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EMBEDDINGS IN RIEMANN–LIOUVILLE FRACTIONAL SOBOLEV SPACES AND APPLICATIONS

SAADI ABDERACHID* AND BENLATRACHE KENZA

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Abstract. In this work, we present results on the embeddings of fractional Riemann-Liouville Sobolev spaces, using an important relationship between Riemann-Liouville Sobolev spaces and ordinary Sobolev spaces. This relationship allows us to prove compact embeddings after establishing continuous embeddings based on the continuity of the Riemann-Liouville fractional integral operators between Lebesgue spaces under certain conditions. We provide an example of a boundary problem where existence and uniqueness are addressed using two methods: the fixed point method and the Faedo-Galerkin method. Both methods require specific fractional type embeddings.

1. Introduction

In the late 20th century, research on fractional-order differentiation of the Riemann-Liouville type and other types increased significantly. The research focused on the properties of this type of differentiation and extended to differential equations and boundary value problems of the fractional-order type. Initially, the research revolved around strong solutions and has recently extended to weak solutions as well.

It is known that classical Sobolev spaces provide a suitable framework for weak solutions of differential equations and partial differential equations (see, for example, [3]). Therefore, it is appropriate to search for similar spaces that can provide a suitable framework for this new type of equations related to fractional-order differentiation. Research began by finding variational formulations related to these problems and then finding appropriate spaces that include weak solutions.

In [6, 7], the authors used the variational method to prove the existence of solutions for nonlinear Dirichlet boundary value problems of the Riemann-Liouville type on a bounded real interval $[0,T]$. For this purpose, a new space denoted by $E_0^{\alpha,p}$ $(0<\alpha<1,\ 1\leqslant p<\infty)$ was introduced, defined as the closure of the space $C_c^\infty(0,T)$ with respect to the norm

$$\|.\|_{\alpha,p}=\left(\int_0^T|u(t)|^p dt+\int_0^T|{}_0D_t\alpha u(t)|^p dt\right),$$

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* Corresponding author.

which is a norm in a Banach space later identified as a fractional-order Sobolev space of the Riemann-Liouville type. This space is a reflexive and separable Banach space, where the subscript 0 indicates the vanishing of the function on the boundary, implying the Caputo derivative in particular. A Poincaré-type inequality was proven, as well as a continuous embedding in the space $L^{\infty}(0,T)$ and a compact embedding in the space $C[0,T]$ for $\alpha > \frac{1}{p}$. L. Bourdin [2] denoted this space by $E^{\alpha,p}$ and provided an equivalent definition (see, for example, [8, 10, 13]).

A comprehensive definition of fractional Sobolev spaces of the Riemann-Liouville type was introduced by Idczak et al. [5] and denoted by $W_{a^+}^{\alpha,p}$ $(0 < \alpha < 1,\ 1 \leqslant p < \infty)$, following the method used to introduce ordinary spaces (see, for example, [1, 3]). Two equivalent norms were presented, and it was proven that these spaces are Banach, reflexive, and separable under the same conditions that p satisfies in ordinary spaces. Several types of embeddings for these spaces were provided, including continuous and compact embeddings in the spaces L^q, presented briefly based on [15, Lemma 1.1].

The authors of [4] defined a Sobolev space of order $0 < \alpha < 1$ on an interval I as the space of functions f from $L^p(I)$ such that the Riemann-Liouville derivative of order $1-\alpha$ of f belongs to the ordinary Sobolev space $W^{1,p}(I)$, which is equivalent to the definition provided in [5]. The norm presented in [4] is also equivalent to the norms presented in [5]. The authors also provided some embeddings of fractional Sobolev spaces in $L^r(I)$ spaces where r is a real number greater than 1 and satisfies specific conditions.

The authors of [11] also introduced right fractional spaces $E_R^{\alpha}(a,b)$ and left fractional spaces $E_L^{\alpha}(a,b)$ where $a,b \in \mathbb{R}$ for $p=2$. It should be noted that the space coincides with the space $W_{a^+}^{\alpha,2}$ presented in [5], and the norm is equivalent to one of the norms. The subspace $E_0^{\alpha,2}$ was introduced under the symbol $E_{L,0}^{\alpha}$, and some properties related to it were proven, whether concerning the traces of functions belonging to this space on the boundary of (a,b) or concerning continuous and compact embeddings of these subspaces in $L^q(a,b)$ spaces under specific conditions that the real number $1 \leqslant q \leqslant \infty$ satisfies, as well as Hölder spaces.

In our paper, we established a relationship between ordinary Sobolev spaces $W^{1,p}(a,b)$ and fractional Sobolev spaces ${}^{RL}W_{a^+}^{\alpha,p}(a,b)$, thereby proving continuous and compact embeddings that generalize those found in [3, Theorem 8.8] in detail, considering the conditions that p and α must satisfy, similar to the conditions of the Rellich-Kondrachov theorem (see, for example, [3, Theorem 9.16]).

We also proved continuous and compact embeddings for subspaces of fractional Sobolev spaces, which satisfy specific boundary conditions. These spaces play a significant role in certain fractional-order boundary value problems.

Finally, we presented an example of a nonlinear fractional boundary value problem of the form:

$$\begin{cases} D_{b^-}^{\alpha}(D_{a^+}^{\alpha}u)(x) = f(x,u) : \text{in } (a,b), \\ I_{a^+}^{1-\alpha}u(a) = u(b) = 0, \end{cases}$$

where we proved the existence and uniqueness under certain conditions satisfied by the function f using two methods: the fixed-point method and the Faedo-Galerkin method.

We divided this work as follows: In the second section, we presented some basic principles of fractional-order calculus of the Riemann-Liouville type. The third section was dedicated to fractional Sobolev spaces. The fourth section dealt with continuous and compact embeddings of the fractional-order space ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$ and the subspace ${}^{RL}_{0}W^{\alpha,p}_{a^+}(a,b)$ in the spaces $L^p(a,b)$ as well as the space $C([a,b])$. Finally, we studied the above-mentioned boundary value problem using the fixed-point method and the Faedo-Galerkin method.

2. Preliminaries

Consider the parameters $1 \leqslant p \leqslant +\infty$, $0 < \alpha < 1$, and $-\infty < a,b < +\infty$. $L^p(a,b)$ is the usual Lebesgue space with norm $\|.\|_{L^p}$. The Euler Gamma function is denoted by $\Gamma(.)$. $AC^p(a,b)$ denotes the space og all measurable functions f such there exist $c \in \mathbb{R}$ and $\varphi \in L^p(a,b)$ satisfying $f(x) = c + \int_a^x \varphi(t)dt$, for all $x \in [a,b]$.

We give some definitions and properties related to fractional calculus.

DEFINITION 1. [9, 17] The Riemann-Liouville Fractional integral $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order α and a function $f \in L^p(a,b)$ are defined by:

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (a < x \leqslant b),$$

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \quad (a \leqslant x < b).$$

THEOREM 1. [17, p. 48] *The Riemann-Liouville integral $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ are well defined for all $f \in L^p(a,b)$. Moreover, we have:*

$$\|I^{\alpha}_{a+}f\|_{L^p} \leqslant \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L^p}, \tag{1}$$

$$\|I^{\alpha}_{b-}f\|_{L^p} \leqslant \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L^p}. \tag{2}$$

DEFINITION 2. [9, 17] The Riemann-Liouville Fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order α of the function $f \in AC^p(a,b)$ are defined by:

$$\begin{aligned}\left(D^{\alpha}_{a+}f\right)(x) &= \frac{d}{dx}(I^{1-\alpha}_{a+}f)(x) \quad (a < x \leqslant b),\\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)dt,\end{aligned} \tag{3}$$

$$\begin{aligned}(D^{\alpha}_{b-}f)(x) &= -\frac{d}{dx}(I^{1-\alpha}_{b-}f)(x) \quad (a \leqslant x < b),\\ &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)dt.\end{aligned} \tag{4}$$

THEOREM 2. [17, p. 34] *Let $f \in L^p(a,b)$ and $g \in L^q(a,b)$ such that $\frac{1}{p}+\frac{1}{q} \leqslant 1+\alpha$. Then, we have:*

$$\int_a^b f(x) I_{b^-}^{\alpha} g(x)dx = \int_a^b g(x) I_{a^+}^{\alpha} f(x)dx. \tag{5}$$

DEFINITION 3. [5] We introduce the following spaces

i) $AC_{a^+}^{\alpha,p}(a,b)$, the set of all functions $f:[a,b]\to\mathbb{R}$ such that:

$$f(x) = \frac{I_{a^+}^{1-\alpha}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(x), \quad x\in[a,b], \tag{6}$$

ii) $AC_{b^-}^{\alpha,p}(a,b)$ the set of all functions $g:[a,b]\to\mathbb{R}$ such that:

$$g(x) = \frac{I_{b^-}^{1-\alpha}u(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1} + I_{b^-}^{\alpha} D_{b^-}^{\alpha} g(x), \quad t\in[a,b]. \tag{7}$$

THEOREM 3. [5] *Let $f\in AC_{a^+}^{\alpha,p}(a,b)$, $g\in AC_{b^-}^{\alpha,p}(a,b)$ and $\varphi\in C^1([a,b])$ such that $\varphi(a)=\varphi(b)=0$. Then,*

$$\int_a^b f(x)(D_{b^-}^{\alpha}\varphi)(x)dx = \int_a^b \varphi(x)(D_{a^+}^{\alpha} f)(x)dx, \tag{8}$$

$$\int_a^b g(x)(D_{a^+}^{\alpha}\varphi)(x)dx = \int_a^b \varphi(x)(D_{b^-}^{\alpha} g)(x)dx. \tag{9}$$

COROLLARY 1. *The above results remain true if we replace $\varphi\in C^1([a,b])$, $\varphi(a)=\varphi(b)=0$ with $\varphi\in C_0^{\infty}(a,b)$, the space of infinitely differentiable functions, with compact support included in (a,b), which is important in definitions of fractional Sobolev spaces.*

THEOREM 4. *Assume that $p>\frac{1}{\alpha}$, then for all $u\in L^p(a,b)$ we have $I_{a^+}^{\alpha}u\in C^{0,\alpha-\frac{1}{p}}((a,b])$ and $I_{b^-}^{\alpha}u\in C^{0,\alpha-\frac{1}{p}}([a,b))$. Therefore, $I_{a^+}^{\alpha}u\in C((a,b])$ and $I_{b^-}^{\alpha}u\in C([a,b))$.*

$C^{0,\alpha-\frac{1}{p}}(I)$ denotes the Hölder's space of order $\left(\alpha-\frac{1}{p}\right)$ on the interval I.

Proof. We will adapt the proof from [2, Property 4]. Let $u\in L^p(a,b)$, with $p>\frac{1}{\alpha}$ and $a<y<x\leqslant b$. Putting,

$$\begin{aligned} |G(x,y)| &= |I_{a^+}^{\alpha}u(x) - I_{a^+}^{\alpha}u(y)| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_a^x (x-t)^{\alpha-1}u(t)dt - \int_a^y (y-t)^{\alpha-1}u(t)dt\right|. \end{aligned}$$

So,

$$\begin{aligned}
|G(x,y)| &\leqslant \frac{1}{\Gamma(\alpha)}\left|\int_a^y [(x-t)^{\alpha-1}-(y-t)^{\alpha-1}]u(t)dt\right| \\
&\quad+\frac{1}{\Gamma(\alpha)}\left|\int_y^x (x-t)^{\alpha-1}u(t)dt\right| \\
&\leqslant \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_a^y |(x-t)^{\alpha-1}-(y-t)^{\alpha-1}|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
&\quad+\frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_y^x (x-t)^{\frac{(\alpha-1)p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
&\leqslant \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_a^y [(y-t)^{\frac{(\alpha-1)p}{p-1}}-(x-t)^{\frac{(\alpha-1)p}{p-1}}]dt\right)^{\frac{p-1}{p}} \\
&\quad+\frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\int_y^x (x-t)^{\frac{(\alpha-1)p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
&\leqslant \frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left((y-a)^{\frac{\alpha p-1}{p-1}}-(x-a)^{\frac{\alpha p-1}{p-1}}+(x-y)^{\frac{\alpha p-1}{p-1}}\right)^{\frac{p-1}{p}} \\
&\quad+\frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}} \\
&\leqslant \frac{2^p\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left((y-a)^{\frac{\alpha p-1}{p-1}}-(x-a)^{\frac{\alpha p-1}{p-1}}\right)^{\frac{p-1}{p}} \\
&\quad+\frac{2^p\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}} \\
&\quad+\frac{\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}} \\
&\leqslant \frac{2^p\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}} \\
&\quad+\frac{1+2^p\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}} \\
&= \frac{2^{p+1}\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\frac{\alpha p-1}{p}} \\
&\leqslant \frac{2^{p+1}\|u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}(x-y)^{\alpha-\frac{1}{p}}.
\end{aligned}$$

Hence, $I_{a^+}^{\alpha}u \in C^{0,\alpha-\frac{1}{p}}((a,b])$. Therefore $I_{a^+}^{\alpha}u \in C((a,b])$.

Using the same reasoning for $I_{b^-}^{\alpha}u$. $\square$

3. Fractional Sobolev spaces

Let $0<\alpha<1$, $1\leqslant p\leqslant\infty$ and $a,b\in\mathbb{R}$.

DEFINITION 4. [5] We introduce the spaces

$$^{RL}W^{\alpha,p}_{a^+}(a,b)=\left\{\begin{array}{l} u\in L^p(a,b),\exists g_a\in L^p(a,b),\forall\varphi\in C^\infty_c(a,b):\\ \int_a^b u(x)D^\alpha_{b^-}\varphi(x)dx=\int_a^b g_a(x)\varphi(x)dx\end{array}\right\},$$

$$^{RL}W^{\alpha,p}_{b^-}(a,b)=\left\{\begin{array}{l} u\in L^p(a,b),\exists g_b\in L^p(a,b),\forall\varphi\in C^\infty_c(a,b):\\ \int_a^b u(x)D^\alpha_{a^+}\varphi(x)dx=\int_a^b g_b(x)\varphi(x)dx\end{array}\right\}.$$

The function g_a,a_b given above will be called the weak left and right fractional derivatives of order α of u, let us denote them by $D^\alpha_{a^+}u, D^\alpha_{b^-}u$.

We denote by $^{RL}H^\alpha_{a^+}(a,b)$, $^{RL}H^\alpha_{b^-}(a,b)$ the space $^{RL}W^{\alpha,2}_{a^+}(a,b)$, $^{RL}W^{\alpha,2}_{b^-}(a,b)$.

THEOREM 5. [5] *For* $1<p<\infty$ *we have:*

$$^{RL}W^{\alpha,p}_{a^+}=AC^{\alpha,p}_{a^+}(a,b)\cap L^p(a,b),$$
$$^{RL}W^{\alpha,p}_{b^-}=AC^{\alpha,p}_{b^-}(a,b)\cap L^p(a,b).$$

It follows that

COROLLARY 2. *If* $u\in {}^{RL}W^{\alpha,p}_{a^+}(a,b)$, $v\in {}^{RL}W^{\alpha,p}_{b^-}(a,b)$ *then,*

$$u(x)=\frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I^\alpha_{a^+}D^\alpha_{a^+}u(x), \tag{10}$$

$$v(x)=\frac{I^{1-\alpha}_{b^-}u(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I^\alpha_{b^-}D^\alpha_{b^-}v(x). \tag{11}$$

REMARK 1. It follows from Corollary 2 that

1. If $p<\frac{1}{1-\alpha}$ then, $AC^{\alpha,p}_{a^+}(a,b)$, $AC^{\alpha,p}_{b^-}(a,b)\subset L^p$.
   So, $^{RL}W^{\alpha,p}_{a^+}(a,b)=AC^{\alpha,p}_{a^+}(a,b)$, $^{RL}W^{\alpha,p}_{b^-}(a,b)=AC^{\alpha,p}_{b^-}(a,b)$.
2. If $p\geqslant\frac{1}{1-\alpha}$ then, $^{RL}W^{\alpha,p}_{a^+}(a,b)$ is the set of all functions belonging to $AC^{\alpha,p}_{a^+}(a,b)$, satisfy the condition $I^{1-\alpha}_{a^+}u(a)=0$.

THEOREM 6. (Poincaré inequality) *Let* $u\in {}^{RL}W^{\alpha,p}_{a^+}(a,b)$, $v\in {}^{RL}W^{\alpha,p}_{b^-}(a,b)$. *Then,*

$$\left\|u-\frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\right\|_{L^p}\leqslant\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|D^\alpha_{a^+}u\|_{L^p}, \tag{12}$$

$$\left\| u - \frac{I_{b^-}^{1-\alpha} v(a)}{\Gamma(\alpha)} (b-x)^{\alpha-1} \right\|_{L^p} \leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|D_{b^-}^\alpha v\|_{L^p}. \tag{13}$$

In particular, if $I_{a^+}^{1-\alpha} u(a) = I_{b^-}^{1-\alpha} v(b) = 0$ *we get*

$$\|u\|_{L^p} \leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|D_{a^+}^\alpha u\|_{L^p}, \tag{14}$$

$$\|v\|_{L^p} \leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|D_{b^-}^\alpha v\|_{L^p}. \tag{15}$$

Proof. From (6), we have

$$u(x) - \frac{(x-a)^{\alpha-1} I_{a^+}^\alpha u(a)}{\Gamma(\alpha)} = I_{a^+}^\alpha D_{a^+}^\alpha u.$$

So, from (1) we obtain

$$\begin{aligned} \left\| u - \frac{I_{a^+}^{1-\alpha} u(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1} \right\|_{L^p} &= \|I_{a^+}^\alpha D_{a^+}^\alpha u\| \\ &\leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|D_{a^+}^\alpha u\|_{L^p}. \quad \square \end{aligned}$$

DEFINITION 5. [5] We consider in the space ${}^{RL}W_{a^+}^{\alpha,p}(a,b)$ two norms ${}^1\|.\|_{W_{a^+}^{\alpha,p}}$ and ${}^2\|.\|_{W_{a^+}^{\alpha,p}}$ given by:

$$ {}^1\|u\|_{W_{a^+}^{\alpha,p}} = (\|u\|_{L^p}^p + \|D_{a^+}^\alpha u\|_{L^p}^p)^{\frac{1}{p}}, \tag{16}$$

$$ {}^2\|u\|_{W_{a^+}^{\alpha,p}} = (|I_{a^+}^{1-\alpha} u(a)|^p + \|D_{a^+}^\alpha u\|_{L^p}^p)^{\frac{1}{p}}. \tag{17}$$

In the same way, we define in the space ${}^{RL}W_{b^-}^{\alpha,p}(a,b)$ two norms ${}^1\|.\|_{W_{b^-}^{\alpha,p}}$ and ${}^2\|.\|_{W_{b^-}^{\alpha,p}}$ given by:

$$ {}^1\|u\|_{W_{b^-}^{\alpha,p}} = (\|u\|_{L^p}^p + \|D_{b^-}^\alpha u\|_{L^p}^p)^{\frac{1}{p}}, \tag{18}$$

$$ {}^2\|u\|_{W_{b^-}^{\alpha,p}} = (|I_{b^-}^{1-\alpha} u(b)|^p + \|D_{b^-}^\alpha u\|_{L^p}^p)^{\frac{1}{p}}. \tag{19}$$

THEOREM 7. [5] *The norm* ${}^1\|.\|_{W_{a^+}^{\alpha,p}}$ *is equivalent to the norm* ${}^2\|u\|_{W_{a^+}^{\alpha,p}}$. *Likewise, the norm* ${}^1\|.\|_{W_{b^-}^{\alpha,p}}$ *is equivalent to the norm* ${}^2\|u\|_{W_{b^-}^{\alpha,p}}$

THEOREM 8. [5] *The spaces* ${}^{RL}W_{a^+}^{\alpha,p}(a,b)$ *and* ${}^{RL}W_{b^-}^{\alpha,p}(a,b)$ *are Banach spaces, reflexives for* $1 < p < \infty$ *and separable for* $1 \leqslant p < \infty$.

REMARK 2. The spaces ${}^{RL}H^{\alpha}_{a^+}(a,b)$, ${}^{RL}H^{\alpha}_{b^-}(a,b)$ are reflexive and separable Hilbert spaces, with the inner products

$$\langle u,v\rangle_{H^{\alpha}_{a^+}} = \int_a^b u(x)v(x)dx + \int_a^b D^{\alpha,p}_{a^+}u(x).D^{\alpha,p}_{a^+}v(x)dx \;\; u,v \in^{RL} H^{\alpha}_{a^+}(a,b),$$
$$\langle u,v\rangle_{H^{\alpha}_{b^-}} = \int_a^b u(x)v(x)dx + \int_a^b D^{\alpha,p}_{b^-}u(x).D^{\alpha,p}_{b^-}v(x)dx \;\; u,v \in^{RL} H^{\alpha}_{b^-}(a,b).$$

The following theorem gives a version of integration by parts in Riemann-Liouville fractional Sobolev spaces.

THEOREM 9. [5] *Let $p,q \leqslant 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then, for all $u \in {}^{RL}W^{\alpha,p}_{a^+}(a,b)$, $v \in {}^{RL}W^{\alpha,p}_{b^-}(a,b)$ we have*

$$\int_a^b u(x)\left(D^{\alpha}_{b^-}v\right)(x)dx = \left(I^{1-\alpha}_{a^+}u\right)(a)v(a) - u(b)\left(I^{1-\alpha}_{b^-}v\right)(b) + \int_a^b \left(D^{\alpha}_{a^+}u\right)(x)v(x)dx. \tag{20}$$

Now, we present a relationship between the fractional and classical Sobolev spaces. For this, we introduce the following operator

$$\begin{aligned} T^{\alpha}_a : {}^{RL}W^{\alpha,p}_{a^+}(a,b) &\longrightarrow W^{1,p}(a,b) \\ u \qquad &\longmapsto v = T^{\alpha}_a(u) = I^{1-\alpha}_{a^+}u, \end{aligned}$$

where $W^{1,p}(a,b)$ is the usual Sobolev space on (a,b).

We have the following theorem.

THEOREM 10. *The operator T^{α}_a is an isomorphism:*

i) *from ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$ to $W^{1,p}(a,b)$ if $p < \frac{1}{1-\alpha}$,*

ii) *from ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$ to $\{v \in W^{1,p}(a,b) : v(a) = 0\}$ if $p \geqslant \frac{1}{1-\alpha}$.*

Proof. The proof is conducted in sequential steps

- The operator T^{α}_a is well defined and injective.

  Let $u \in {}^{RL}W^{\alpha,p}(a,b)$, set $v(x) = I^{1-\alpha}_{a^+}u(x)$. Then,

$$\begin{aligned} \|v\|_{L^p(a,b)} + \|v'\|_{L^p(a,b)} &= \|I^{1-\alpha}_{a^+}u\|_{L^p} + \|D^{\alpha}_{a^+}u\|_{L^p} \\ &\leqslant \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\|u\|_{L^p(a,b)} + \|D^{\alpha}_{a^+}u\|_{L^p(a,b)} \\ &\leqslant C.^1\|u\|_{W^{\alpha,p}_{a^+}} < \infty. \end{aligned}$$

  So, $v \in {}^{RL}W^{1,p}_{a^+}(a,b)$.

  Moreover, $u \in KerT^{\alpha}_a$ if and only if $I^{1-\alpha}_{a^+}u = 0$, i.e. $\int_a^x u(t)dt = I^{\alpha}_{a^+}0 = 0$, which leads to $u = 0$.

  Then, $I^{1-\alpha}_{a^+}$ is injective.

- The operator T is surjective:

  i) from ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$ to $W^{1,p}(a,b)$ if $p < \frac{1}{1-\alpha}$,

  ii) from ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$ to $\{v \in W^{1,p}(a,b) : v(a) = 0\}$ if $p \geqslant \frac{1}{1-\alpha}$.

  Let $u \in {}^{RL}W^{\alpha,p}_{a^+}(a,b)$. Then, $v = I^{1-\alpha}_{a^+}u$ if and only if $u = \frac{d}{dx}I^{\alpha}_{a^+}v = D^{1-\alpha}_{a^+}v$.

  Note that

$$\begin{aligned} I^{\alpha}_{a^+}v &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}v(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\left(\left[\frac{-(x-t)^{\alpha}}{\alpha}v(t)\right]_a^x + \int_a^x \frac{(x-t)^{\alpha}}{\alpha}v'(t)dt\right) \\ &= \frac{(x-a)^{\alpha}}{\alpha\Gamma(\alpha)}v(a) + \frac{1}{\alpha\Gamma(\alpha)}\int_a^x (x-t)^{\alpha}v'(t)dt. \end{aligned}$$

  So,

$$\begin{aligned} u(x) &= \frac{d}{dx}I^{\alpha}_{a^+}v \\ &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + \frac{1}{\alpha\Gamma(\alpha)}\left[(x-t)^{\alpha}v'(t)dt\right]_{\{x=t\}} \\ &\quad + \frac{1}{\alpha\Gamma(\alpha)}\int_a^x \frac{\partial}{\partial x}\left[(x-t)^{\alpha}v'(t)\right]dt \\ &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}v'(t)dt \\ &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) + I^{\alpha}_{a^+}v'(x). \end{aligned}$$

  We debusses two cases

  1. if $p < \frac{1}{1-\alpha}$ then, $\dfrac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}v(a) \in L^p(a,b)$ and $I^{\alpha}_{a^+}v' \in L^p(a,b)$.

     So, $u \in {}^{RL}W^{\alpha,p}_{a^+}(a,b)$. Therefore, $T^{\alpha}_a :{}^{RL} W^{\alpha,p}_{a^+}(a,b) \longrightarrow W^{1,p}_{a^+}(a,b)$ is surjective.
  2. if $p \geqslant \frac{1}{1-\alpha}$ then, $v(a) = I^{1-\alpha}_{a^+}u(a) = 0$, $u = I^{\alpha}_{a^+}v' \in L^p(a,b)$ and $D^{\alpha}_{a^+}u = v' \in L^p(a,b)$.

     So, $T^{\alpha}_a :\ {}^{RL}W^{\alpha,p}_{a^+} \longrightarrow \left\{v \in W^{1,p}_{a^+}(a,b) : v(a) = 0\right\}$ is surjective.

- The operator T^{α}_a is an isomorphism.

  Let $u \in {}^{RL}W^{\alpha,p}_{a^+}(a,b)$. From the first step, we have

$$\|T^{\alpha}_a u\|_{W^{\alpha,p}_{a^+}} \leqslant C.^1\|u\|_{W^{\alpha,p}_{a^+}}.$$

  Then, T^{α}_a is continuous.

  Now, let $v \in W^{1,p}(a,b)$.

1. if $p < \frac{1}{1-\alpha}$ then,

$$\begin{aligned}
\|(T_a^\alpha)^{-1}v\|_{W_{a^+}^{\alpha,p}} &= \|(T_a^\alpha)^{-1}v\|_{L^p} + \|D_{a^+}^\alpha (T_a^\alpha)^{-1}v\|_{L^p} \\
&= \left\| \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} v(a) + I_{a^+}^\alpha v'(x) \right\|_{L^p} + \|D_{a^+}^\alpha D_{a^+}^{1-\alpha} v\|_{L^p} \\
&\leqslant \frac{1}{\Gamma(\alpha)} |v(a)|.\|(x-a)^{\alpha-1}\|_{L^p} + \|I_{a^+}^\alpha v'(x)\|_{L^p} \\
&\quad + \|D_{a^+}^\alpha D_{a^+}^{1-\alpha} v\|_{L^p} \\
&= \frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{[(\alpha-1)p+1]^{\frac{1}{p}}\Gamma(\alpha)} |v(a)| + \|I_{a^+}^\alpha v'(x)\|_{L^p} + \|v'\|_{L^p}.
\end{aligned}$$

From the continuous embedding of $W^{1,p}(a,b)$ into $L^\infty(a,b)$, we obtain:

$$\frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{[(\alpha-1)p+1]^{\frac{1}{p}}\Gamma(\alpha)} |v(a)| \leqslant \frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{[(\alpha-1)p+1]^{\frac{1}{p}}\Gamma(\alpha)} \|v\|_{L^\infty} \leqslant C_1 \|v\|_{W^{1,p}}.$$

So,

$$\begin{aligned}
\|(T_a^\alpha)^{-1}v\|_{W_{a^+}^{\alpha,p}} &\leqslant C_1\|v\|_{W^{1,p}} + \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|v'\|_{L^p} + \|v'\|_{L^p} \\
&\leqslant C_2 \|v\|_{W^{1,p}}.
\end{aligned}$$

2. If $p \geqslant \frac{1}{1-\alpha}$ we have $v(a) = 0$. Then,

$$\begin{aligned}
\|(T_a^\alpha)^{-1}v\|_{W_{a^+}^{\alpha,p}} &= \|I_{a^+}^\alpha v'\|_{L^p} + \|D_{a^+}^\alpha I_{a^+}^\alpha v'\|_{L^p} \\
&= \|I_{a^+}^\alpha v'\|_{L^p} + \|v'\|_{L^p} \\
&\leqslant \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|v'\|_{L^p} + \|v'\|_{L^p} \\
&\leqslant C_3 \|v\|_{W^{1,p}}.
\end{aligned}$$

Therefore, $(T_a^\alpha)^{-1}$ is an isomorphism. □

Using similar arguments, we prove the following theorem

THEOREM 11. *The operator*

$$\begin{aligned}
T_b^\alpha : {}^{RL}W_{b^-}^{\alpha,p}(a,b) &\longrightarrow W^{1,p}(a,b) \\
u &\longmapsto v = T_b^\alpha(u) = I_{b^-}^{1-\alpha} u,
\end{aligned}$$

is an isomorphism:

i) *from* ${}^{RL}W_{b^-}^{\alpha,p}(a,b)$ *to* $W^{1,p}(a,b)$ *if* $p < \frac{1}{1-\alpha}$,

ii) *from* ${}^{RL}W_{b^-}^{\alpha,p}(a,b)$ *to* $\{v \in W^{1,p}(a,b) : v(b) = 0\}$ *if* $p \geqslant \frac{1}{1-\alpha}$.

4. Embeddings in Riemann-Liouville fractional Sobolev spaces

Let $0<\alpha<1$, $1\leqslant p\leqslant\infty$ and $a,b\in\mathbb{R}$.

The following theorem ensure the continuous and compact embeddings of Riemann-Liouville fractional Sobolev spaces into $L^q(a,b)$ and $C([a,b])$.

We will only prove the embeddings of ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$. The proofs of the embeddings of ${}^{RL}W^{\alpha,p}_{b^-}(a,b)$ are done in the same way.

Setting $p^\alpha_* = \frac{p}{1-\alpha p}$ for $p<\frac{1}{\alpha}$.

THEOREM 12. *Assume that $\alpha<\frac{1}{2}$. Then, we have the following embeddings*

1. *If $1\leqslant p<\frac{1}{1-\alpha}$ then, ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}W^{\alpha,p}_{b^-}(a,b)\hookrightarrow L^q(a,b)$ for all $q\in[1,\frac{1}{1-\alpha})$.*
2. *If $\frac{1}{1-\alpha}<p<\frac{1}{\alpha}$ then, ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}W^{\alpha,p}_{b^-}(a,b)\hookrightarrow L^q(a,b)$ for all $q\in[1,p^\alpha_*]$.*
3. *If $p=\frac{1}{\alpha}$ then, ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}W^{\alpha,p}_{b^-}(a,b)\hookrightarrow L^q(a,b)$ for all $q\in[1,+\infty)$.*
4. *If $p>\frac{1}{\alpha}$ then, ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}W^{\alpha,p}_{b^-}(a,b)\hookrightarrow L^q(a,b)$ for all $q\in[1,+\infty]$. In particular, ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}W^{\alpha,p}_{b^-}(a,b)\hookrightarrow C([a,b])$.*

Proof. Since $\alpha<\frac{1}{2}$, we deduce that $\frac{1}{1-\alpha}<\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}<p^\alpha_*$.

Let $u\in^{RL}W^{\alpha,p}_{a^+}(a,b)$. We know according to (6) that

$$u(x)=\frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I^\alpha_{a^+}D^\alpha_{a^+}u(x).$$

Note that $\frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\in L^p(a,b)$ if only if $p<\frac{1}{1-\alpha}$ or $I^{1-\alpha}_{a^+}u(a)=0$.

So, for $q\geqslant 1$ we have

$$\begin{aligned}\|u\|_{L^q} &= \left\|\frac{I^{1-\alpha}_{a^+}u(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I^\alpha_{a^+}D^\alpha_{a^+}u\right\|_{L^q}\\ &\leqslant \frac{|I^{1-\alpha}_{a^+}u(a)|}{\Gamma(\alpha)}\left\|(x-a)^{\alpha-1}\right\|_{L^q}+\left\|I^\alpha_{a^+}D^\alpha_{a^+}u\right\|_{L^q}.\end{aligned}$$

1. If $1\leqslant p<\frac{1}{1-\alpha}$ then $1\leqslant p<\frac{1}{\alpha}$. Since $D^\alpha_{a^+}u\in L^p(a,b)$, from [12, Theorem 0.2] there exists $c>0$ such that $\left\|I^\alpha_{a^+}D^\alpha_{a^+}u\right\|_{L^q}\leqslant c.\left\|D^\alpha_{a^+}u\right\|_{L^p}$, for all $q\in[1,p^\alpha_*]$.

   On the other hand, $(x-a)^{\alpha-1}\in L^q(a,b)$ if only if $q<\frac{1}{1-\alpha}$. In this case we get

$$\frac{|I^{1-\alpha}_{a^+}u(a)|}{\Gamma(\alpha)}\left\|(x-a)^{\alpha-1}\right\|_{L^q}\leqslant\frac{(b-a)^{1-\alpha+\frac{1}{q}}}{\Gamma(\alpha).[(1-\alpha)q+1]^{\frac{1}{q}}}|I^{1-\alpha}_{a^+}u(a)|.$$

Hence, for $q \in [1, \frac{1}{1-\alpha}) \cap [1, p_*^\alpha] = [1, \frac{1}{1-\alpha})$ there exists $M > 0$ such that

$$\|u\|_{L^q} \leqslant M \left(|I_{a^+}^{1-\alpha} u(a)|^p + \left\| D_{a^+}^\alpha u \right\|_{L^p}^p \right)^{\frac{1}{p}} = M \|u\|_{{}^2W_{a^+}^{\alpha,p}} .$$

So, ${}^{RL}W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{1}{1-\alpha})$.

2. If $\frac{1}{1-\alpha} < p < \frac{1}{\alpha}$ then, $I_{a^+}^{1-\alpha} u(a) = 0$. Therefore, from [12, Theorem 0.2] there exists $c > 0$ such that for all $q \in [1, p_*^\alpha]$ we have

$$\begin{aligned} \|u\|_{L^q} &= \left\| I_{a^+}^\alpha D_{a^+}^\alpha u \right\|_{L^q} \\ &\leqslant c \left\| D_{a^+}^\alpha u \right\|_{L^p} \\ &= c \|u\|_{W_{a^+}^{\alpha,p}} . \end{aligned}$$

Then, ${}^{RL}W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, p_*^\alpha]$.

3. If $p = \frac{1}{\alpha}$ then, $I_{a^+}^{1-\alpha} u(a) = 0$. So, from [12, Theorem 0.3] there exists $c > 0$ such that for all $q \in [1, \infty)$ we have

$$\begin{aligned} \|u\|_{L^q} &= \left\| I_{a^+}^\alpha D_{a^+}^\alpha u \right\|_{L^q} \\ &\leqslant c \left\| D_{a^+}^\alpha u \right\|_{L^p} \\ &= c \|u\|_{W_{a^+}^{\alpha,p}} . \end{aligned}$$

So, ${}^{RL}W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, \infty)$.

4. If $p > \frac{1}{\alpha}$ then, $I_{a^+}^{1-\alpha} u(a) = 0$. So, from [12, Theorem 0.4] there exists $c > 0$ such that for all $q \in [p, \infty]$ we have

$$\|u\|_{L^q} \leqslant c \|u\|_{W_{a^+}^{\alpha,p}} .$$

So, ${}^{RL}W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [p, \infty]$.

In particular, since $p > \frac{1}{\alpha}$, using same arguments as in Theorem 4, we deduce that $u \in C([a,b])$. So,

$$\|u\|_{C([a,b])} = \|u\|_{L^\infty} \leqslant c_1 \|u\|_{W_{a^+}^{\alpha,p}} .$$

Hence, ${}^{RL}W_{a^+}^{\alpha,p} \hookrightarrow C([a,b])$. □

In the same context, we can prove the following theorems

THEOREM 13. *Assume that $\alpha > \frac{1}{2}$. Then, we have the following embeddings.*

1. *If $1 \leqslant p \leqslant \frac{1}{\alpha}$ then, ${}^{RL}W_{a^+}^{\alpha,p}(a,b), {}^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{1}{1-\alpha})$.*

2. *If $\frac{1}{\alpha} < p < \frac{1}{1-\alpha}$ then, ${}^{RL}W_{a^+}^{\alpha,p}(a,b), {}^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ for all $q \in [1, \frac{1}{1-\alpha})$.*

3. *If* $p \geqslant \frac{1}{1-\alpha}$ *then,* ${}^{RL}W^{\alpha,p}_{a^+}(a,b), {}^{RL}W^{\alpha,p}_{b^-}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [p,+\infty]$.
   In particular, ${}^{RL}W^{\alpha,p}_{a^+}(a,b), {}^{RL}W^{\alpha,p}_{b^-}(a,b) \hookrightarrow C([a,b])$.

THEOREM 14. *Assume that* $\alpha = \frac{1}{2}$. *Then, we have the following embeddings*

1. *If* $1 \leqslant p \leqslant 2$ *then,* ${}^{RL}W^{\frac{1}{2},p}_{a^+}(a,b), {}^{RL}W^{\frac{1}{2},p}_{b^-}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1,2)$.

2. *If* $p = 2$ *then,* ${}^{RL}H^{\frac{1}{2}}_{a^+}(a,b)$, ${}^{RL}H^{\frac{1}{2}}_{b^-}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1,+\infty)$.

3. *If* $p > 2$ *then,* ${}^{RL}W^{\frac{1}{2},p}_{a^+,0}(a,b)$, ${}^{RL}W^{\frac{1}{2},p}_{b^-,0}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [p,+\infty]$.

   In particular, ${}^{RL}W^{\frac{1}{2},p}_{a^+,0}(a,b)$, ${}^{RL}W^{\frac{1}{2},p}_{b^-,0}(a,b) \hookrightarrow C([a,b])$.

Now, we will present the conditions concerning the compactness of the previous embeddings.

THEOREM 15. *If the embeddings* ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}W^{\alpha,p}_{a^+}(a,b) \hookrightarrow L^q(a,b)$ $(q < +\infty)$ *are satisfied, then they are compacts.*

Proof. Let (u_n) be a bounded sequence in ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$. Then, $(v_n) = (T^\alpha_a u_n)$ is bounded in $W^{1,p}(a,b)$. So, we can extract a subsequence $(v_{n\ell})$ weakly convergence to $v = T^\alpha_a u$ in $W^{1,p}(a,b)$.

From usually Sobolev embeddings, we can extract a subsequence (v_{nk}) convergence to $T^\alpha_a u$ in $L^q(a,b)$, i.e, $\|v_{nk} - v\|_{L^q} \to 0$.

Now, we have

$$\begin{aligned}
\|u_{nk} - u\|_{L^q} &= \|(T^\alpha_a)^{-1}(v_{nk} - v)\|_{L^q} \\
&= \left\| \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}(v_{nk}(a) - v(a)) + I^\alpha_{a^+}(v'_{nk} - v') \right\|_{L^q} \\
&\leqslant \frac{|v_{nk}(a) - v(a)|}{\Gamma(\alpha)} \|(x-a)^{\alpha-1}\|_{L^q} + \|I^\alpha_{a^+}(v'_{nk} - v')\|_{L^q}.
\end{aligned}$$

From (1), we obtain

$$\begin{aligned}
\|I^\alpha_{a^+}(v'_{nk} - v')\|_{L^q} &\leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|v'_{nk} - v'\|_{L^q} \\
&\leqslant M\|v'_{nk} - v'\|_{L^p} \to 0.
\end{aligned}$$

- If $I^{1-\alpha}_{a^+}u(a) = 0$, then we obtain directly the convergence of (u_{nk}) to u in $L^q(a,b)$.

- If $I^{1-\alpha}_{a^+}u(a) \neq 0$ and $q < \frac{1}{1-\alpha}$ then, we have

$$\begin{aligned}
\|u_{nk} - u\|_{L^q} &\leqslant C_1\|v_{nk} - v\|_{L^\infty} + C_2\|I^\alpha_{a^+}(v'_{nk} - v')\|_{L^q} \\
&\leqslant C\left(\|v_{nk} - v\|_{W^{1,p}} + \|I^\alpha_{a^+}(v'_{nk} - v')\|_{L^q}\right) \to 0.
\end{aligned}$$

  So, the convergence of (u_{nk}) to u in $L^q(a,b)$.

Thus, the compactness of the embedding. □

THEOREM 16. *If* $\max\{\frac{1}{\alpha},\frac{1}{1-\alpha}\}<p<+\infty$ *then, the embedding* ${}^{RL}W^{\alpha,p}_{a^+}(a,b)\hookrightarrow C([a,b])$ *is compact.*

Proof. Since ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$ is reflexive, we only have to prove that for all sequence $(u_n)\subset {}^{RL}W^{\alpha,p}_{a^+}(a,b)$, weakly converges to u in ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$, we obtain that (u_n) is strongly converge to u in $C([a,b])$, i.e $\|u_n-u\|_{L^\infty}\to 0$.

Let $(u_n)\subset {}^{RL}W^{\alpha,p}_{a^+}(a,b)$, be a sequence weakly converge to u in ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$. Since ${}^{RL}W^{\alpha,p}_{a^+}(a,b)\hookrightarrow C([a,b])$, (u_n) weakly converges to u in $C([a,b])$. Moreover, (u_n) is bounded in ${}^{RL}W^{\alpha,p}_{a^+}(a,b)$.

Hence, there exists a constant $C>0$ such that $\|D^\alpha_{a^+}u_n\|_{L^p}\leqslant C$.

Since $p>\frac{1}{1-\alpha}$, we obtain $I^{1-\alpha}_{a^+}u(a)=0$. So, $u=I^\alpha_{a^+}D^\alpha_{a^+}u$. Hence, from Theorem 4 we get for all $x,y\in[a,b]$:

$$\begin{aligned}|u(x)-u(y)| &= |I^\alpha_{a^+}D^\alpha_{a^+}u(x)-I^\alpha_{a^+}D^\alpha_{a^+}u(y)|\\ &\leqslant \frac{2^{p+1}\|D^\alpha_{a^+}u\|_{L^p}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}|x-y|^{\alpha-\frac{1}{p}}\\ &\leqslant \frac{2^{p+1}C}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}|x-y|^{\alpha-\frac{1}{p}}\\ &= M|x-y|^{\alpha-\frac{1}{p}}.\end{aligned}$$

Hence, is uniformly Lipschitz on $[a,b]$. From Ascoli's theorem, (u_n) is relatively compact in $C([a,b])$. Consequently, there exists a subsequence (u_{nk}) of (u_n) converging strongly in $C([a,b])$ to u by uniqueness of the weak limit. □

In the following, we give injections of subspaces play an important role in the study in some boundary problems of fractional order.

DEFINITION 6. The subspaces ${}^{RL}_{0}W^{\alpha,p}_{a^+}(a,b)$ and ${}^{RL}_{0}W^{\alpha,p}_{b^-}(a,b)$ are the sets defined by

$$\begin{aligned}{}^{RL}_{0}W^{\alpha,p}_{a^+}(a,b) &= \{u\in W^{\alpha,p}_{a^+}(a,b): I^{1-\alpha}_{a^+}u(a)=u(b)=0\},\\ {}^{RL}_{0}W^{\alpha,p}_{b^-}(a,b) &= \{u\in W^{\alpha,p}_{a^+}(a,b): u(a)=I^{1-\alpha}_{b^-}u(b)=0\}.\end{aligned}$$

Setting: ${}^{RL}_{0}H^{\alpha}_{a^+}(a,b)={}^{RL}_{0}W^{\alpha,2}_{a^+}(a,b)$, ${}^{RL}_{0}H^{\alpha}_{b^-}(a,b)={}^{RL}_{0}W^{\alpha,2}_{b^-}(a,b)$.

REMARK 3. According to Poincaré-inequality, the quantities $\|D^\alpha_{a^+}u\|_{L^p}$ and $\|D^\alpha_{b^-}u\|_{L^p}$ define norms on ${}^{RL}_{0}W^{\alpha,p}_{a^+}(a,b)$ and ${}^{RL}_{0}W^{\alpha,p}_{b^-}(a,b)$, equivalent to norms ${}^{1}\|.\|$ and ${}^{2}\|.\|$. These norms are denoted by $\|.\|_{{}_0W^{\alpha,p}_{a^+}}$ and $\|.\|_{{}_0W^{\alpha,p}_{b^-}}$.

THEOREM 17. *We have the following embeddings.*

1. *If* $1\leqslant p<\frac{1}{\alpha}$ *then,* ${}^{RL}_{0}W^{\alpha,p}_{a^+}(a,b)$, ${}^{RL}_{0}W^{\alpha,p}_{b^-}(a,b)\hookrightarrow L^q(a,b)$ *for all* $q\in[1,p^\alpha_*]$.

2. *If* $p = \frac{1}{\alpha}$ *then,* ${}_0^{RL}W_{a^+}^{\alpha,p}(a,b)$, ${}_0^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1,+\infty)$.
3. *If* $p > \frac{1}{\alpha}$ *then,* ${}_0^{RL}W_{a^+}^{\alpha,p}(a,b)$, ${}_0^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ *for all* $q \in [1,+\infty]$.
   In particular, ${}_0^{RL}W_{a^+}^{\alpha,p}(a,b)$, ${}_0^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow C([a,b])$.

Proof. Let $u \in {}_0^{RL}W_{a^+}^{\alpha,p}(a,b)$. According to (6) we have

$$u(x) = I_{a^+}^{\alpha} D_{a^+}^{\alpha} u(x).$$

So, for $q \geqslant 1$ we have

$$\|u\|_{L^q} = \left\|I_{a^+}^{\alpha} D_{a^+}^{\alpha} u\right\|_{L^q}.$$

1. If $1 \leqslant p < \frac{1}{\alpha}$, from [12, Theorem 0.2] there exists $c > 0$ such that for all $q \in [1, p_*^{\alpha}]$ we have

   $$\|u\|_{L^q} = \left\|I_{a^+}^{\alpha} D_{a^+}^{\alpha} u\right\|_{L^q} \leqslant c.\|D_{a^+}^{\alpha} u\|_{L^p} = c.\|D_{a^+}^{\alpha} u\|_{{}_0W_{a^+}^{\alpha,p}}.$$

   So, ${}_0^{RL}W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1, p_*^{\alpha}]$.

2. If $p = \frac{1}{\alpha}$ then, from [12, Theorem 0.3] there exists $c > 0$ such that for all $q \in [1,+\infty)$ we have

   $$\begin{aligned}\|u\|_{L^q} &= \left\|I_{a^+}^{\alpha} D_{a^+}^{\alpha} u\right\|_{L^q} \\ &\leqslant c\left\|D_{a^+}^{\alpha} u\right\|_{L^p} \\ &= c\|u\|_{{}_0W_{a^+}^{\alpha,p}}.\end{aligned}$$

   So, $RL_0W_{a^+,0}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [1,\infty)$.

3. If $p > \frac{1}{\alpha}$ then, from [12, Theorem 0.4] there exists $c > 0$ such that for all $q \in [p,+\infty]$ we have

   $$\|u\|_{L^q} \leqslant c\|u\|_{{}_0W_{a^+}^{\alpha,p}}.$$

   So, ${}_0^{RL}W_{a^+}^{\alpha,p} \hookrightarrow L^q(a,b)$ for all $q \in [p,\infty]$.

   In particular, since $p > \frac{1}{\alpha}$, using same arguments as in Theorem 16, we deduce that $u \in C([a,b])$. So,

   $$\|u\|_{C([a,b])} = \|u\|_{L^\infty} \leqslant c\|u\|_{{}_0W_{a^+}^{\alpha,p}}.$$

   Hence, ${}_0^{RL}W_{a^+}^{\alpha,p} \hookrightarrow C([a,b])$. □

Arguing as in Theorem 15 and Theorem 16, we can prove the following compact embeddings

THEOREM 18. *If the embeddings* ${}_0^{RL}W_{a^+}^{\alpha,p}(a,b)$, ${}_0^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow L^q(a,b)$ $(q < +\infty)$ *are satisfied, then they are compacts.*

THEOREM 19. *If* $p > \max\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\}$ *then, the embeddings* ${}_0^{RL}W_{a^+}^{\alpha,p}(a,b)$, ${}_0^{RL}W_{b^-}^{\alpha,p}(a,b) \hookrightarrow C([a,b])$ *are compacts.*

5. Application

Assume that $0 < \alpha < 1$ and let $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e

$$\begin{cases} f(.,u) \text{ is measurable on } (a,b), \text{for all } u \in \mathbb{R}, \\ f(x,.) \text{ is continuous on } \mathbb{R}, \text{a.e } x \in (a,b). \end{cases} \tag{21}$$

Consider the following problem

$$\begin{cases} D^{\alpha}_{b^-} D^{\alpha}_{a^+} u(x) = f(x,u) : \text{in } (a,b), \\ I^{1-\alpha}_{a^+} u(a) = u(b) = 0. \end{cases} \tag{22}$$

Taking into consideration that each weak solution of (22) belongs to ${}^{RL}_{0}H^{\alpha}_{a^+}(a,b)$. To find the variational formulation it is necessary to follow the following steps:

- We multiply the first equation of (22) by a test function v smooth enough, we get

$$D^{\alpha}_{b^-} D^{\alpha}_{a^+} u(x) v(x) = f(x,u) v(x).$$

- We apply the integration by parts (20), we obtain

$$\int_a^b D^{\alpha}_{a^+} u(x) D^{\alpha}_{a^+} v(x) dx + I^{1-\alpha}_{a^+} v(a).D^{\alpha}_{a^+} u(a) - v(b) I^{1-\alpha}_{a^+} D^{\alpha}_{a^+} u(b) = \int_a^b f(x,u) v(x) dx.$$

- Assume that $v \in {}^{RL}_{0}H^{\alpha}_{a^+}(a,b)$, we obtain the variational formulation of (22)

$$\int_a^b D^{\alpha}_{a^+} u(x) D^{\alpha}_{a^+} v(x) dx = \int_a^b f(x,u) v dx, \quad \forall v \in {}^{RL}_{0}H^{\alpha,p}_{a^+}(a,b). \tag{23}$$

We need to make sure that the above formulation (23) is well defined.

THEOREM 20. *Assume that there exists* $\mu \in L^2(a,b)$, $\lambda \in L^{\infty}(a,b)$ *such that*

$$|f(x,u)| \leqslant \mu(x) + \lambda(x)|u(x)|, \quad a.e \ \ x \in (a,b). \tag{24}$$

Then, the problem (23) *is well defined.*

Proof. Let $u, v \in {}^{RL}_{0}H^{\alpha}_{a^+}(a,b)$. First, we have

$$\left| \int_a^b D^{\alpha}_{a^+} u D^{\alpha}_{a^+} v dx \right| \leqslant \|D^{\alpha}_{a^+} u\|_{L^2} \|D^{\alpha}_{a^+} v\|_{L^2} < \infty.$$

Then, the left side of (23) is well defined.

Moreover, we have

$$\left| \int_a^b f(x,u) v(x) dx \right| \leqslant \|\mu\|_{L^2} \|v\|_{L^2} + \|\lambda\|_{L^{\infty}} \|u\|_{L^2} \|v\|_{L^2} < \infty.$$

Therefore, the right side of (23) is well defined. □

The following theorem ensure the existence of a solution of the problem (23).

THEOREM 21. *Assume that f is a Carathéodory function, satisfying the condition (24). If*

$$\Gamma^2(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha} > 0. \tag{25}$$

Then, the problem (23) admits at least one solution.

Proof. To prove this theorem, we apply two methods.

Fixed point method

We will demonstrate this through the following steps.

- *Linearization of the problem:*
  Let $w \in {}_0^{RL}H^\alpha_{a^+}(a,b)$. Consider the following linear problem

$$\int_a^b D^\alpha_{a^+} u D^\alpha_{a^+} v dx = \int_a^b f(x,w) v dx, \quad \forall v \in {}_0^{RL}H^\alpha_{a^+}(a,b). \tag{26}$$

Putting:

$$A(u,v) = \int_a^b D^\alpha_{a^+} u D^\alpha_{a^+} v dx, \qquad \ell(v) = \int_a^b f(x,w) v dx.$$

A is continuous: Let $u, v \in {}_0^{RL}H^\alpha_{a^+}(a,b)$. Then,

$$\begin{aligned} |A(u,v)| &= \left| \int_a^b D^\alpha_{a^+} u(x) D^\alpha_{a^+} v(x) dx \right| \\ &\leqslant \int_a^b |D^\alpha_{a^+} u| |D^\alpha_{a^+} v| dx \\ &\leqslant \left(\int_a^b |D^\alpha_{a^+} u|^2 dx \right)^{\frac{1}{2}} \left(\int_a^b |D^\alpha_{a^+} v|^2 dx \right)^{\frac{1}{2}} \\ &= \|u\|_{{}_0H^\alpha_{a^+}} \|v\|_{{}_0H^\alpha_{a^+}}. \end{aligned}$$

So, A is continuous.
A is coercive: Let $u \in {}_0^{RL}H^\alpha_{a^+}$. Then,

$$\begin{aligned} A(u,u) &= \int_a^b |D^\alpha_{a^+} u(x)|^2 dx \\ &= \|u\|^2_{{}_0H^\alpha_{a^+}}. \end{aligned}$$

So, A coercive.
ℓ is continuous: Let $v \in {}_0^{RL}H^\alpha_{a^+}(a,b)$. Then, from (24) we get

$$\begin{aligned} |\ell(v)| &= \left| \int_a^b f(x,w) v(x) dx \right| \\ &\leqslant \|\mu\|_{L^2} \|v\|_{L^2} + \|\lambda\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\ &\leqslant \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} (\|\mu\|_{L^2} + \|\lambda\|_{L^\infty} \|w\|_{L^2}) \|v\|_{{}_0H^\alpha_{a^+}}. \end{aligned}$$

So, ℓ is continuous.

Consequently, from lax-Milgram theorem the linear problem (26) admits a unique solution in ${}_0^{RL}H_{a^+}^{\alpha}(a,b)$.

- Let T the operator given as

$$T : L^2(a,b) \longrightarrow {}_0^{RL}H_{a^+}^{\alpha}(a,b),$$
$$w \longmapsto u,$$

where u is the unique solution of linear problem (26).

Let $K = \bar{B}(0,R)$ be a ball from ${}_0^{RL}H_{a^+}^{\alpha}(a,b)$. For $w \in K$, we have

$$\begin{aligned}\|T(w)\|_{{}_0H_{a^+}^{\alpha}}^2 &= \|D_{a^+}^{\alpha}T(w)\|_{L^2}^2 \\ &= \|D_{a^+}^{\alpha}u\|_{L^2}^2 \\ &= \int_a^b f(x,T(w))udx.\end{aligned}$$

Using the inequalities (1) and (24), we obtain

$$\begin{aligned}\|T(w)\|_{{}_0H_{a^+}^{\alpha}}^2 &\leqslant \|\mu\|_{L^2}\|T(w)\|_{L^2} + \|\lambda\|_{L^\infty}\|T(w)\|_{L^2}^2 \\ &\leqslant \frac{\|\mu\|_{L^2}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|T(w)\|_{{}_0H_{a^+}^{\alpha}} + \frac{\|\lambda\|_{L^\infty}(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\|T(w)\|_{{}_0H_{a^+}^{\alpha}}^2.\end{aligned}$$

So,

$$\left(1 - \frac{\|\lambda\|_{L^\infty}(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|T(w)\|_{{}_0H_{a^+}^{\alpha}}^2 \leqslant \frac{\|\mu\|_{L^2}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|T(w)\|_{{}_0H_{a^+}^{\alpha}}$$

which can be written

$$\left(\Gamma^2(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha}\right)\|T(w)\|_{{}_0H_{a^+}^{\alpha}}^2 \leqslant \|\mu\|_{L^2}(b-a)^{\alpha}.\Gamma(\alpha+1)\|T(w)\|_{{}_0H_{a^+}^{\alpha}}.$$

Thus, from (1) we obtain

$$\|T(w)\|_{{}_0H_{a^+}^{\alpha}} \leqslant \frac{\|\mu\|_{L^2}(b-a)^{\alpha}}{\Gamma(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha}}.$$

So, for $R = \dfrac{\|\mu\|_{L^2}(b-a)^{\alpha}}{\Gamma(\alpha+1) - \|\lambda\|_{L^\infty}(b-a)^{2\alpha}}$, we can write

$$T : \bar{B}(0,R) \longrightarrow \bar{B}(0,R),$$

where $\bar{B}(0,R) = \left\{w \in {}_0^{RL}H_{a^+}^{\alpha}(a,b) : \|w\|_{{}_0H_{a^+}^{\alpha}} \leqslant R\right\}$.

K *is convex* (Ball).

K is closed in $L^2(a,b)$:

Let $(w_n) \subseteq K$ converge to w in $L^2(a,b)$, we will prove that $v \in K$.

Since (w_n) is a bounded sequence then, from the compactness embedding of ${}_0^{RL}H^{\alpha}_{a^+}(a,b)$ into $L^2(a,b)$, we can extract a subsequence (w_{nk}) weakly convergence to v. Hence,

$$\|v\|_{{}_0H^{\alpha}_{a^+}(a,b)} \leqslant \liminf_{nk\to+\infty} \|v_{nk}\|_{{}_0H^{\alpha}_{a^+}} \leqslant R.$$

Therefore, $v \in K$.

- *T is continuous:*

  Consider the sequence $(w_n) \subset K$, converge to w in $L^2(a,b)$. We denote $u_n = T(w_n)$. So,
  $$\|u_n\| = \|T(w_n)\|_{{}_0H^{\alpha}_{a^+}} \leqslant R.$$
  Therefore, (u_n) is bounded in ${}_0H^{\alpha}_{a^+}(a,b)$, which is reflexive space. Then, we can extract a subsequence $u_{nk} \rightharpoonup u$. From the compactness embedding of ${}_0^{RL}H^{\alpha}_{a^+}(a,b)$ into $L^2(a,b)$, we have $u_{nk} \to u$ in $L^2(a,b)$.

  Hence, for all $v \in {}_0^{RL}H^{\alpha}_{a^+}(a,b)$ we have

$$\begin{array}{ccc} \int_a^b D^{\alpha}_{a^+}u_{nk}(x)D^{\alpha}_{a^+}v(x)dx & = & \int_a^b f(x,w_n)v(x)dx, \\ \text{weakly convergence} & & \text{Lebesgue theorem,} \\ \downarrow & & \downarrow \\ \int_a^b D^{\alpha}_{a^+}uD^{\alpha}_{a^+}v(x)dx & = & \int_a^b f(x,w)v(x)dx. \end{array}$$

  Then, $u = T(w)$, which deduce that $T(K)$ is relative compact.

From the all above, T admits a fixed point, a solution of the problem (23).

Faedo-Galerkin's method

We will demonstrate Theorem 21 through the following steps.

- *Approximation of the space ${}_0^{RL}H^{\alpha}_{a^+}(a,b)$:*

  Since ${}_0^{RL}H^{\alpha}_{a^+}(a,b)$ is a separable Hilbert space, there exists a countable basis $\{V_m\}_{m=1}^{\infty}$ such that $V_m = Vect\left\{v_j\right\}_{j=1}^{m}$ and ${}_0^{RL}H^{\alpha}_{a^+}(a,b) = \overline{\bigcup_{m=1}^{+\infty} V_m}$.

  Using the dot product

$$\langle v_i, v_j\rangle = \int_a^b v_i.v_j\, dx, \quad v_i, v_j \in V_m \subseteq V_{m+1}.$$

- *Approximate problem:* For $u_m \in V_m$, we consider the following approximate problem
$$\int_a^b D_{a^+}^{\alpha} u_m D_{a^+}^{\alpha} v \, dx = \int_a^b f(x,u_m) v \, dx, \;\; \forall v \in V_m. \tag{27}$$
Let $P_m(u_m)$ be the function from V_m to V_m, given by
$$\langle P_m(u_m), v\rangle = \int_a^b D_{a^+}^{\alpha} u_m D_{a^+}^{\alpha} v \, dx - \int_a^b f(x,u_m) v \, dx, \quad \forall v \in V_m.$$
So, if u_m is a solution of (27) then, $P_m(u_m) = 0$.
From previous, P is continuous and we have
$$\begin{aligned}\langle P_m(u_m), u_m\rangle &= \int_a^b |D_{a^+}^{\alpha} u_m|^2 \, dx - \int_a^b f(x,u_m) u_m \, dx\\ &= \|D_{a^+}^{\alpha} u_m\|_{L^2}^2 - \int_a^b f(x,u_m) u_m \, dx\\ &\geqslant \|D_{a^+}^{\alpha} u_m\|_{L^2}^2 - \|\mu\|_{L^2}\|u_m\|_{L^2} - \|\lambda\|_{L^\infty}\|u_m\|_{L^2}^2.\end{aligned}$$
Using the Poincaré inequality (14), we obtain
$$\begin{aligned}\langle P_m(u_m), u_m\rangle &\geqslant \|D_{a^+}^{\alpha} u_m\|_{L^2}^2 - \frac{\|\mu\|_{L^2}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|D_{a^+}^{\alpha} u_m\|_{L^2}\\ &\quad - \frac{\|\lambda\|_{L^\infty}(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\|D_{a^+}^{\alpha} u_m\|_{L^2}^2\\ &= M\|D_{a^+}^{\alpha} u_m\|_{L^2}\left(\|D_{a^+}^{\alpha} u_m\|_{L^2} - r\right),\end{aligned}$$
where
$$M = \frac{\Gamma^2(\alpha+1) - (b-a)^{2\alpha}\|\lambda\|_{L^\infty}}{\Gamma^2(\alpha+1)}, \qquad r = \frac{\Gamma(\alpha+1)(b-a)^{\alpha}\|\mu\|_{L^2}}{\Gamma^2(\alpha+1) - (b-a)^{\alpha}\|\lambda\|_{L^\infty}}.$$
So, for u belongs to the sphere of radius r, we get $\langle P_m(u_m), u_m\rangle \geqslant 0$.
From [14, Theorem 2.7], there exists $u \in {}_0^{RL}H_{a^+}^{\alpha}$ such that $\|u_m\|_{{}_0H_{a^+}^{\alpha}} \leqslant r$ and $P_m(u_m) = 0$, i.e u_m is a solution of the problem (27).

- *Prior estimate:* We have
$$\begin{aligned}\|u_m\|_{{}_0H_{a^+}^{\alpha}}^2 &= \|D_{a^+}^{\alpha} u_m\|_{L^2}^2\\ &= \int_a^b |D_{a^+}^{\alpha} u_m|^2 \, dx\\ &= \int_a^b f(x,u_m) u_m \, dx\\ &\leqslant \|\mu\|_{L^2}\|u_m\|_{L^2} + \|\lambda\|_{L^\infty}\|u_m\|_{L^2}^2\\ &\leqslant \frac{(b-a)^{\alpha}\|\mu\|_{L^2}}{\Gamma(\alpha+1)}\|D_{a^+}^{\alpha} u_m\|_{L^2} + \frac{(b-a)^{2\alpha}\|\lambda\|_{L^\infty}}{\Gamma^2(\alpha+1)}\|D_{a^+}^{\alpha} u_m\|_{L^2}.\end{aligned}$$

So, $M\|u_m\|^2_{{}_0H^{\alpha}_{a^+}} \leqslant \dfrac{(b-a)^{\alpha}\|\mu\|_{L^2}}{\Gamma(\alpha+1)}\|u_m\|_{{}_0H^{\alpha}_{a^+}}$.

Hence, $\|u_m\|_{{}_0H^{\alpha}_{a^+}} \leqslant r$.

Therefore, (u_m) is bounded in ${}_0^{RL}H^{\alpha}_{a^+}(a,b)$.

- *Passage to limit:*

Since (u_m) is bounded in ${}_0^{RL}H^{\alpha}_{a^+}(a,b)$, there exists a subsequence (u_{mk}) such that

$$u_{mk} \rightharpoonup u \ \text{ in } {}_0^{RL}H^{\alpha}_{a^+}(a,b), \qquad \text{and} \qquad D^{\alpha}_{a^+}u_{mk} \rightharpoonup D^{\alpha}_{a^+}u \ \text{ in } L^2(a,b).$$

Therefore, for $m \geqslant j$ we obtain

$$\text{for all } v_j : \int_a^b D^{\alpha}_{a^+}u_{mk}D^{\alpha}_{a^+}v_j\,dx \longrightarrow \int_a^b D^{\alpha}_{a^+}uD^{\alpha}_{a^+}v_j\,dx.$$

Using the fact that ${}_0^{RL}H^{\alpha}_{a^+}(a,b) \hookrightarrow L^2(a,b)$ with compactness, we get

$$u_{mk} \longrightarrow u \qquad \text{in } L^2(a,b).$$

Hence, from [16, Proposition 3], we obtain

$$f(x,u_{mk}) \longrightarrow f(x,u) \qquad \text{in } L^2(a,b).$$

So,

$$f(x,u_{mk}) \rightharpoonup f(x,u) \qquad \text{in } L^2(a,b),$$

which lead to

$$\int_a^b f(u_{mk})v_j\,dx \to \int_a^b f(x,u)v_j\,dx.$$

Hence,

$$\int_a^b D^{\alpha}_{a^+}uD^{\alpha}_{a^+}v_j\,dx = \int_a^b f(x,u)v_j\,dx, \ \ \forall v_j.$$

Setting $W = \bigcup_{m=1}^{\infty} v_j$, then each $w \in W$ can be written $w = \sum_{m=1}^{\infty}\alpha_j v_j$.

Therefore,

$$\int_a^b D^{\alpha}_{a^+}uD^{\alpha}_{a^+}w\,dx = \int_a^b f(x,u)w\,dx, \quad \forall w \in \bigcup_{m=1}^{\infty} v_j.$$

Taking into account that $\overline{\bigcup_{m=1}^{+\infty} V_m} = {}_0^{RL}H^{\alpha}_{a^+}(a,b)$, we obtain

$$\int_a^b D^{\alpha}_{a^+}uD^{\alpha}_{a^+}v\,dx = \int_a^b f(x,u)v\,dx, \ \ \forall v \in {}_0^{RL}H^{\alpha}_{a^+}(a,b).$$

So, u is a solution of problem (23). $\square$

The following theorem give the condition for the uniqueness of solution of problem (23).

THEOREM 22. *Further the assumptions of Theorem 21, if f is nonincreasing then, the solution to problem (23) is unique.*

Proof. Let u_1 and u_2 be two solutions of (23). Then, for $v \in {}_0^{RL}H_{a^+}^{\alpha}(a,b)$ we have

$$\int_a^b \left(D_{a^+}^{\alpha}u_1(x) - D_{a^+}^{\alpha}u_2(x)\right) \cdot D_{a^+}^{\alpha}v(x)dx = \int_a^b [f(x,u_1) - f(x,u_2)]v(x)dx.$$

Setting $v = u_1 - u_2$, we get

$$\int_a^b \left(D_{a^+}^{\alpha}u_1(x) - D_{a^+}^{\alpha}u_2(x)\right)^2(x)dx = \int_a^b [f(x,u_1) - f(x,u_2)](u_1 - u_2)(x)dx.$$

So,

$$\begin{aligned} \|u_1 - u_2\|_{{}_0H_{a^+}^{\alpha}}^2 &= \int_a^b \left(D_{a^+}^{\alpha}u_1(x) - D_{a^+}^{\alpha}u_2(x)\right)^2(x)dx \\ &= \int_a^b [f(x,u_1) - f(x,u_2)](u_1 - u_2)(x)dx \\ &\leqslant 0. \end{aligned}$$

Hence, $\|u_1 - u_2\|_{{}_0H_{a^+}^{\alpha}}^2 = 0$, which deduce that $u_1 = u_2$. □

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Saadi Abderachid
National Higher School of Mathematics
P.O. Box 75, Mahelma 16093, Sidi Abdellah (Algiers), Algeria
and
Laboratory of Non Linear PDE
ENS Kouba, Algiers, Algeria
e-mail: abderachid.saadi@nhsm.edu.dz
rachidsaadi81@gmail.com

Benlatrache Kenza
Department of Mathematics
University of M'sila, University pole
Road Bordj Bou Arreridj, M'sila 28000, Algeria
e-mail: benlatrachekenza@gmail.com