\alpha+1) \tag{4.1}$$

such that

$$c_1\Gamma(\alpha+1) + c_2\phi(\phi(b)) < \Gamma(\alpha+1)$$

where $F_i^* = \max\{F_i(\theta,0) : \theta \in I\}$; $i = 1,2$.

We can re-write equation (3.1) in the form for further purposes i.e.

$$x = \Theta x, \tag{4.2}$$

where

$$(\Theta x)(\theta) = (\widehat{F_1}x)(\theta) + (\widehat{F_2}x)(\theta).(Kx)(\theta), \qquad \theta \in I \tag{4.3}$$

also, the Erdelyi-Kober integral operator is K given by

$$(Kx)(\theta) = \frac{\beta}{\Gamma(\alpha)}\int_0^\theta \frac{s^{\beta-1}k(\theta,s,x(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds, \qquad \theta \in I, \quad 0 < \alpha < 1, \quad \beta > 0. \tag{4.4}$$

In the proof of our main theorem, we will need the following two lemmas [10,11].

LEMMA 4.1. *If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave mapping with $f(0) = 0$. Then F is subadditive, i.e. $f(u+v) < f(u) + f(v)$ for any $u,v \in \mathbb{R}_+$.*

LEMMA 4.2. *If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the mapping defined by $f(u) = u^\gamma$.*
(i) If $0 < \alpha < 1$ and $\theta_1, \theta_2 \in I$ with $\theta_2 > \theta_1$. Then $\theta_2^\gamma - \theta_1^\gamma \leqslant (\theta_2 - \theta_1)^\gamma$.
(ii) If $\gamma \geqslant 1$ and $\theta_1, \theta_2 \in I$ with $\theta_2 > \theta_1$. Then $\theta_2^\gamma - \theta_1^\gamma \leqslant \gamma(\theta_2 - \theta_1)$.

Therefore, define $\widehat{B_b} \subseteq B_b$ in the form

$$\widehat{B_b} = \{x \in B_b : x(\theta) \geqslant 0 \text{ for } \theta \in l\}.$$

So, $\widehat{B_b}$ is non-empty, closed and convex.

THEOREM 4.3. *Under the assumption (1)–(4), the equation* (3.1) *has at least one solution $x \in \mathbb{C}(I)$ which is non-decreasing and non-negative on I.*

Proof. Step 1^{st}, we shall prove that $\Theta : \mathbb{C}(I) \to \mathbb{C}(I)$. In order to make this claim, it is sufficient to prove that $x \in \mathbb{C}(I)$ implies that Kx is continuous mapping on I, due to (i). Then, take the arbitrary numbers $\theta_1, \theta_2 \in I$ and set $\varepsilon > 0$ with $|\theta_2 - \theta_1| \leqslant \varepsilon$. Now, assume that $\theta_2 > \theta_1$, we have

$$\begin{aligned}&|(Kx)(\theta_2) - (Kx)(\theta_1)|\\ &= \Big|\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_2} \frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^\beta - s^\beta)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1} \frac{s^{\beta-1}k(\theta_1,s,x(s))}{(\theta_1^\beta - s^\beta)^{1-\alpha}}ds\Big|\end{aligned}$$

$$
\begin{aligned}
&\leqslant \Big|\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_2}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds-\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds\Big|\\
&\quad+\Big|\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds-\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_1,s,x(s))}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds\Big|\\
&\quad+\Big|\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_1,s,x(s))}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds-\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_1,s,x(s))}{(\theta_1^\beta-s^\beta)^{1-\alpha}}ds\Big|\\
&\leqslant \frac{\beta}{\Gamma(\alpha)}\int_{\theta_1}^{\theta_2}\frac{s^{\beta-1}|k(\theta_2,s,x(s))|}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}|k(\theta_2,s,x(s))-k(\theta_1,s,x(s))|}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}s^{\beta-1}|k(\theta_2,s,x(s))|\left[\frac{1}{(\theta_1^\beta-s^\beta)^{1-\alpha}}-\frac{1}{(\theta_2^\beta-s^\beta)^{1-\alpha}}\right]ds\\
&\leqslant \frac{\beta}{\Gamma(\alpha)}\int_{\theta_1}^{\theta_2}\frac{s^{\beta-1}\phi(|x|)}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds+\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}\frac{s^{\beta-1}\omega_{\phi(|x|)}(k,\varepsilon)}{(\theta_2^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta}{\Gamma(\alpha)}\int_0^{\theta_1}s^{\beta-1}\phi(|x|)\Big[(\theta_1^\beta-s^\beta)^{\alpha-1}-(\theta_2^\beta-s^\beta)^{\alpha-1}\Big]ds\\
&\leqslant \frac{\phi(|x|)}{\Gamma(\alpha+1)}(\theta_2^\beta-\theta_1^\beta)^\alpha+\frac{\omega_{\phi(|x|)}(k,\varepsilon)}{\Gamma(\alpha+1)}\theta_2^{\alpha\beta}\\
&\quad+\frac{\phi(|x|)}{\Gamma(\alpha+1)}\Big[\theta_1^{\alpha\beta}-\theta_2^{\alpha\beta}+(\theta_2^\beta-\theta_1^\beta)^\alpha\Big]\\
&\leqslant \frac{\omega_{\phi(|x|)}(k,\varepsilon)}{\Gamma(\alpha+1)}\theta_2^{\alpha\beta}+\frac{2\phi(|x|)}{\Gamma(\alpha+1)}(\theta_2^\beta-\theta_1^\beta)^\alpha. \qquad (4.5)
\end{aligned}
$$

Where, we defined

$$
\begin{aligned}
\omega_b(k,\varepsilon)=\sup[&|k(\theta_2,s,x)-k(\theta_1,s,x)| : s,\theta_1,\theta_2\in l,\theta_1\geqslant s,\ \theta_2\geqslant s,\ x\in[-b,b]\\
&\text{and } |\theta_2-\theta_1|\leqslant\varepsilon].
\end{aligned}
$$

We must now differentiate between two cases.

Case 1: $0<\beta<1$

Using Lemma 4.2, $(\theta_2^\beta-\theta_1^\beta)^\alpha\leqslant(\theta_2-\theta_1)^{\alpha\beta}$ and from inequality (4.5), we get

$$
\begin{aligned}
|(Kx)(\theta_2)-(Kx)(\theta_1)| &\leqslant \frac{\omega_{\phi(|x|)}(k,\varepsilon)}{\Gamma(\alpha+1)}+\frac{2\phi(|x|)}{\Gamma(\alpha+1)}(\theta_2-\theta_1)^{\alpha\beta}\\
&\leqslant \frac{\omega_{\phi(|x|)}(k,\varepsilon)}{\Gamma(\alpha+1)}+\frac{2\phi(|x|)}{\Gamma(\alpha+1)}\varepsilon^{\alpha\beta}.
\end{aligned}
$$

Case 2: $\beta \geqslant 1$

By Lemma 4.2, $(\theta_2^\beta - \theta_1^\beta)^\alpha \leqslant \alpha\beta(\theta_2 - \theta_1)^\alpha$ and from (4.5), we obtain

$$|(Kx)(\theta_2) - (Kx)(\theta_1)| \leqslant \frac{\omega_{\phi(|x|)}(k,\varepsilon)}{\Gamma(\alpha+1)} + \frac{2\phi(|x|)}{\Gamma(\alpha+1)}\alpha\beta(\theta_2-\theta_1)^\alpha$$
$$\leqslant \frac{\omega_{\phi(|x|)}(k,\varepsilon)}{\Gamma(\alpha+1)} + \frac{2\phi(|x|)}{\Gamma(\alpha+1)}\varepsilon^\alpha.$$

In both cases, via the uniform continuity of the mapping k on $I \times I \times [-\phi(|x|), \phi(|x|)]$, we obtain that $\omega_{\phi(|x|)}(k,\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, Kx is a continuous on I.

Step 2^{nd}, we shall claim that $\widehat{F} : B_{r_0} \to B_{r_0}$. For $x \in \mathbb{C}(I)$ with $\theta \in l$, we get

$$|(\Theta x)(\theta)| \leqslant \left|F_1(\theta,x(\theta)) + \frac{\beta F_2(\theta,x(\theta))}{\Gamma(\alpha)} \int_0^\theta \frac{s^{\beta-1}k(\theta,s,x(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds\right|$$
$$\leqslant |F_1(\theta,x(\theta)) - F_1(\theta,0)| + |F_1(\theta,0)|$$
$$+ \frac{\beta[|F_2(\theta,x(\theta)) - F_2(\theta,0)| + |F_2(\theta,0)|}{\Gamma(\alpha)} \int_0^\theta \frac{s^{\beta-1}|k(\theta,s,x(s))|}{(\theta^\beta - s^\beta)^{1-\alpha}}ds$$
$$\leqslant c_1|x-0| + F_1^* + \frac{c_2|x| + F_2^*}{\Gamma(\alpha+1)}\phi(|x|).$$

Therefore,

$$|\Theta x| \leqslant c_1|x| + \frac{c_2|x| + F_2^*}{\Gamma(\alpha+1)}\phi(|x|).$$

From this analysis, we conclude that the ball B_b is applied by the operator Θ.

Step 3^{rd}, based on our assumptions (i), (ii) and (iv) that $\widehat{B_b}$ is NBCC, we conclude that Θ applies the ball $\widehat{B_b}$ into itself.

Step 4^{th}, we shall prove that Θ is a continuous mapping on $\widehat{B_b}$. For this purpose, set $\varepsilon > 0$ and $x \in \widehat{B_b}$. We take $y \in \widehat{B_b}$ and $\| x - y \| < \varepsilon$. So, for $\theta \in I$, we get

$$|(\Theta x)(\theta) - (\Theta y)(\theta)|$$
$$\leqslant |F_1(\theta,x(\theta)) - F_1(\theta,y(\theta))| + \left|\frac{\beta F_2(\theta,x(\theta))}{\Gamma(\alpha)} \int_0^\theta \frac{s^{\beta-1}k(\theta,s,x(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds\right.$$
$$\left. - \frac{\beta F_2(\theta,y(\theta))}{\Gamma(\alpha)} \int_0^\theta \frac{s^{\beta-1}k(\theta,s,y(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds\right|$$
$$\leqslant c_1|x(\theta) - y(\theta)|$$
$$+ \frac{\beta}{\Gamma(\alpha)}\left|F_2(\theta,x(\theta)) \int_0^\theta \frac{s^{\beta-1}k(\theta,s,x(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds - F_2(\theta,y(\theta)) \int_0^\theta \frac{s^{\beta-1}k(\theta,s,x(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds\right|$$
$$+ \frac{\beta}{\Gamma(\alpha)}\left|F_2(\theta,y(\theta)) \int_0^\theta \frac{s^{\beta-1}k(\theta,s,x(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds - F_2(\theta,y(\theta)) \int_0^\theta \frac{s^{\beta-1}k(\theta,s,y(s))}{(\theta^\beta - s^\beta)^{1-\alpha}}ds\right|$$

$$
\begin{aligned}
&\leqslant c_1|x(\theta)-y(\theta)| \\
&\quad+\frac{\beta|F_2(\theta,x(\theta))-F_2(\theta,y(\theta))|}{\Gamma(\alpha)}\int_0^{\theta}\frac{s^{\beta-1}|k(\theta,s,x(s))|}{(\theta^{\beta}-s^{\beta})^{1-\alpha}}ds \\
&\quad+\frac{\beta[|F_2(\theta,x(\theta))-F_2(\theta,0)|+|F_2(\theta,0)|}{\Gamma(\alpha)}\int_0^{\theta}\frac{s^{\beta-1}|k(\theta,s,x(s))-k(\theta,s,y(s))|}{(\theta^{\beta}-s^{\beta})^{1-\alpha}}ds \\
&\leqslant c_1|x(\theta)-y(\theta)|+\frac{\beta c_2|x(\theta)-y(\theta)|}{\Gamma(\alpha)}\int_0^{\theta}\frac{s^{\beta-1}\phi(|x|)}{(\theta^{\beta}-s^{\beta})^{1-\alpha}}ds \\
&\quad+\frac{\beta[c_2|x(\theta)|+|F_2(\theta,0)|}{\Gamma(\alpha)}\int_0^{\theta}\frac{s^{\beta-1}\delta_k(\varepsilon)}{(\theta^{\beta}-s^{\beta})^{1-\alpha}}ds.
\end{aligned}
$$

Where, we denote

$$
\delta_k(\varepsilon)=\sup\left\{\begin{array}{c}|k(\theta,s,x)-k(\theta,s,y)|:\theta,s\in I,\\ \|x-y\|\leqslant\varepsilon,x,y\in[0,\phi(r_0)]\end{array}\right\}.
$$

Thus,

$$
\|\Theta x-\Theta y\|\leqslant\left\{c_1+\frac{c_2\phi(r_0)}{\Gamma(\alpha+1)}\right\}\|x-y\|+\frac{c_2r_0+F_2^*}{\Gamma(\alpha+1)}\delta_k(\varepsilon). \tag{4.6}
$$

Since, k is uniformly continuous function on $I\times I\times[0,\phi(b)]$, so $\delta_k(\varepsilon)\to 0$. Therefore, from eq. (4.6), Θ is a continuous operator on $\widehat{B_b}$.

We consider the subset Q of $\widehat{B_b}$ identified by the subsequent

$$
Q=\{x\in\widehat{B_b}:x\ \text{ is a non-decreasing on }\ I\}.
$$

So, Q is NBCC.

Step 5^{th}, we show that Θ applies Q into itself. Taking into account (i) assumptions, it is sufficient to show that K applies Q to itself to establish our claim. For this, we take $x\in Q$, it is clear from our assumption that $Kx\geqslant 0$. Set $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. So, given our assumptions, we get

$$
\begin{aligned}
(Kx)(\theta_2)-(Kx)(\theta_1)&=\frac{\beta}{\Gamma(\alpha)}\Big[\int_0^{\theta_2}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^{\beta}-s^{\beta})^{1-\alpha}}ds-\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_1,s,x(s))}{(\theta_1^{\beta}-s^{\beta})^{1-\alpha}}ds\Big] \\
&=\frac{\beta}{\Gamma(\alpha)}\Big[\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^{\beta}-s^{\beta})^{1-\alpha}}ds-\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_1,s,x(s))}{(\theta_1^{\beta}-s^{\beta})^{1-\alpha}}ds \\
&\quad+\int_{\theta_1}^{\theta_2}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^{\beta}-s^{\beta})^{1-\alpha}}ds\Big] \\
&\geqslant\frac{\beta}{\Gamma(\alpha)}\Big[\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^{\beta}-s^{\beta})^{1-\alpha}}ds-\int_0^{\theta_1}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_1^{\beta}-s^{\beta})^{1-\alpha}}ds \\
&\quad+\int_{\theta_1}^{\theta_2}\frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^{\beta}-s^{\beta})^{1-\alpha}}ds\Big]
\end{aligned}
$$

$$= \frac{\beta}{\Gamma(\alpha)}\Big[\int_0^{\theta_1} s^{\beta-1}k(\theta_2,s,x(s))[(\theta_2^\beta - s^\beta)^{\alpha-1} - (\theta_1^\beta - s^\beta)^{\alpha-1}]ds + \int_{\theta_1}^{\theta_2} \frac{s^{\beta-1}k(\theta_2,s,x(s))}{(\theta_2^\beta - s^\beta)^{1-\alpha}}ds\Big]. \tag{4.7}$$

Since, $(\theta_2^\beta - s^\beta)^{\alpha-1} - (\theta_1^\beta - s^\beta)^{\alpha-1} \leqslant 0$ and x is non-decreasing,therefore given our assumptions (iii) and (iv) from eq. (4.7), we get

$$\begin{aligned}
&(Kx)(\theta_2) - (Kx)(\theta_1)\\
&\geqslant \frac{\beta}{\Gamma(\alpha)}\Big[\int_0^{\theta_1} s^{\beta-1}k(\theta_2,\theta_1,x(\theta_1))[(\theta_2^\beta - s^\beta)^{\alpha-1} - (\theta_1^\beta - s^\beta)^{\alpha-1}]ds\\
&\quad + \int_{\theta_1}^{\theta_2} \frac{s^{\beta-1}k(\theta_2,\theta_1,x(\theta_1))}{(\theta_2^\beta - s^\beta)^{1-\alpha}}ds\Big]\\
&= \frac{\gamma k(\theta_2,\theta_1,x(\theta_1))}{\Gamma(\alpha)}\Big(\int_0^{\theta_2} \frac{s^{\beta-1}}{(\theta_2^\beta - s^\beta)^{1-\alpha}}ds - \int_0^{\theta_1} \frac{s^{\beta-1}}{(\theta_1^\beta - s^\beta)^{1-\alpha}}ds\Big)\\
&= \frac{k(\theta_2,\theta_1,x(\theta_1))}{\Gamma(\alpha+1)}\Big(\theta_2^{\alpha\beta} - \theta_1^{\alpha\beta}\Big)\\
&\geqslant 0.
\end{aligned}$$

This shows that, the function Kx is non-decreasing. Hence, K maps Q into itself also Θ maps Q into itself.

Step 6^{th}, for $\emptyset \neq X \subset Q$, We're going to create an estimate of $\omega_0(\Theta X)$. This is to be obtained, set $\varepsilon > 0$ with $x \in X$. We're taking $\theta_1, \theta_2 \in I$ and $|\theta_2 - \theta_1| \leqslant \varepsilon$. We can assume that without the loss of generality, $\theta_1 \leqslant \theta_2$. So, using our assumptions, we obtain

$$\begin{aligned}
&|(\Theta x)(\theta_2) - (\Theta x)(\theta_1)|\\
&\leqslant |(\widehat{F}_1 x)(\theta_2) - (\widehat{F}_1 x)(\theta_1)| + |(\widehat{F}_2 x)(\theta_2)||(Kx)(\theta_2) - (Kx)(\theta_1)|\\
&\quad + |(\widehat{F}_2 x)(\theta_2) - (\widehat{F}_2 x)(\theta_1)||(Kx)(\theta_1)|\\
&\leqslant |F_1(\theta_2,x(\theta_2)) - F_1(\theta_2,x(\theta_1))| + |F_1(\theta_2,x(\theta_1)) - F_1(\theta_1,x(\theta_1))|\\
&\quad + \frac{|F_2(\theta_2,x(\theta_2)) - F_2(\theta_2,0)| + |F_2(\theta_2,0)|}{\Gamma(\alpha+1)}\Big[\omega_{\phi(b)}(k,\varepsilon)\theta_2^{\alpha\beta} + 2\phi(|b|)(\theta_2^\beta - \theta_1^\beta)^\alpha\Big]\\
&\quad + [|F_2(\theta_2,x(\theta_2)) - F_2(\theta_2,x(\theta_1))| + |F_2(\theta_2,x(\theta_1)) - F_2(\theta_1,x(\theta_1))|]\frac{\phi(b)}{\Gamma(\alpha+1)}\\
&\leqslant c_1|x(\theta_2) - x(\theta_1)| + \delta_{F_1}(\varepsilon) + \frac{c_2|x(\theta_2)| + F_2^*}{\Gamma(\alpha+1)}[\omega_{\phi(b)}(k,\varepsilon)\theta_2^{\alpha\beta} + 2\phi(b)(\theta_2^\beta - \theta_1^\beta)^\alpha]\\
&\quad + [c_2|x(\theta_2) - x(\theta_1)| + \delta_{F_2}(\varepsilon)]\frac{\phi(b)}{\Gamma(\alpha+1)}
\end{aligned}$$

$$\leqslant c_1\omega(x,\varepsilon)+\delta_{F_1}(\varepsilon)+\frac{c_2b+F_2^*}{\Gamma(\alpha+1)}[\omega_{\phi(b)}(k,\varepsilon)\theta_2^{\alpha\beta}+2\phi(b)(\theta_2^{\beta}-\theta_1^{\beta})^{\alpha}]$$
$$+[c_2\omega(x,\varepsilon)+\delta_{F_2}(\varepsilon)]\frac{\phi(b)}{\Gamma(\alpha+1)}. \tag{4.8}$$

Where, we defined

$$\delta_g(\varepsilon)=\sup\{|g(\theta_2,v)-g(\theta_1,v)|:\theta_1,\theta_2\in I, v\in[0,b], |\theta_2-\theta_1\leqslant\varepsilon\}.$$

Again, we have to differentiate between two cases in the first case $0<\beta<1$ and $\beta\geqslant 1$ respectively. Using Lemma 4.2, eq. (4.8) gives

$$\omega(\Theta X,\varepsilon)\leqslant\Big(c_1+\frac{c_2\phi(b)}{\Gamma(\alpha+1)}\Big)\omega(X,\varepsilon)+\delta_{F_1}(\varepsilon)+\delta_{F_2}(\varepsilon)\frac{\phi(b)}{\Gamma(\alpha+1)}$$
$$+\frac{c_2b+F_2^*}{\Gamma(\alpha+1)}\omega_{\phi(b)}(k,\varepsilon)+\frac{2[c_2b+F_2^*]}{\Gamma(\alpha+1)}\phi(b)\varepsilon^{\alpha\beta}$$

and

$$\omega(\Theta X,\varepsilon)\leqslant\Big(c_1+\frac{c_2\phi(b)}{\Gamma(\alpha+1)}\Big)\omega(X,\varepsilon)+\delta_{F_1}(\varepsilon)+\delta_{F_2}(\varepsilon)\frac{\phi(b)}{\Gamma(\alpha+1)}$$
$$+\frac{c_2b+F_2^*}{\Gamma(\alpha+1)}\omega_{\phi(b)}(k,\varepsilon)+\frac{2\alpha\beta[c_2b+F_2^*]}{\Gamma(\alpha+1)}\phi(b)\varepsilon^{\alpha}$$

respectively. Therefore, the last two inequalities implies that

$$\omega_0(\Theta X)\leqslant\Big(c_1+\frac{c_2\phi(b)}{\Gamma(\alpha+1)}\Big)\omega_0(X). \tag{4.9}$$

Step 7^{th}, we now apply Darbo fixed point theorem. Currently, the fact that $c_1+\frac{c_2\phi(b)}{\Gamma(\alpha+1)}<1$ (assumption (iv)) hostage an application of Theorem 1.3 with the operator Θ has a fixed point in Q. Hence, eq. (3.1) has atleast one nondecreasing and nonnegative solution $x\in\mathbb{C}(I)$. This is completing the proof. □

EXAMPLE 4.4. Consider the following integral equation to illustrate our investigations:

$$x(\theta)=\frac{\theta x(\theta)}{\theta+9}+\frac{3x(\theta)}{48\Gamma(\frac{1}{5})}\int_0^{\theta}\frac{\sin^{-1}\left(\frac{\theta x^2(\theta)}{1-s^2}\right)}{\sqrt{s}(\theta^{\frac{3}{2}}-s^{\frac{3}{2}})^{\frac{2}{3}}}ds \tag{4.10}$$

which is a special case of eq. (3.1).

Here, we have

$$\beta=\frac{3}{2},\quad \alpha=\frac{1}{3},$$

$$F_1(\theta,x)=\frac{\theta x}{\theta+9},\quad F_2(\theta,x)=\frac{x}{24}$$

and

$$k(\theta,s,x) = \sin^{-1}\left(\frac{\theta x^2}{1-s^2}\right).$$

The function $F_1(\theta,x) = \frac{\theta x}{\theta+9}$ is continuous of $F_1 : I\times\mathbb{R}\to\mathbb{R}$ and $F_1 : I\times\mathbb{R}_+\to\mathbb{R}_+$ and $|F_1(\theta,x)-F_1(\theta,y)|\leqslant c_1|x-y|$, where $c_1=\frac{1}{10}$. Again, the function $F_2(\theta,x)=\frac{x}{24}$ is continuous and $F_2 : I\times\mathbb{R}\to\mathbb{R}$. Also $|F_2(\theta,x)-F_2(\theta,y)|\leqslant c_2|x-y|$, where $c_2=\frac{1}{24}$. So, $F_i(\theta,x)$ $(i=1,2)$ satisfied assumption (i) with $F_i^*=0$, $i=1,2$.

Now, the function $k(\theta,s,x)=\sin^{-1}(\frac{\theta x^2}{1-s^2})$ is continuous and $k : I\times I\times\mathbb{R}\to\mathbb{R}$. Also, in each variable, it is non-decreasing, i.e. $k(\theta,s,x)$ checks assumption (ii). forby,

$$|k(\theta,s,x)|\leqslant|x^2|,\quad \text{for all}\quad \theta,s\in I\quad x\in\mathbb{R}.$$

Thus, $k(\theta,s,x)$ satisfied assumption (iii) and $\phi(b)=b^2$.

Using all the above results in the inequality 4.1, we get the form

$$\frac{b}{10}\Gamma\left(\frac{4}{3}\right)+\frac{b^3}{24}\leqslant b\Gamma\left(\frac{4}{3}\right)$$

or

$$8b\Gamma\left(\frac{1}{3}\right)+10b^3\leqslant 80b\Gamma\left(\frac{1}{3}\right).$$

The final inequality concedes a solution $b\in(0,4.39)$. Forby, $c_1\Gamma(\frac{4}{3})+c_2\phi(b)<\Gamma(\frac{4}{3})$ for all $b\in(0,4.39)$. In fact, take $b=4.38$ we have

$$\begin{aligned}
c_1\Gamma\left(\frac{4}{3}\right)+c_2\phi(b) &= \frac{1}{10}\Gamma\left(\frac{4}{3}\right)+\frac{1}{24}(4.38)^2\\
&\cong 0.089+0.799\\
&= 0.888\\
&< 0.893\\
&\cong \Gamma\left(\frac{4}{3}\right).
\end{aligned}$$

Hence, Theorem 4.3 proofs that the eq. (4.10) has atleast one non-negative and nondecreasing and continuous solution $x(\theta)$ such that $|x|\leqslant 4.39$.

Data Availability Statement. No data were used to support this study.

Competing Interest. The authors declare that there is not any competing interest regarding the publication of this manuscript.

Author contributions. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding. This research received no external funding.

Acknowledgements. Not applicable.

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(Received June 17, 2024)

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