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SOLVABILITY OF A FRACTIONAL DIFFERENTIAL PROBLEM FOR BEAM EQUILIBRIUM

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Abstract. In this work, we investigate a novel nonlinear differential problem involving Caputo derivatives. The problem features sequential derivatives that do not adhere to the semi-group and commutativity properties. Under certain specific conditions, the problem simplifies to a fourth-order ordinary problem, representing the static equilibrium of an elastic beam. Using the Banach contraction principle followed by Schaefer's fixed point theorem, we establish two key results regarding the uniqueness of solutions and the existence of at least one solution. We provide an example to validate one of these results. Additionally, we discuss Ulam-Hyers stability, including noteworthy limiting-case examples.

1. Introduction

Fractional calculus generalizes the concepts of classic differential calculus to non-integer order, so fractional differential equations are a generalization of classical differential equations, where the order of differentiation [17] can be a fractional or non-integer. These equations have attracted the interest of many researchers due to their ability for modelling complex mechanisms displaying innate qualities and memory, when integer-order differential equations are not entirely descriptive [1, 2, 3, 4, 5, 13, 21]. One of the most used concepts in fractional calculus, the Caputo derivative, which is exceptionally useful [9, 14] for starting value problems, and that makes it appropriate for physical applications. A significant use of fractional differential equations is in the analysis of beam deflection systems and/or equations. The potential of beams to resist forces imposed across to their axis makes them structural elements. The classical beam theory explores how beams bend and deflect under different weights; it is generally modelled using fourth-order ordinary differential equations [27]. But the presence of fractional derivatives in these models can provide a more precise description of materials possessing viscoelastic characteristics and non-local behaviour, which are frequently found in intricate engineering materials and complex structures. This results in fractional differential models that provide profound understanding and better design skills in applied physics and engineering by more precisely modelling the deflection and dynamic response of beams. Let's mention some of the published articles that are

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related to the work underway presently. First in [27], the authors studied the existence of positive solutions for the following nonlinear system. It is employed to characterise the buckling of an elastic beam that has endpoint support.

$$\begin{aligned} &u^{(4)}(t)+\beta_1 u''(t)-\alpha_1 u(t)=f_1(t,u(t),v(t)),\ t\in(0,1)\\ &v^{(4)}(t)+\beta_2 v''(t)-\alpha_2 v(t)=f_2(t,u(t),v(t)),\ t\in(0,1)\\ &u(0)=u(1)=u''(0)=u''(1)=0,\\ &v(0)=v(1)=v''(0)=v''(1)=0, \end{aligned} \tag{1.1}$$

where, $f_i\in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$, $\mathbb{R}^+=[0,+\infty)$ and $\beta_i,\alpha_i\in\mathbb{R}$ verify $\beta_i<2\pi^2$, $-\beta_i^2\leqslant\alpha_i$, $\alpha_i/\pi^4+\beta_i/\pi^2<1$, $i=1,2$.

By giving a cone P in $C([0,1])\times C([0,1])$; where $C([0,1])$ is the space of all continuous functions from $[0,1]$ to $\mathbb{R}$, the authors proved an existence of positive solution results, then by constructing over a product cone, they estabilished another positive solution result.

In [28], the authors studied the classical problem of elastic beam differential equation with two parameters, when the nonlinear term satisfies some growth conditions only near the origin.

$$\begin{cases} u^{(4)}(x)+2h(x)u''(x)+(h^2(x)+h'(x))u''(x)=\lambda f(x,u(x)),\ x\in[0,1],\\ u(0)=u'(0)=u''(1)=u'''(1)-\mu g(u(1))=0, \end{cases} \tag{1.2}$$

where $\lambda>0$, $\mu\in\mathbb{R}$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ is a continuous function, $h\in C^1[0,1]$ is nonnegative, for $h=0$, the problem describes the static equilibrium of an elastic beam which is fixed at the left end of $x=0$ and is attached to a bearing device at the right end of $x=1$.

In [26], the authors investigated the existence and uniqueness of solutions for the following system which contains sequential Caputo derivatives; the Caputo derivative is a particular case of the new mixed fractional derivative introduced by K. Hattaf [12].

$$\begin{cases} ({}^cD^{\alpha}+\lambda\,{}^cD^{\alpha-1})x(t)=f(t,x(t),y(t),I_{0^+}^{p_1}x(t),I_{0^+}^{p_2}y(t)),\ t\in(0,1)\\ ({}^cD^{\beta}+\mu\,{}^cD^{\beta-1})y(t)=f(t,x(t),y(t),I_{0^+}^{q_1}x(t),I_{0^+}^{q_2}y(t)),\ t\in(0,1), \end{cases} \tag{1.3}$$

with

$$\begin{cases} x(0)=x'(0)=0,x'(1)=0,x(1)=\int_0^1 x(s)d\aleph_1(s)+\int_0^1 y(s)d\aleph_2(s),\\ y(0=y'(0)=0,y'(1)=0,y(1)=\int_0^1 x(s)d\daleth_1(s)+\int_0^1 y(s)d\daleth_2(s), \end{cases}$$

where, $I_{0^+}^J$ is the Riemann-Liouville integral of order ν, with $\nu=(p_1,q_1,p_2,q_2)$ and the Riemann-Stieltjes integrals with given bounded variation functions $\aleph_1,\aleph_2,\daleth_1,\daleth_2$. Such systems can be used in bio sciences, see [1, 2] and their references. The authors obtained existence and uniqueness results for solutions of the system.

In [4], the authors investigated the existence of solutions and their Ulam-stability for the problem.

$$\begin{cases} D^{\alpha}u(x)=f_1(x,u(x),v(x))+a_1g_1(x,u(x))+b_1h_1(x,u''(x)),\\ D^{\beta}v(x)=f_2(x,u(x),v(x))+a_2g_2(x,v(x))+b_2h_2(x,v''(x)), \end{cases} \tag{1.4}$$

under the conditions.

$$\begin{cases} u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

where, $x \in [0,1]$, $3 < \alpha, \beta \leqslant 4$, D^{α} and D^{β} denote the fractional derivatives in the sense of Caputo, and $f_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g_i, h_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $a_i, b_i \in \mathbb{R}$.

In [7], K. Bensaassa et al. studied existence and uniqueness of solutions and stability in the sense of Ulam Hyers of the system.

$$\begin{cases} D^{\alpha_1} D^{\alpha_2} = f_1(x, u(x), v(x)) + a_1 g_1(x, u(x)) + b_1 h_1(x, D^{\delta} u(x)), \\ D^{\beta_1} D^{\beta_2} v(x) = f_2(x, u(x), v(x)) + a_2 g_2(x, u(x)) + b_2 h_2(x, D^{\delta} u(x)), \end{cases} \tag{1.5}$$

under the conditions.

$$\begin{cases} u(0) = u(1) = a, \\ u'(0) = u'(1) = 0, \\ v(0) = v(1) = b, \\ v'(0) = v'(1) = 0, \end{cases}$$

where, $a_i, b_i \in \mathbb{R}$, for $i = 1,2$, $D^{\alpha_i}, D^{\beta_i}, D^{\delta}$ are some fractional derivatives, $0 < \delta \leqslant 1$, $f_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g_i, h_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

In [6], the authors studied with the existence and uniqueness of solutions for the following coupled system with several seqential derivatives.

$$\begin{cases} D^{\alpha_1} D^{\alpha_2} D^{\alpha_3} D^{\alpha_4} x(t) = H_1(t, x(t), y(t)) + a_1 f_1(x(t)) + b_1 g_1(D^{\alpha_1} D^{\alpha_2} x(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J = [0,1] \\ D^{\beta_1} D^{\beta_2} D^{\beta_3} D^{\beta_4} y(t) = H_2(t, x(t), y(t)) + a_2 f_2(y(t)) + b_2 g_2(D^{\beta_1} D^{\beta_2} y(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J = [0,1] \\ x(0) = x(1) = D^{\alpha_1} D^{\alpha_2} x(1) = D^{\alpha_4} x(0) = 0, \\ y(0) = y(1) = D^{\beta_1} D^{\beta_2} y(1) = D^{\beta_4} y(0) = 0, \end{cases} \tag{1.6}$$

where, $D^{\alpha_1}, D^{\alpha_2}, D^{\alpha_3}, D^{\alpha_4}, D^{\beta_1}, D^{\beta_2}, D^{\beta_3}, D^{\beta_4}$ are Caputo fractional derivatives, $0 < \alpha_i \leqslant 1$, $0 < \beta_i \leqslant 1$, $i = 1, \cdots, 4$, $\alpha_2 + \alpha_1 < \alpha_4$, $\beta_1 + \beta_2 < \beta_4$, $f_j : \mathbb{R} \to \mathbb{R}$, $g_j : \mathbb{R} \to \mathbb{R}$ and $H_j : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $j = 1,2$ are continuous functions, and $H_i(t,0,0) \neq 0$, $f_i(0) \neq 0$, $g_i(0) \neq 0$, $i = 1,2$.

Very recently, Y. Boulatriour et Z. Dahmani [8] studied the following generalized problem associated with elastic beam system in which the authors introduced a supplementary derivative in the sense of Caputo of order $\delta - 1$; $1 < \delta < 2$ and two general nonlinearities F_1, F_2.

$$D^{\alpha_1} D^{\beta_1} u_1(x) = F_1(x, u_1(x), u_2(x), D^{\delta-1} u_1(x), D^{\delta} u_1(x)), \tag{1.7}$$

$$D^{\alpha_2} D^{\beta_2} u_2(x) = F_2(x, u_1(x), u_2(x), D^{\delta-1} u_2(x), D^{\delta} u_2(x)), \tag{1.8}$$

under the conditions

$$\begin{cases} u_1(0) = u_1(1) = \theta_1, \\ u_1'(0) = u_1'(1) = 0, \\ u_2(0) = u_2(1) = \theta_2, \\ u_2'(0) = u_2'(1) = 0, \end{cases}$$

where, for $i = 1,2$, $\theta_i \in \mathbb{R}^+$, $F_i \in C([0,1] \times \mathbb{R}^4, \mathbb{R})$ and $D^{\alpha_i}, D^{\beta_i}, D^{\delta}$ are Caputo fractional derivatives, with $0 < \alpha_1, \alpha_2 \leqslant 1$, $2 < \beta_1, \beta_2 \leqslant 3$. The authors explored the system by using the Banach contraction principle and the Schauder fixed point theorem to prove the uniqueness of solutions and the existence of at least one solution. Ulam-Hyers was also discussed by the authors.

In the present paper and with the intention of advancing the research mentioned previously and done in [29] in the fractional case, we investigate the existence of solutions and Ulam-stability of the following Caputo fractional problem including sequential derivatives:

$$\begin{cases} D^{\alpha_1} D^{\alpha_2} D^{\alpha_3} D^{\alpha_4} \phi(t) = h_1(t, D^{\alpha_2} D^{\alpha_3} D^{\alpha_4} \phi(t)) + h_2(t, D^{\alpha_4} \phi(t), D^{\alpha_3} D^{\alpha_4} \phi(t)) \\ \qquad\qquad\qquad\qquad\qquad + h_3(t, \phi(t)), \quad t \in J, \\ \phi(0) = 0 \\ D^{\alpha_4} \phi(0) = 0 \\ D^{\alpha_3} D^{\alpha_4} \phi(1) = 0 \\ D^{\alpha_2} D^{\alpha_3} D^{\alpha_4} \phi(1) - \mu g(\phi(1)) = 0 \\ 0 < \alpha_i \leqslant 1, \quad i = 1,2,3,4, \quad \mu \in \mathbb{R}, \end{cases} \tag{1.9}$$

where $J := [0,1]$, D^{α_i} are Caputo fractional derivatives, $\alpha_1 + \alpha_2 > 1$; $\alpha_2 + \alpha_3 > 1$; $\alpha_3 + \alpha_4 > 1$. For $i = 1,3$, $h_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $h_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ represents force of the bearing device.

The reader can observe that our problem is more general than the above problems, they also can see that when α_i; $i = 1,2,3$ are close to 1, we approach the particular case [28]. The sequential idea on the derivatives of the system's left side and the occurrence of derivative terms on both sides of the same system constitute the novelty of our problem.

The paper is organized as follows: In Section 2, we introduce some notions and notation that will be used throughout the manuscript. In Section 3, we transform our problem to a fixed point problem. Section 4 is concerned with the main existence results for the problem (1.9) and his Ulam Hayers stability. Finally, in the last section, we give several examples to reinforce the results obtained.

2. Preliminaries and background material

We need to introduce the Caputo derivatives and some other auxiliary lemmas, we refer to the reference [14, 17].

DEFINITION 1. Let $\alpha > 0$ and $f : J \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral is defined by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

DEFINITION 2. For any $f \in C^n(J, \mathbb{R})$ and $n-1 < \alpha \leqslant n$, the Caputo derivative is defined by:

$$D^{\alpha} f(t) = I^{n-\alpha} \frac{d^n}{dt^n}(f(t)).$$

To study (1.9), we need the following two results [14, 17]:

LEMMA 1. *Let $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$. Then, the set of solutions of $D^{\alpha} y(t) = 0$; $t \in J$ is given by the polynomial expression:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i,$$

where, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$.

LEMMA 2. *In the case of $n \in \mathbb{N}^*$ and $n-1 < \alpha < n$, we have the property*

$$I^{\alpha} D^{\alpha} y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i,$$

with $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$.

THEOREM 1. (Schauder Fixed Point Theorem, [10]) *If U is a nonempty, closed, bounded convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has a fixed point in U.*

THEOREM 2. (Arzela Ascoli's theorem, [22]) *Suppose U is a compact space, $C(U)$ is the supnormed Banach space of all continuous complex functions on U, and $\varphi \subset C(U)$ is pointwise bounded and equicontinuous.*

Then φ is totally bounded in $C(U)$.

Now, we prove the following equivalence between the above differential problem and its integral equation:

LEMMA 3. *The differential problem*

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t) = H(t), \\ \phi(0) = 0 \\ D^{\alpha_4}\phi(0) = 0 \\ D^{\alpha_3}D^{\alpha_4}\phi(1) = 0 \\ D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1) - \mu g(\phi(1)) = 0 \\ 0 < \alpha_i \leqslant 1, \ \ i = 1,2,3,4, \ \ \mu \in \mathbb{R} \end{cases} \tag{2.1}$$

is equivalent to the integral problem

$$\phi(t) = I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}H(t) + [\mu g(\phi(1)) - I^{\alpha_1}H(1)]\frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} \tag{2.2}$$

$$+\left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H(1)\right]\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}. \tag{2.3}$$

Proof. First and thanks to Lemma 2, we have

$$\begin{aligned} D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t) &= I^{\alpha_1}H(t) + \rho_1 \\ D^{\alpha_3}D^{\alpha_4}\phi(t) &= I^{\alpha_1+\alpha_2}H(t) + \rho_1\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \rho_2 \\ D^{\alpha_4}\phi(t) &= I^{\alpha_1+\alpha_2+\alpha_3}H(t) + \rho_1\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + \rho_2\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)} + \rho_3 \\ \phi(t) &= I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}H(t) + \rho_1\frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} + \rho_2\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} \\ &\quad + \rho_3\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)} + \rho_4. \end{aligned} \tag{2.4}$$

We obtain the two first constants as follows:

$$\phi(0) = 0 \Rightarrow \rho_4 = 0,$$

$$D^{\alpha_4}\phi(0) = 0 \Rightarrow \rho_3 = 0.$$

By considering the remaining conditions, we get

$$D^{\alpha_3}D^{\alpha_4}\phi(1) = 0$$

and

$$D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1) - \mu g(\phi(1)) = 0.$$

Hence, it yields that

$$\rho_1 = \mu g(\phi(1)) - I^{\alpha_1}H(1),$$

and
$$\rho_2 = \frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H(1).$$
For the second implication, we can see that
$$\begin{aligned}
\phi(0) &= 0\\
D^{\alpha_4}\phi(t) &= I^{\alpha_1+\alpha_2+\alpha_3}H(t) + [\mu g(\phi(1)) - I^{\alpha_1}H(1)]\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)}\\
&\quad + \left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H(1)\right]\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)},
\end{aligned}$$
and
$$D^{\alpha_4}\phi(0) = 0,$$
$$\begin{aligned}
D^{\alpha_3}D^{\alpha_4}\phi(t) &= I^{\alpha_1+\alpha_2}H(t) + [\mu g(\phi(1)) - I^{\alpha_1}H(1)]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}\\
&\quad + \left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H(1)\right],
\end{aligned}$$
and
$$D^{\alpha_3}D^{\alpha_4}\phi(1) = 0,$$
$$D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t) = I^{\alpha_1}H(t) + [\mu g(\phi(1)) - I^{\alpha_1}H(1)];$$
and
$$D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1) - \mu g(\phi(1)) = 0.$$
The proof is thus achieved. $\square$

3. Transformation of the problem

In what follows, we use fixed point theory to study the above problem. Thus, we can transform our problem into a fixed point problem.

Let introduce the space:
$$\Psi := \{\phi \in C(J,\mathbb{R}), D^{\alpha_4}\phi \in C(J,\mathbb{R}), D^{\alpha_3}D^{\alpha_4}\phi \in C(J,\mathbb{R}), D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi \in C(J,\mathbb{R})\},$$
we can easily verify that Ψ is a vector space and
$$\|\phi\|_\Psi = \|\phi\|_\infty + \|D^{\alpha_4}\phi\|_\infty + \|D^{\alpha_3}D^{\alpha_4}\phi\|_\infty + \|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty$$
such that,
$$\|\phi\|_\infty = \sup_{t\in J}|\phi(t)|, \quad \|D^{\alpha_4}\phi\|_\infty = \sup_{t\in J}|D^{\alpha_4}\phi(t)|,$$
$$\|D^{\alpha_3}D^{\alpha_4}\phi\|_\infty = \sup_{t\in J}|D^{\alpha_3}D^{\alpha_4}\phi(t)|$$

and

$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty = \sup_{t\in J}|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t)|,$$

is a norm.

Let us verify that Ψ is complete. We take (ϕ_n) as a Cauchy sequence in Ψ, so

$$\|\phi_n - \phi_m\|_\Psi \xrightarrow[n,m\to+\infty]{} 0.$$

But,

$$\begin{aligned}\|\phi_n - \phi_m\|_\Psi = \|\phi_n - \phi_m\|_\infty + \|D^{\alpha_4}\phi_n - D^{\alpha_4}\phi_m\|_\infty + \|D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_3}D^{\alpha_4}\phi_m\|_\infty \\ +\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_m\|_\infty,\end{aligned}$$

and

$$\begin{aligned}&\|\phi_n - \phi_m\|_\Psi \geqslant \|\phi_n - \phi_m\|_\infty,\\ &\|\phi_n - \phi_m\|_\Psi \geqslant \|D^{\alpha_4}\phi_n - D^{\alpha_4}\phi_m\|_\infty,\\ &\|\phi_n - \phi_m\|_\Psi \geqslant \|D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_3}D^{\alpha_4}\phi_m\|_\infty,\\ &\|\phi_n - \phi_m\|_\Psi \geqslant \|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_m\|_\infty,\end{aligned}$$

therefore,

$$\|\phi_n - \phi_m\|_\infty \xrightarrow[n,m\to+\infty]{} 0, \quad \|D^{\alpha_4}\phi_n - D^{\alpha_4}\phi_m\|_\infty \xrightarrow[n,m\to+\infty]{} 0,$$
$$\|D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_3}D^{\alpha_4}\phi_m\|_\infty \xrightarrow[n,m\to+\infty]{} 0$$

and

$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_m\|_\infty \xrightarrow[n,m\to+\infty]{} 0.$$

Then (ϕ_n), $(D^{\alpha_4}\phi_n)$, $(D^{\alpha_3}D^{\alpha_4}\phi_n)$ and $(D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n)$ are Cauchy sequences in $\left(C(J,\mathbb{R}),\|.\|_\infty\right)$, which is complete. Therefore, there exists ϕ in $C(J,\mathbb{R})$ such that

$$\|\phi_n - \phi\|_\infty \xrightarrow[n\to+\infty]{} 0,$$

and

$$\|D^{\alpha_4}\phi_n - D^{\alpha_4}\phi\|_\infty \xrightarrow[n\to+\infty]{} 0, \quad \|D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_3}D^{\alpha_4}\phi\|_\infty \xrightarrow[n\to+\infty]{} 0,$$
$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n - D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty \xrightarrow[n\to+\infty]{} 0,$$

then,

$$\|\phi_n - \phi\|_\Psi \xrightarrow[n\to+\infty]{} 0.$$

We conclude that $(\Psi,\|.\|_\Psi)$ is a Banach space.

We take the nonlinear operator $B:\Psi\to\Psi$ such that for all $t\in J$:

$$B\phi(t)=I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}H_\phi^*(t)+[\mu g(\phi(1))-I^{\alpha_1}H_\phi^*(1)]\frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)}$$
$$+\left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H_\phi^*(1)-\mu g(\phi(1)))-I^{\alpha_1+\alpha_2}H_\phi^*(1)\right]\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)},$$

where,

$$H_\phi^*(t)=h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t))+h_2(t,D^{\alpha_4}\phi(t),D^{\alpha_3}D^{\alpha_4}\phi(t))+h_3(t,\phi(t)).$$

4. Main results

We consider the following hypotheses:

$(\Xi 1)$: The functions h_1 and h_3 defined on $[0,1]\times\mathbb{R}$ are continuous. The function h_2 defined on $[0,1]\times\mathbb{R}^2$ is continuous. The function g defined on $\mathbb{R}$ is continuous.

$(\Xi 2)$: Suppose the existence of nonnegative constants r_{h1}, such that for any $t\in J$, $\phi,\phi^*\in\mathbb{R}$, we have

$$|h_1(t,\phi)-h_1(t,\phi^*)|\leqslant r_{h1}|\phi-\phi^*|.$$

Suppose also that there are nonnegative constants $r_{(h2)_1}$, $r_{(h2)_2}$, such that for any $t\in J$, $\phi,\phi^*\in\mathbb{R}$, we have

$$|h_2(t,\phi_1,\phi_2)-h_2(t,\phi_1{}^*,\phi_2{}^*)|\leqslant\sum_{i=1}^{2}r_{(h2)_i}|\phi_i-\phi_i{}^*|,$$

with;

$$r_{h2}=\mathrm{Max}\,(r_{(h2)_1},r_{(h2)_2}),$$

and, for any $t\in J$, $\phi,\phi^*\in\mathbb{R}$, we have

$$|h_3(t,\phi)-h_3(t,\phi^*)|\leqslant r_{h3}|\phi-\phi^*|,$$
$$|g(\phi)-g(\phi^*)|\leqslant r_g|\phi-\phi^*|.$$

The following quantities are to be considered:

$$\varphi_1:=R\Gamma_1^4+(\mu r_g+R\Gamma_1^1)\Gamma_2^4+(\mu r_g\Gamma_2^2+R\Gamma_1^1\Gamma_2^2+R\Gamma_1^2)\Gamma_3^4,$$

$$\varphi_2:=R\Gamma_1^3+(\mu r_g+R\Gamma_1^1)\Gamma_2^3+(\mu r_g\Gamma_2^2+R\Gamma_1^1\Gamma_2^2+R\Gamma_1^2)\Gamma_3^3,$$

$$\varphi_3:=R\Gamma_1^2+(\mu r_g+R\Gamma_1^1)\Gamma_2^2+(\mu r_g\Gamma_2^2+R\Gamma_1^1\Gamma_2^2+R\Gamma_1^2),$$

$$\varphi_4:=2R\Gamma_1^1+\mu r_g,$$

where,

$$\Gamma_k^n:=\frac{1}{\Gamma(\sum_{i=k}^{n}\alpha_i+1)},\quad R=r_{h1}+r_{h2}+r_{h3}.$$

4.1. An existence and uniqueness result

Our first main result is the following theorem in which we establish a unique solution for our problem. The proof is based on the principle of contraction of Banach:

THEOREM 3. *Assume that the two hypotheses* (Ξ_1) *and* (Ξ_2) *are satisfied. Then, the problem* (1.9) *has a unique solution, provided that* $\sum_{i=1}^{4} \varphi_i < 1$.

Proof. According to Lemma 3, the fixed points of the operator B are solutions of problem (1.9),

Let $(\phi, \phi') \in \Psi^2$. For $t \in J$, we have

$$\begin{aligned}
|B\phi(t) - B\phi'(t)| \leqslant\ & I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}|h_1(t, D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t)) - h_1(t, D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi'(t))| \\
&+ I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}|h_2(t, D^{\alpha_4}\phi(t)) - h_2(t, D^{\alpha_4}\phi'(t))| \\
&+ I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}|h_3(t, \phi(t)) - h_3(t, \phi'(t))| \\
&+ \Gamma_2^4|\mu||g(\phi(1)) - g(\phi'(1))| + \Gamma_2^4 I^{\alpha_1}|H_\phi^*(1) - H_{\phi'}^*(1)| \\
&+ \Gamma_3^4\Gamma_2^2 I^{\alpha_1}|H_\phi^*(1) - H_{\phi'}^*(1)| + \Gamma_3^4\Gamma_2^2|\mu||g(\phi(1)) - g(\phi'(1))| \\
&+ \Gamma_3^4 I^{\alpha_1+\alpha_2}|H_\phi^*(1) - H_{\phi'}^*(1)|,
\end{aligned}$$

thus,

$$\|B\phi - B\phi'\|_\infty \leqslant [R\Gamma_1^4 + (\mu r_g + R\Gamma_1^1)\Gamma_2^4 + (\mu r_g\Gamma_2^2 + R\Gamma_1^1\Gamma_2^2 + R\Gamma_1^2)\Gamma_3^4]\|\phi - \phi'\|_\Psi.$$

On the other hand, we have

$$\begin{aligned}
D^{\alpha_4}B\phi(t) =\ & I^{\alpha_1+\alpha_2+\alpha_3}H_\phi^*(t) + [\mu g(\phi(1)) - I^{\alpha_1}H_\phi^*(1)]\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} \\
&+ \left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H_\phi^*(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H_\phi^*(1)\right]\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)},
\end{aligned}$$

and

$$\|D^{\alpha_4}B\phi - D^{\alpha_4}B\phi'\|_\infty \leqslant [R\Gamma_1^3 + (\mu r_g + R\Gamma_1^1)\Gamma_2^3 + (\mu r_g\Gamma_2^2 + R\Gamma_1^1\Gamma_2^2 + R\Gamma_1^2)\Gamma_3^3]\|\phi - \phi'\|_\Psi.$$

Also, we get

$$\begin{aligned}
D^{\alpha_3}D^{\alpha_4}B\phi(t) =\ & I^{\alpha_1+\alpha_2}H_\phi^*(t) + [\mu g(\phi(1)) - I^{\alpha_1}H_\phi^*(1)]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&+ \left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H_\phi^*(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H_\phi^*(1)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|D^{\alpha_3}D^{\alpha_4}B\phi - D^{\alpha_3}D^{\alpha_4}B\phi'\|_\infty \\
&\leqslant [R\Gamma_1^2 + (\mu r_g + R\Gamma_1^1)\Gamma_2^2 + (\mu r_g\Gamma_2^2 + R\Gamma_1^1\Gamma_2^2 + R\Gamma_1^2)]\|\phi - \phi'\|_\Psi.
\end{aligned}$$

On the other hand, we can write

$$D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi(t) = I^{\alpha_1}H_\phi^*(t) + [\mu g(\phi(1)) - I^{\alpha_1}H_\phi^*(1)].$$

Hence, we get

$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi - D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi'\|_\infty \leqslant [2R\Gamma_1^1 + \mu r_g]\|\phi - \phi'\|_\Psi.$$

Consequently,

$$\|B\phi - B\phi'\|_\Psi \leqslant (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)\|\phi - \phi'\|_\Psi.$$

The above condition imposed on $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ and by the Banach contraction principle, B has a unique fixed point which is a unique solution of the problem (1.9). □

4.2. An example

EXAMPLE 1. We consider the following problem:

$$\begin{cases} D^{0.5}D^{0.6}D^{0.7}D^{0.8}\phi(t) = \dfrac{1}{200e^{t+2}}\sin(D^{0.6}D^{0.7}D^{0.8}\phi(t)) \\ \qquad\qquad + \dfrac{1}{100e^{t+3}} \dfrac{(D^{0.8}\phi + D^{0.7}D^{0.8}\phi(t))^2 + |D^{0.8}\phi(t) + D^{0.7}D^{0.8}\phi(t)|}{1 + |D^{0.8}\phi(t) + D^{0.7}D^{0.8}\phi(t)|} \\ \qquad\qquad + \dfrac{1}{50\pi^3 + t^2}\cos(\phi(t)) \\ \phi(0) = 0 \\ D^{0.8}\phi(0) = 0 \\ D^{0.7}D^{0.8}\phi(1) = 0 \\ D^{0.6}D^{0.7}D^{0.8}\phi(1) - \dfrac{3}{400}\cos(\phi(1)) = 0, \end{cases} \tag{4.1}$$

where,

$$\begin{aligned} h_1(t,\phi) &= \frac{1}{200e^{t+2}}\sin(\phi), \\ h_2(t,\phi,\phi^*) &= \frac{1}{100e^{t+3}} \frac{(\phi+\phi^*)^2 + |\phi+\phi^*|}{1 + |\phi+\phi^*|}, \\ h_3(t,\phi) &= \frac{1}{50\pi^3 + t^2}\cos(\phi), \\ g(t) &= \cos(t), \quad \mu = \frac{3}{400}. \end{aligned}$$

We have:

$$\begin{aligned} &\varphi_1 = 0.0131589, \quad \varphi_2 = 0.0212733, \\ &\varphi_3 = 0.0229992, \quad \varphi_4 = 0.0106595, \\ &\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 0.0680909, \end{aligned}$$

The conditions of Theorem 3 hold. Therefore, problem (4.1) has a unique solution over $[0,1]$.

4.3. Stability results

DEFINITION 3. The problem (1.9) has the Ulam Hyers stability if there exists a real number $\Omega > 0$, such that for each $\upsilon > 0$, $t \in J$ and for each $\phi \in \Psi$ solution of the inequality

$$\begin{aligned}
&\big|D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t) - h_1(t, D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t)) \\
&\qquad\qquad - h_2(t, D^{\alpha_4}\phi(t), D^{\alpha_3}D^{\alpha_4}\phi(t)) - h_3(t,\phi(t))\big| \leqslant \upsilon, \\
&\phi(0) = 0 \\
&D^{\alpha_4}\phi(0) = 0 \\
&D^{\alpha_3}D^{\alpha_4}\phi(1) = 0 \\
&D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1) - \mu g(\phi(1)) = 0
\end{aligned} \tag{4.2}$$

there exists $\phi^* \in \Psi$ solution of (1.9), such that

$$\|\phi - \phi^*\|_\Psi \leqslant \Omega\upsilon.$$

DEFINITION 4. The problem (1.9) has the Ulam Hyers stability in the generalized sense if there exists $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $f(0) = 0$, such that for each $\upsilon > 0$, and for any $\phi \in \Psi$ solution of (4.2), there exists a solution $\phi^* \in \Psi$ of (1.9), such that

$$\|\phi - \phi^*\|_\Psi < f(\upsilon).$$

Now, we are able to prove the second main result.

THEOREM 4. *Under the conditions of Theorem 3, problem (1.9) is Ulam Hyers stable.*

Proof. Let $\phi \in \Psi$ be a solution of inequality (4.2) and assume (by Theorem 3) that $\phi^* \in \Psi$ is the unique solution of problem (1.9).

By integration of the inequality (4.2), we obtain

$$\begin{aligned}
&\Bigg|\phi(t) - I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}H^*_\phi(t) + [\mu g(\phi(1)) - I^{\alpha_1}H^*_\phi(1)]\frac{t^{\alpha_2+\alpha_3+\alpha_4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} \\
&+ \left[\frac{1}{\Gamma(\alpha_2+1)}(I^{\alpha_1}H^*_\phi(1) - \mu g(\phi(1))) - I^{\alpha_1+\alpha_2}H^*_\phi(1)\right]\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\Bigg| \\
&\leqslant \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)}.
\end{aligned} \tag{4.3}$$

By adding and subtracting ϕ^*, using (4.2) and (4.3), we obtain the following estimate

$$\begin{aligned}
\|\phi - \phi^*\|_\infty \leqslant{}& \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)} \\
&+ [R\Gamma_1^4 + (\mu r_g + R\Gamma_1^1)\Gamma_2^4 + (\mu r_g\Gamma_2^2 + R\Gamma_1^1\Gamma_2^2 + R\Gamma_1^2)\Gamma_3^4]\|\phi - \phi^*\|_\infty,
\end{aligned} \tag{4.4}$$

which implies that

$$\|\phi-\phi^*\|_\infty \leqslant \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)}+\varphi_1\|\phi-\phi^*\|_\infty.$$

Therefore,

$$\|\phi-\phi^*\|_\infty \leqslant \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)(1-\varphi_1)} \leqslant \upsilon\sigma.$$

With the same arguments as before, we can write

$$\|D^{\alpha_4}(\phi-\phi^*)\|_\infty \leqslant \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)(1-\varphi_2)} \leqslant \upsilon\sigma^*,$$

$$\|D^{\alpha_3}D^{\alpha_4}(\phi-\phi^*)\|_\infty \leqslant \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)(1-\varphi_3)} \leqslant \upsilon\varpi,$$

and

$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}(\phi-\phi^*)\|_\infty \leqslant \frac{\upsilon}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4+1)(1-\varphi_4)} \leqslant \upsilon\varpi^*.$$

Thus,

$$\|\phi-\phi^*\|_\Psi \leqslant \upsilon\big(\,\sigma+\sigma^*+\varpi+\varpi^*\big).$$

And this implies, by definition, that (1.9) has the Ulam Hyers stability. $\square$

REMARK 1. Taking $f(\upsilon) := \upsilon\big(\sigma+\sigma^*+\varpi+\varpi^*\big)$, we obtain the generalised Ulam Hyers stability of (1.9).

4.4. Existence of solution

Our second existence result of (1.9) is based on the fixed point theorem of Schauder, We consider the following hypothesis.

$(\Xi 3)$: For any $u,v\in\mathbb{R}$ and each $t\in J$,

$$|h_1(t,u)| \leqslant r_{h1}|u|+r^*_{h1};\; r^*_{h1}=\sup_{t\in J} h_1(t,0),$$

$$|h_2(t,u,v)| \leqslant r_{(h2)_1}|u|+r_{(h2)_2}|v|+r^*_{h2};\; r^*_{h_2}=\sup_{t\in J} h_2(t,0,0),$$

$$|h_3(t,u)| \leqslant r_{h3}|u|+r^*_{h3};\; r^*_{h3}=\sup_{t\in J} h_3(t,0),$$

$$|g(t,u)| \leqslant r_g|u|+r^*_g;\; r^*_g=\sup_{t\in J} g(t,0).$$

The following quantities are to be considered.

$$R^*=r^*_{h1}+r^*_{h2}+r^*_{h3};\; \Gamma_4=\Gamma_1^4+\Gamma_2^4\Gamma_1^1+\Gamma_3^4(\Gamma_1^2+\Gamma_2^2\Gamma_1^1);$$

$$\Gamma_3=\Gamma_1^3+\Gamma_2^3\Gamma_1^1+\Gamma_3^3(\Gamma_1^2+\Gamma_2^2\Gamma_1^1);\; \Gamma_2=\Gamma_1^2+\Gamma_2^2,$$

$$\vartheta_1=\Gamma_4+\Gamma_3+2\Gamma_2+4\Gamma_1^1;\; \vartheta_2=\Gamma_2^4+\Gamma_3^4\Gamma_2^2+\Gamma_2^3+\Gamma_3^3\Gamma_2^2+2\Gamma_2^2+2.$$

THEOREM 5. *Assume that* $(\Xi_1),(\Xi_3)$ *hold. If*

$$\vartheta_1 R+\vartheta_2|\mu|r_g<1, \tag{4.5}$$

then problem (1.9) *has at least one solution.*

Proof. We'll take several steps to establish the proof.

Step 1. B is continuous.

Let (ϕ_n) be a sequence in ψ such that $\|\phi_n-\phi\|_\psi\to 0$, as $n\to\infty$. For $t\in J$, we have

$$\begin{aligned}
&|B\phi_n(t)-B\phi(t)|\\
\leqslant\ & I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\Big(|h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_2(t,D^{\alpha_4}\phi_n(t),D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_2(t,D^{\alpha_4}\phi(t),D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_3(t,\phi_n(t))-h_3(t,\phi(t))|\Big)+\Gamma_2^4|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+\Gamma_2^4 I^{\alpha_1}\Big[\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)\Big]\\
&+\Gamma_3^4\Gamma_2^2 I^{\alpha_1}\Big[\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)\Big]+\Gamma_3^4\Gamma_2^2|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+\Gamma_3^4 I^{\alpha_1+\alpha_2}\Big[\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)\Big].
\end{aligned}$$

Since, $\|\phi_n-\phi\|_\psi\to 0$ and by continuity of h_i, $i=1,2,3$ and g, we get for $t\in J$; $|B\phi_n(t)-B\phi(t)|\to 0$, as $n\to\infty$, and hence

$$\|B\phi_n-B\phi\|_\infty\to 0 \text{ as } n\to\infty.$$

Similarly, for $t\in J$ we have.

$$\begin{aligned}
&|D^{\alpha_4}B\phi_n(t)-D^{\alpha_4}B\phi(t)|\\
\leqslant\ & I^{\alpha_1+\alpha_2+\alpha_3}\Big(|h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_2(t,D^{\alpha_4}\phi_n(t),D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_2(t,D^{\alpha_4}\phi(t),D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_3(t,\phi_n(t))-h_3(t,\phi(t))|\Big)+\Gamma_2^3|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+\Gamma_2^3 I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|
\end{aligned}$$

$$
\begin{aligned}
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)\\
&+\Gamma_3^3\Gamma_2^2 I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)+\Gamma_3^3\Gamma_2^2|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+\Gamma_3^3 I^{\alpha_1+\alpha_2}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big).
\end{aligned}
$$

With the same arguments as before, we get

$$\|D^{\alpha_4}B\phi_n-D^{\alpha_4}B\phi\|_\infty\to 0,\ \text{ as }\ n\to\infty,$$

and

$$
\begin{aligned}
&|D^{\alpha_3}D^{\alpha_4}B\phi_n(t)-D^{\alpha_4}B\phi(t)|\\
\leqslant\ &I^{\alpha_1+\alpha_2}\Big(|h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_2(t,D^{\alpha_4}\phi_n(t),D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_2(t,D^{\alpha_4}\phi(t),D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_3(t,\phi_n(t))-h_3(t,\phi(t))|\Big)+\Gamma_2^2|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+\Gamma_2^2 I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)\\
&+\Gamma_2^2 I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big)+\Gamma_2^2|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+I^{\alpha_1+\alpha_2}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big).
\end{aligned}
$$

Hence,

$$\|D^{\alpha_3}D^{\alpha_4}B\phi_n-D^{\alpha_3}D^{\alpha_4}B\phi\|_\infty\to 0\ \text{ as }\ n\to\infty.$$

Finaly we have

$$
\begin{aligned}
&|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi_n(t)-D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi(t)|\\
\leqslant\ & I^{\alpha_1}\Big(|h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_2(t,D^{\alpha_4}\phi_n(t),D^{\alpha_3}D^{\alpha_4}\phi_n(t))-h_2(t,D^{\alpha_4}\phi(t),D^{\alpha_3}D^{\alpha_4}\phi(t))|\\
&+|h_3(t,\phi_n(t))-h_3(t,\phi(t))|\Big)+|\mu||g(\phi_n(1))-g(\phi(1))|\\
&+I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_2(1,D^{\alpha_4}\phi_n(1),D^{\alpha_3}D^{\alpha_4}\phi_n(1))-h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|\\
&+|h_3(1,\phi_n(1))-h_3(1,\phi(1))|\Big).
\end{aligned}
$$

So,

$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi_n-D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi\|_\infty\to 0 \ \text{ as } \ n\to\infty.$$

Then,

$$\|B\phi_n-B\phi\|_\psi\to 0 \ \text{ as } \ n\to\infty.$$

As a result, B is continuous.

Step 2. Let us take $\rho>0$, such that

$$\rho\geqslant\frac{\vartheta_1R^*+\vartheta_2|\mu|r_g^*}{1-\vartheta},$$

with

$$\vartheta=\vartheta_1R+\vartheta_2|\mu|r_g<1,$$

and $\mathscr{C}_\rho=\{\phi\in\psi:\|\phi\|_\psi\leqslant\rho\}$.

For $\phi\in\mathscr{C}_\rho$ and for all $t\in J$, using the hypothese $(\Xi 3)$ we obtain,

$$
\begin{aligned}
&|B\phi(t)|\\
\leqslant\ & I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\Big(|h_1(t,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t))|+|h_2(t,D^{\alpha_4}\phi(t),D^{\alpha_3}D^{\alpha_4}\phi(t))|+|h_3(t,\phi(t))|\Big)\\
&+\Gamma_2^4|\mu||g(\phi(1))|\\
&+\Gamma_2^4I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|+|h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|+|h_3(1,\phi(1))|\Big)\\
&+\Gamma_3^4\Gamma_2^2I^{\alpha_1}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|+|h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|+|h_3(1,\phi(1))|\Big)\\
&+\Gamma_3^4\Gamma_2^2|\mu||g(\phi(1))|\\
&+\Gamma_3^4I^{\alpha_1+\alpha_2}\Big(|h_1(1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(1))|+|h_2(1,D^{\alpha_4}\phi(1),D^{\alpha_3}D^{\alpha_4}\phi(1))|+|h_3(1,\phi(1))|\Big)\\
\leqslant\ & \Gamma_1^4\Big(r_{h1}\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h1}^*+r_{(h2)_1}\|D^{\alpha_4}\phi\|_\infty+r_{(h2)_2}\|D^{\alpha_3}D^{\alpha_4}\phi\|_\infty\\
&+r_{h2}^*+r_{h3}\|\phi\|_\infty+r_{h3}^*\Big)\\
&+\Gamma_2^4|\mu|\Big(r_g\|\phi\|_\infty+r_g^*\Big)+\Gamma_2^4\Gamma_1^1\Big(r_{h1}\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h1}^*+r_{(h2)_1}\|D^{\alpha_4}\phi\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
&+r_{(h2)_2}\|D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h2}^*+r_{h3}\|\phi\|_\infty+r_{h3}^*\Big)\\
&+\Gamma_3^4\Gamma_2^2\Gamma_1^1\Big(r_{h1}\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h1}^*+r_{(h2)_1}\|D^{\alpha_4}\phi\|_\infty\\
&+r_{(h2)_2}\|D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h2}^*+r_{h3}\|\phi\|_\infty+r_{h3}^*\Big)\\
&+\Gamma_3^4\Gamma_2^2|\mu|\Big(r_g\|\phi\|_\infty+r_g^*\Big)+\Gamma_3^4\Gamma_1^2\Big(r_{h1}\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h1}^*+r_{(h2)_1}\|D^{\alpha_4}\phi\|_\infty\\
&+r_{(h2)_2}\|D^{\alpha_3}D^{\alpha_4}\phi\|_\infty+r_{h2}^*+r_{h3}\|\phi\|_\infty+r_{h3}^*\Big).
\end{aligned}
$$

Hence

$$\|B\phi\|_\infty \leqslant \rho\Big(\Gamma_4 R+(\Gamma_2^4+\Gamma_3^4\Gamma_2^2)|\mu|r_g\Big)+\Gamma_4 R^*+(\Gamma_2^4+\Gamma_3^4\Gamma_2^2)|\mu|r_g^*.$$

In the same way, we find

$$\|D^{\alpha_4}B\phi\|_\infty \leqslant \rho\Big(\Gamma_3 R+(\Gamma_2^3+\Gamma_3^3\Gamma_2^2)|\mu|r_g\Big)+\Gamma_3 R^*+(\Gamma_2^3+\Gamma_3^3\Gamma_2^2)|\mu|r_g^*,$$

and

$$\|D^{\alpha_3}D^{\alpha_4}B\phi\|_\infty \leqslant 2\rho(\Gamma_2 R+\Gamma_2^2|\mu|r_g)+2\Gamma_2 R^*+2\Gamma_2^2|\mu|r_g^*.$$

Finaly, we obtain

$$\|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi\|_\infty \leqslant 2\rho(2\Gamma_1^1 R+|\mu|r_g)+4\Gamma_1^1 R^*+2|\mu|r_g^*,$$

and

$$\|B\phi\|_\psi \leqslant \rho\vartheta+\vartheta_1 R^*+\vartheta_2|\mu|r_g^*. \tag{4.6}$$

This implies that $\|B\phi\|_\psi \leqslant \rho$, then $B(\mathscr{C}_\rho)\subset\mathscr{C}_\rho$, hence we get the stability.

Step 3. We prove that $B(\mathscr{C}_\rho)$ is equicontinuous. Let $t_1,t_2\in J$; $t_1<t_2$ and $\phi\in\mathscr{C}_\rho$. Then,

$$
\begin{aligned}
&|B\phi(t_2)-B\phi(t_1)|\\
&\leqslant I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\Big(|h_1(t_2,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_1(t_1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_1))|\\
&\quad+|h_2(t_2,D^{\alpha_4}\phi(t_2),D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_2(t_1,D^{\alpha_4}\phi(t_1),D^{\alpha_3}D^{\alpha_4}\phi(t_1))|\\
&\quad+|h_3(t_2,\phi(t_2))-h_3(t_1,\phi(t_1))|\Big)\leqslant\Gamma_1^4\lambda\,(t_2^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}-t_1^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}).
\end{aligned}
\tag{4.7}
$$

We have also

$$
\begin{aligned}
&|D^{\alpha_4}B\phi(t_2)-D^{\alpha_4}B\phi(t_1)|\\
&\leqslant I^{\alpha_1+\alpha_2+\alpha_3}\Big(|h_1(t_2,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_1(t_1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_1))|\\
&\quad+|h_2(t_2,D^{\alpha_4}\phi(t_2),D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_2(t_1,D^{\alpha_4}\phi(t_1),D^{\alpha_3}D^{\alpha_4}\phi(t_1))|\\
&\quad+|h_3(t_2,\phi(t_2))-h_3(t_1,\phi(t_1))|\Big)\leqslant\Gamma_1^3\lambda\,(t_2^{\alpha_1+\alpha_2+\alpha_3}-t_1^{\alpha_1+\alpha_2+\alpha_3}),
\end{aligned}
\tag{4.8}
$$

and

$$\begin{aligned}
&|D^{\alpha_3}D^{\alpha_4}B\phi(t_2)-D^{\alpha_3}D^{\alpha_4}B\phi(t_1)| \\
\leqslant\ & I^{\alpha_1+\alpha_2}\Big(|h_1(t_2,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_1(t_1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_1))| \\
&+|h_2(t_2,D^{\alpha_4}\phi(t_2),D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_2(t_1,D^{\alpha_4}\phi(t_1),D^{\alpha_3}D^{\alpha_4}\phi(t_1))| \\
&+|h_3(t_2,\phi(t_2))-h_3(t_1,\phi(t_1))|\Big)\leqslant \Gamma_1^2\lambda(t_2^{\alpha_1+\alpha_2}-t_1^{\alpha_1+\alpha_2}).
\end{aligned} \tag{4.9}$$

Finaly, we get

$$\begin{aligned}
&|D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi(t_2)-D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}B\phi(t_1)| \\
\leqslant\ & I^{\alpha_1}\Big(|h_1(t_2,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_1(t_1,D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}\phi(t_1))| \\
&+|h_2(t_2,D^{\alpha_4}\phi(t_2),D^{\alpha_3}D^{\alpha_4}\phi(t_2))-h_2(t_1,D^{\alpha_4}\phi(t_1),D^{\alpha_3}D^{\alpha_4}\phi(t_1))| \\
&+|h_3(t_2,\phi(t_2))-h_3(t_1,\phi(t_1))|\Big)\leqslant \Gamma_1^{\lambda}(t_2^{\alpha_1}-t_1^{\alpha_1}),
\end{aligned} \tag{4.10}$$

The right hand sides of (4.7), (4.8), (4.9) and (4.10) are independent of ϕ and tend to zero as $\vartheta_2 \to \vartheta_1$.

From Step 1 to Step 3 and the Arzela-Ascoli theorem, we conclued that B is completely continuous. Then by Schauder theorem, B has a fixed point in ψ. □

5. Examples

EXAMPLE 2. We consider the following problem:

$$\begin{cases}
D^{0.55}D^{0.68}D^{0.7}D^{0.75}\phi(t)=\dfrac{1}{300e^{t+3}}\sin\left(D^{0.68}D^{0.7}D^{0.75}\phi\left(t+\dfrac{\pi}{4}\right)\right) \\
\qquad\qquad +\dfrac{e^t+2t+1}{500e^{t+5}}\dfrac{1}{1+|D^{0.75}\phi(t)+3|+|\sqrt[3]{D^{0.7}D^{0.75}\phi(t)}|} \\
+\dfrac{1}{200e^{t+2}}\dfrac{1}{95(1+|\phi|)} \\
\phi(0)=0 \\
D^{0.75}\phi(0)=0 \\
D^{0.7}D^{0.75}\phi(1)=0 \\
D^{0.68}D^{0.7}D^{0.8}\phi(1)-\dfrac{5}{700}\cos(\phi(1))=0,
\end{cases} \tag{5.1}$$

where,

$$\begin{aligned}
h_1(t,\phi)&=\frac{1}{300e^{t+3}}\sin\left(\phi+\frac{\pi}{4}\right), \\
h_2(t,\phi,\phi^*)&=\frac{e^t+2t+1}{500e^{t+5}}\frac{1}{1+|\phi+3|+|\sqrt[3]{\phi^*}|},
\end{aligned}$$

$$h_3(t,\phi) = \frac{1}{200e^{t+2}}\left[\frac{1}{95(1+|\phi|)}\right],$$
$$g(t) = \cos(t), \quad \mu = \frac{5}{700}.$$

The condition (4.5) is verified,

$$\vartheta = \vartheta_1 R + \vartheta_2|\mu|r_g = 0.0703008 < 1$$

then (5.1) has at least one solution.

EXAMPLE 3. We consider the following problem:

$$\begin{cases} D^{0.8}D^{0.5}D^{0.6}D^{0.7}\phi(t) = \dfrac{\sin(t)}{200e^{t+5}}\dfrac{1}{1+|D^{0.5}D^{0.6}D^{0.7}\phi(t)|} \\ \qquad\qquad +\dfrac{1}{233+33e^{3-t}}\left[1+\dfrac{|D^{0.7}\phi(t)|}{1+D^{0.7}\phi(t)|}+\dfrac{|D^{0.6}D^{0.7}\phi(t)|}{1+D^{0.6}D^{0.7}\phi(t)|}\right] \\ \qquad\qquad +\dfrac{e^t+1}{100e^{t+1}}\dfrac{1}{1+|\phi(t)|} \\ \phi(0) = 0 \\ D^{0.7}\phi(0) = 0 \\ D^{0.6}D^{0.7}\phi(1) = 0 \\ D^{0.5}D^{0.6}D^{0.7}\phi(1) - \dfrac{2}{233}\cos(\phi(1)) = 0, \end{cases} \tag{5.2}$$

where,

$$h_1(t,\phi) = \frac{\sin t}{200e^{t+5}}\frac{1}{1+|\phi|},$$
$$h_2(t,\phi,\phi^*) = \frac{1}{33e^{3-t}+233}\left[1+\frac{|\phi|}{1+|\phi|}+\frac{\phi^*}{1+|\phi^*|}\right],$$
$$h_3(t,\phi) = \frac{e^t+1}{100e^{t+1}}\frac{1}{1+|\phi|},$$
$$g(t) = \cos(t), \quad \mu = \frac{2}{233}.$$

The condition (4.5) is verified,

$$\vartheta = \vartheta_1 R + \vartheta_2|\mu|r_g = 0.2321664 < 1,$$

then (5.2) has at least one solution.

6. Conclusion

In conclusion, this study introduced a novel nonlinear differential problem characterized by Caputo derivatives and sequential derivatives, which do not conform to the semi-group and commutativity properties. Our findings demonstrate that under certain conditions, the problem simplified to a fourth-order ordinary differential problem, effectively modeling the static equilibrium of an elastic beam. By applying the Banach contraction principle and Schauder's fixed point theorem, we have proven the uniqueness of solutions and the existence of at least one solution. The presented example substantiated one of these key results, further validating our theoretical approach. Additionally, the exploration of Ulam-Hyers stability, complemented by limiting-case examples, added robustness to our analysis.

Future work will focus on extending these results to more complex systems involving fractional differential equations. Specifically, we aim to explore the applicability of different types of fractional derivatives, such as Hattaf fractal and Riesz fractional derivatives [11, 14], to broaden the scope of the problem. Furthermore, investigating the impact of variable coefficients and non-homogeneous boundary conditions could provide deeper insights into real-world applications. Finally, numerical simulations and practical implementations in engineering contexts, such as structural analysis and materials science, will be pursued to bridge the gap between theoretical findings and practical applications.

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