Fractional
Differential
Calculus
Volume 15, Number 1 (2025), 1–14
doi:10.7153/fdc-2025-15-01

OPERATORS OF FRACTIONAL CALCULUS AND ASSOCIATED INTEGRAL TRANSFORMS OF THE (p,q)–EXTENDED HURWITZ–LERCH ZETA FUNCTION

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(*Communicated by H. M. Srivastava*)

Abstract. In this paper, our aim is to establish certain fractional integral and derivative formulas of the generalized (p,q)-extended Hurwitz-Lerch zeta function by using generalized Marichev-Saigo-Maeda fractional operators which involve, in their kernel, Appell's two-variable hypergeometric function $F_3(.)$. These results are expressed in terms of the Hadamard product (or the convolution) of two analytic functions in terms of (p,q)-extended Hurwitz-Lerch zeta function and Fox-Wright hypergeometric function ${}_r\Psi_s(.)$. We then obtain their composition formulas by using fractional integral and derivative formulas and certain Integral transforms associated with Beta, Laplace and Whittaker transforms involving generalized (p,q)-extended Hurwitz-Lerch Zeta function.

1. Introduction

Operators of fractional calculus (such as the Riemann-Liouville, Weyl, Liouville-Caputo, and other operators of fractional integration and fractional derivative) have been developed and investigated widely and extensively, because mainly of their importance and potential for applications (see for details, [15], [24] and [38]). We first recall a general pair of fractional integral operators popularly known as Marichev-Saigo-Maeda which involve, in their kernel, Appell's two-variable hypergeometric function $F_3(.)$ of the third kind, which is defined by (see, for details, [17, 26, 27]):

DEFINITION 1. Let $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta \in \mathrm{C}$ and $x > 0$, then for $Re(\eta) > 0$,

$$\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) = \frac{x^{-\sigma_1}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\sigma_1'} \times F_3\left(\sigma_1, \sigma_1', \nu_1, \nu_1'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, \mathrm{d}t. \quad (1)$$

Mathematics subject classification (2020): 26A33, 33B15, 33C05, 33C99, 44A20.
Keywords and phrases: Hurwitz-Lerch zeta function, (p,q)-extended Hurwitz-Lerch zeta function, fractional calculus operators.

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and

$$\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) = \frac{x^{-\sigma_1'}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\sigma_1} \times F_3\left(\sigma_1,\sigma_1',\nu_1,\nu_1';\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,\mathrm{d}t. \quad (2)$$

Here $F_3(.)$ denotes the Appell's hypergometic function of two variables [37].

DEFINITION 2. Let $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta \in \mathrm{C}$ and $x>0$, then for $Re(\eta)>0$,

$$\begin{aligned}\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) &= \left(I_{0+}^{-\sigma_1',-\sigma_1,-\nu_1',-\nu_1,-\eta} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\sigma_1',-\sigma_1,-\nu_1'+n,-\nu_1,-\eta+n} f\right)(x) \quad (n=[Re(\eta)]+1) \\ &= \frac{1}{\Gamma(n-\eta)}\left(\frac{d}{dx}\right)^n x^{\sigma_1'} \int_0^x (x-t)^{n-\eta-1} t^{\sigma} \\ &\quad \times F_3\left(-\sigma_1',-\sigma_1,n-\nu_1',-\nu_1;n-\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,\mathrm{d}t. \quad (3)\end{aligned}$$

and

$$\begin{aligned}\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) &= \left(I_{-}^{-\sigma_1',-\sigma_1,-\nu_1',-\nu_1,-\eta} f\right)(x) \\ &= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\sigma_1',-\sigma_1,-\nu_1',-\nu_1'+n,-\eta+n} f\right)(x) \quad (n=[Re(\eta)]+1) \\ &= \frac{1}{\Gamma(n-\eta)}\left(-\frac{d}{dx}\right)^n x^{\sigma_1'} \int_x^\infty (t-x)^{n-\eta-1} t^{\sigma'} \\ &\quad \times F_3\left(-\sigma_1',-\sigma_1,\nu_1',n-\nu_1;n-\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,\mathrm{d}t. \quad (4)\end{aligned}$$

These operators includes Saigo hypergeometric fractional calculus operators, Riemann-Liouville and Erdélyi-Kober fractional calculus operators as special cases for various choice of parameters (see for details, [15], [24] and [38]). In recent papers, various authors established several interesting generalized fractional formulas involving (p,q)-extended Bessel function and modified Bessel function, (p,q)-extended Sturve function and modified Sturve function and (p,q)-extended Mathieu series (see, for details, [9, 13]) and other recent developments on several general families of integral transformations and fractional integrations and differentiations of certain higher transcendental functions [28, 31, 33, 34].

The more generalized form so-called (p,q)-extended Hurwitz-Lerch zeta function has been considered very recently by Luo *et al.* [12] in the following form

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p,q) := \sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \frac{B(\mu+n,\,\nu-\mu\,;p,q)}{B(\mu,\,\nu-\mu)} \frac{z^n}{(n+a)^s} \quad (5)$$

$$\left(\min\{\Re(p),\Re(q)\} \geqslant 0;\ p,q,\lambda,\mu,s \in \mathbb{C};\ \nu,\,a \in \mathbb{C}\setminus\mathbb{Z}_0^-;|z|<1\right).$$

where $\mathrm{B}(x,y;p,q)$ is the (p,q)-extended Beta function introduced by Choi *et al.* [3]

$$\mathrm{B}(x,y;p,q) = \mathrm{B}_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t\,, \tag{6}$$

when $\min\{Re(x),Re(y)\} > 0; \min\{Re(p),Re(q)\} \geqslant 0$. They also introduced (p,q)-extended of hypergeometric function as

$$F_{p,q}(a,b;c;z) = \sum_{n\geqslant 0} (a)_n \frac{\mathrm{B}(b+n,\,c-b;\,p,q)}{\mathrm{B}(b,\,c-b)} \frac{z^n}{n!}, \tag{7}$$

$$(p,q \geqq 0; |z|<1; \Re(c) > \Re(b) > 0).$$

Related properties, various integral representations, differentiation formulae, Mellin transform, recurrence relations, summations are also given in [3]. For $p=q$, reduces immediately to the following p-Gauss hypergeometric function $F_p(a,b;c;z)$ studied by Chaudhry *et al.* [2]:

$$F_p(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B(b+n,c-b;p)}{B(b,c-b)} \frac{z^n}{n!}$$

$$\big(p \geqq 0;\ |z|<1;\ \Re(c) > \Re(b) > 0\big).$$

This (p,q)-extended Hurwitz-Lerch zeta function includes another forms of extended Hurwitz-Lerch zeta function as special cases (see, for details, [9, 12, 13]). A detailed study and survey on some recent developments on higher transcendental functions, incomplete Hurwitz-Lerch zeta function, general families of double and multiple Hurwitz-Lerch zeta function and other related functions of analytic number theory with applications can be seen in recent papers [4, 5, 6, 7, 8, 19, 21, 22, 23, 29, 30, 32, 35, 36].

In our present investigation, we require the definition of the Hadamard product (or the convolution) of two analytic functions [9]. If the R_f and R_g be the radii of convergence of the two power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \ (|z|<R_f) \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} b_n z^n \ (|z|<R_g),$$

respectively. Then the Hadamard product is the new emerged series defined by

$$(f*g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = (g*f)(z) \ \ (|z|<R) \tag{8}$$

where

$$R = \lim_{n\to\infty}\left|\frac{a_n b_n}{a_{n+1} b_{n+1}}\right| = \left(\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\right)\cdot\left(\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|\right) = R_f\,.R_g,$$

so that, in general, we have $R \geqq R_f \cdot R_g$.

In this present note, we aim to develop the compositions of the generalized fractional integral and differential operators (1), (2), (3) and (4) for the (p,q)-extended Hurwitz-Lerch zeta function (5) by using the Hadamard product (8) in terms of (p,q)-extended Hurwitz-Lerch zeta function and Wright hypergeometric function.

2. Fractional formulas of the (p,q)-extended Hurwitz-Lerch zeta function

The Wright hypergeometric function ${}_r\Psi_s(z)$ $(r,\, s \in \mathbb{N}_0)$ having numerator and denominator parameters r and s, respectively, defined for $\alpha_1,\ldots,\ \alpha_r \in \mathbb{C}$ and $\beta_1,\ldots,\ \beta_s \in \mathbb{C}\setminus\mathbb{Z}_0^-$ by (see, for example, [14, 16, 24, 37]):

$$ {}_r\Psi_s\left[\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_r,A_r);\\ (\beta_1,B_1),\cdots,(\beta_s,B_s);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+A_1n)\cdots\Gamma(\alpha_r+A_rn)}{\Gamma(\beta_1+B_1n)\cdots\Gamma(\beta_s+B_sn)}\,\frac{z^n}{n!} \tag{9}$$

$$\left(A_j\in\mathbb{R}^+\ (j=1,\ldots,r);\ B_j\in\mathbb{R}^+\ (j=1,\ldots,s);\ 1+\sum_{j=1}^{s}B_j-\sum_{j=1}^{r}A_j \geqq 0\right),$$

with

$$|z| < \nabla := \left(\prod_{j=1}^{r} A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^{s} B_j^{B_j}\right).$$

Also, if we take $A_j = B_k = 1$ $(j=1,\ldots,r;\ \ k=1,\ldots,s)$ in (9), reduces to the generalized hypergeometric function ${}_rF_s$ $(r,\, s\in\mathbb{N}_0)$ (see, *e.g.*, [37]):

$$ {}_rF_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_r;\\ \beta_1,\ldots,\beta_s;\end{matrix} z\right] = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_s)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_r)}\ {}_r\Psi_s\left[\begin{matrix}(\alpha_1,1),\cdots,(\alpha_r,1);\\ (\beta_1,1),\cdots,(\beta_s,1);\end{matrix} z\right]. \tag{10}$$

The following image formulas or power function are useful in our investigation [1].

LEMMA 1. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho\in C$ *and* $x>0$. *Then the following relation exists*

(a) *If* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\sigma_1+\sigma_1'+\nu_1-\eta),Re(\sigma_1'-\nu_1')\}$, *then*

$$\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma_1-\sigma_1'-\nu_1)\Gamma(\rho+\nu_1'-\sigma_1')}{\Gamma(\rho+\nu_1')\Gamma(\rho+\eta-\sigma_1-\sigma_1')\Gamma(\rho+\eta-\sigma_1'-\nu_1)}x^{\rho+\eta-\sigma_1-\sigma_1'-1} \tag{11}$$

(b) *If* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(-\nu_1),Re(\sigma_1+\sigma_1'-\eta),Re(\sigma_1+\nu_1'-\eta)\}$, *then*

$$\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\nu_1)\Gamma(1-\rho-\eta+\sigma_1+\sigma_1')\Gamma(1-\rho-\eta+\sigma_1+\nu_1')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\sigma_1+\sigma_1'+\nu_1')\Gamma(1-\rho+\sigma_1-\nu_1)} \times x^{\rho+\eta-\sigma_1-\sigma_1'-1}. \tag{12}$$

LEMMA 2. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho\in\mathrm{C}$ *and* $x>0$. *Then the following relation exists*

(a) *If* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\eta-\sigma_1-\sigma_1'+\nu_1'),Re(\nu_1-\sigma_1)\}$, *then*

$$\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}t^{\rho-1}\right)(x)$$
$$=\frac{\Gamma(\rho)\Gamma(\rho-\eta+\sigma_1+\sigma_1'+\nu_1')\Gamma(\rho-\nu_1+\sigma_1)}{\Gamma(\rho-\nu_1)\Gamma(\rho-\eta+\sigma_1+\sigma_1')\Gamma(\rho-\eta+\sigma_1+\nu_1')}x^{\rho-\eta+\sigma_1+\sigma_1'-1} \tag{13}$$

(b) *If* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(\nu_1'),Re(\eta-\sigma_1-\sigma_1',Re(\eta-\sigma_1'-\nu_1)\}$, *then*

$$\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}t^{\rho-1}\right)(x)$$
$$=\frac{\Gamma(1-\rho-\nu_1')\Gamma(1-\rho+\eta-\sigma_1-\sigma_1')\Gamma(1-\rho+\eta-\sigma_1'-\nu_1)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\sigma_1-\sigma_1'-\nu)\Gamma(1-\rho-\sigma_1'-\nu_1')} \tag{14}$$
$$\times x^{\rho-\eta+\sigma_1+\sigma_1'-1}.$$

We begin the main results exposition with presenting the composition formulas of generalized fractional operators (1), (2), (3) and (4) involving the (p,q)-extended Hurwitz-Lerch zeta function by using the Hadamard product (8) in terms of (p,q)-extended Hurwitz-Lerch zeta function (5) and Fox-Wright function (9).

THEOREM 1. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\sigma_1+\sigma_1'+\nu_1-\eta),Re(\sigma_1'-\nu_1')\}$ *with* $|t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following formula for fractional integration holds true:*

$$\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\,\Phi_{\lambda,\mu;\nu}(t^\gamma,s,a;p,q)\right\}\right)(x)$$
$$=x^{\rho+\eta-\sigma_1-\sigma_1'-1}\,\Phi_{\lambda,\mu;\nu}(x^\gamma,s,a;p,q)$$
$$*\,{}_4\Psi_3\left[\begin{matrix}(1,1),(\rho,\gamma),(\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),(\rho+\nu_1'-\sigma_1',\gamma);\\ (\rho+\nu_1',\gamma),(\rho+\eta-\sigma_1-\sigma_1',\gamma),(\rho+\eta-\sigma_1'-\nu_1,\gamma);\end{matrix}\,x^\gamma\right].$$

Proof. Applying the definitions (5), (1) and then we change the order of integration and using the relation (11), we find for $x>0$

$$\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\,\Phi_{\lambda,\mu;\nu}(t^\gamma,s,a;p,q)\right\}\right)(x)$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k\,\mathrm{B}(\mu+k,\,\nu-\mu\,;p,q)}{(k+a)^s\,\mathrm{B}(\mu,\,\nu-\mu)\,k!}\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\{t^{\rho+\gamma k-1}\}\right)(x)$$
$$=x^{\rho+\eta-\sigma_1-\sigma_1'-1}\sum_{k=0}^{\infty}\frac{(\lambda)_k\,\mathrm{B}(\mu+k,\,\nu-\mu\,;p,q)}{(k+a)^s\,\mathrm{B}(\mu,\,\nu-\mu)\,k!}$$
$$\times\frac{\Gamma(\rho+\gamma k)\Gamma(\rho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma k)\Gamma(\rho+\nu_1'-\sigma_1'+\gamma k)}{\Gamma(\rho+\nu_1'+\gamma k)\Gamma(\rho+\eta-\sigma_1-\sigma_1'+\gamma k)\Gamma(\rho+\eta-\sigma_1'-\nu_1+\gamma k)}x^{\gamma k}. \tag{15}$$

Finally by using the Hadamard product (8) in (15), which in view of (5) and (9), yields the desired formula. □

THEOREM 2. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(-\nu_1),Re(\sigma_1+\sigma_1'-\eta),Re(\sigma_1+\nu_1'-\eta)\}$ *with* $|1/t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following formula for fractional integration holds true:*

$$\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\frac{1}{t^\gamma},s,a;p,q\right)\right\}\right)(x)$$
$$=x^{\rho+\eta-\sigma_1-\sigma_1'-1}\Phi_{\lambda,\mu;\nu}\left(\frac{1}{x^\gamma},s,a;p,q\right)$$
$$*\,{}_4\Psi_3\left[\begin{matrix}(1,1),(1-\rho-\nu_1,\gamma),(1-\rho-\eta+\sigma_1+\sigma_1',\gamma),(1-\rho-\eta+\sigma_1+\nu_1',\gamma);\\(1-\rho,\gamma),(1-\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(1-\rho+\sigma_1-\nu_1,\gamma);\end{matrix}\ \frac{1}{x^\gamma}\right].$$

THEOREM 3. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\eta-\sigma_1-\sigma_1'-\nu_1'),Re(\nu_1-\sigma_1)\}$ *with* $|t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following formula for fractional differentiation holds true:*

$$\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}(t^\gamma,s,a;p,q)\right\}\right)(x)$$
$$=x^{\rho-\eta+\sigma_1+\sigma_1'-1}\Phi_{\lambda,\mu;\nu}(x^\gamma,s,a;p,q)$$
$$*\,{}_4\Psi_3\left[\begin{matrix}(1,1),(\rho,\gamma),(\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(\rho-\nu_1+\sigma_1,\gamma);\\(\rho-\nu_1,\gamma),(\rho-\eta+\sigma_1+\sigma_1',\gamma),(\rho-\eta+\sigma_1+\nu_1',\gamma);\end{matrix}\ x^\gamma\right].$$

Proof. Applying the definitions (5), (3) and then we change the order of integration and using the relation (13), we find for $x>0$

$$\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}(x^\gamma,s,a;p,q)\right\}\right)(x)$$
$$=\sum_{k=0}^{\infty}\frac{(\lambda)_k\,\mathrm{B}(\mu+k,\,\nu-\mu\,;p,q)}{(k+a)^s\,\mathrm{B}(\mu,\,\nu-\mu)\,k!}\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\{t^{\rho+\gamma k-1}\}\right)(x)$$
$$=x^{\rho-\eta+\sigma_1+\sigma_1'-1}\sum_{k=0}^{\infty}\frac{(\lambda)_k\,\mathrm{B}(\mu+k,\,\nu-\mu\,;p,q)}{(k+a)^s\,\mathrm{B}(\mu,\,\nu-\mu)\,k!}$$
$$\times\frac{\Gamma(\rho+\gamma k)\Gamma(\rho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma k)\Gamma(\rho-\nu_1+\sigma_1+\gamma k)}{\Gamma(\rho-\nu_1+\gamma k)\Gamma(\rho-\eta+\sigma_1+\sigma_1'+\gamma k)\Gamma(\rho-\eta+\sigma_1+\nu_1'+\gamma k)}x^{\gamma k}. \tag{16}$$

Finally by using the Hadamard product (8) in (16), which in view of (5) and (9), yields the desired formula (3). □

THEOREM 4. *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \rho, p, q, \lambda, \mu, s \in \mathbb{C}$ *with* $\gamma \in \mathbb{R}^+$ *and* $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *such that* $Re(\eta) > 0$ *and* $Re(\rho) < 1 + \min\{Re(\nu_1'), Re(\eta - \sigma_1 - \sigma_1', Re(\eta - \sigma_1' - \nu_1)\}$ *with* $|1/t| < 1$*. Then for* $\min\{\Re(p), \Re(q)\} \geqslant 0$*, the following fractional differentiation formula holds true:*

$$\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\frac{1}{t^\gamma}, s, a; p, q\right)\right\}\right)(x)$$
$$= x^{\rho-\eta+\sigma_1+\sigma_1'-1}\Phi_{\lambda,\mu;\nu}\left(\frac{1}{x^\gamma}, s, a; p, q\right)$$
$$* {}_4\Psi_3\left[\begin{matrix}(1,1),(1-\rho-\nu_1',\gamma),(1-\rho+\eta-\sigma_1-\sigma_1',\gamma),(1-\rho+\eta-\sigma_1'-\nu_1,\gamma); & \dfrac{1}{x^\gamma}\\ (1-\rho,\gamma),(1-\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),(1-\rho-\sigma_1'-\nu_1',\gamma); & \end{matrix}\right].$$

The above Theorem 1 to Theorem 4 includes various special cases for which particular choice of parameters reduces to Saigo hypergeometric fractional calculus operators, Riemann-Liouville and Erdélyi-Kober fractional calculus operators. These special cases can be easily evaluated and are left for interesting readers.

3. Certain integral transforms

With the help of the results established in the previous section, in this section, we shall present certain very interesting results in the form of several Theorems associated with Beta, Laplace and Whittaker transforms. For this, first we would like to define these transforms.

DEFINITION 3. The Euler-Beta transform [25] of the function $f(z)$ is defined, as usual, by

$$\mathscr{B}\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\, dz. \tag{17}$$

DEFINITION 4. The Laplace transform (see, *e.g.*, [25]) of the function $f(z)$ is defined, as usual, by

$$L\{f(z); t\} = \int_0^\infty e^{-tz} f(z)\, dz. \quad (Re(t) > 0) \tag{18}$$

The following integral involving Whittaker function (see Mathai *et al.* [16, p. 79]):

$$\int_0^\infty t^{\rho-1} e^{-\frac{1}{2}at}\, W_{\kappa,\nu}(at)\, dt = a^{-\rho}\frac{\Gamma(\frac{1}{2} \pm \nu + \rho)}{\Gamma(1-\kappa+\rho)} \quad (Re(a) > 0,\ Re(\rho \pm \nu) > -\frac{1}{2}), \tag{19}$$

is useful in this section, where $W_{\kappa,\nu}$ is the Whittaker function [18, p. 334].

The following interesting results in the form of Theorems will be established in this section. As these results are direct consequences of the definitions (17), (18), (19) and Theorems 1 to 4, therefore they are given here without proof.

THEOREM 5. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\sigma_1+\sigma_1'+\nu_1-\eta),Re(\sigma_1'-\nu_1')\}$ *with* $|t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following Beta-transform formula holds true:*

$$B\left\{\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}((tz)^{\gamma},s,a;p,q)\right\}\right)(x):l,m\right\}$$
$$=x^{\rho+\eta-\sigma_1-\sigma_1'-1}\Gamma(m)\Phi_{\lambda,\mu;\nu}(x^{\gamma},s,a;p,q)$$
$$*\,{}_5\Psi_4\left[\begin{matrix}(1,1),(l,\gamma),(\rho,\gamma),(\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),(\rho+\nu_1'-\sigma_1',\gamma);\\(l+m,\gamma),(\rho+\nu_1',\gamma),(\rho+\eta-\sigma_1-\sigma_1',\gamma),(\rho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\end{matrix}x^{\gamma}\right].$$

THEOREM 6. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(-\nu_1),Re(\sigma_1+\sigma_1'-\eta),Re(\sigma_1+\nu_1'-\eta)\}$ *with* $|1/t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following Beta-transform formula holds true:*

$$B\left\{\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{z}{t}\right)^{\gamma},s,a;p,q\right)\right\}\right)(x):l,m\right\}$$
$$=x^{\rho+\eta-\sigma_1-\sigma_1'-1}\Gamma(m)\Phi_{\lambda,\mu;\nu}\left(\frac{1}{x^{\gamma}},s,a;p,q\right)$$
$$*\,{}_5\Psi_4\left[\begin{matrix}(1,1),(l,\gamma),(1-\rho-\nu_1,\gamma),\\(l+m,\gamma),(1-\rho,\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho-\eta+\sigma_1+\sigma_1',\gamma),(1-\rho-\eta+\sigma_1+\nu_1',\gamma);\\(1-\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(1-\rho+\sigma_1-\nu_1,\gamma);\end{matrix}\frac{1}{x^{\gamma}}\right].$$

THEOREM 7. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\eta-\sigma_1-\sigma_1'-\nu_1'),Re(\nu_1-\sigma_1)\}$ *with* $|t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following Beta-transform formula holds true:*

$$B\left\{\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}((tz)^{\gamma}),s,a;p,q\right\}\right)(x):l,m\right\}$$
$$=x^{\rho-\eta+\sigma_1+\sigma_1'-1}\Gamma(m)\Phi_{\lambda,\mu;\nu}(x^{\gamma},s,a;p,q)$$
$$*\,{}_5\Psi_4\left[\begin{matrix}(1,1),(l,\gamma),(\rho,\gamma),(\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(\rho-\nu_1+\sigma_1,\gamma);\\(l+m,\gamma),(\rho-\nu_1,\gamma),(\rho-\eta+\sigma_1+\sigma_1',\gamma),(\rho-\eta+\sigma_1+\nu_1'+\gamma,\gamma);\end{matrix}x^{\gamma}\right].$$

THEOREM 8. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(\nu_1'),Re(\eta-\sigma_1-\sigma_1',Re(\eta-\sigma_1'-\nu_1)\}$ *with* $|1/t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following Beta-transform* *formula holds true:*

$$B\left\{\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{z}{t}\right)^{\gamma},s,a;p,q\right)\right\}\right)(x):l,m\right\}$$
$$=x^{\rho-\eta+\sigma_1+\sigma_1'-1}\,\Gamma(m)\,\Phi_{\lambda,\mu;\nu}\left(\frac{1}{x^{\gamma}},s,a;p,q\right)$$
$$*\ {}_5\Psi_4\left[\begin{matrix}(1,1),(l,\gamma),(1-\rho-\nu_1',\gamma),\\(1-\rho,\gamma),(1-\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho+\eta-\sigma_1-\sigma_1',\gamma),(1-\rho+\eta-\sigma_1'-\nu_1,\gamma);\\(l+m,\gamma),(1-\rho-\sigma_1'-\nu_1',\gamma);\end{matrix}\ \frac{1}{x^{\gamma}}\right].$$

THEOREM 9. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\sigma_1+\sigma_1'+\nu_1-\eta),Re(\sigma_1'-\nu_1')\}$ *with* $|t|<1$ *. Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$*, the following Laplace-transform formula holds true:*

$$L\left\{z^{l-1}\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}((tz)^{\gamma}),s,a;p,q\right\}\right)(x)\right\}$$
$$=\frac{x^{\rho+\eta-\sigma_1-\sigma_1'-1}}{s^{l}}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{x}{s}\right)^{\gamma},s,a;p,q\right)$$
$$*\ {}_5\Psi_3\left[\begin{matrix}(1,1),(l,\gamma),(\rho,\gamma),(\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),(\rho+\nu_1'-\sigma_1',\gamma);\\(\rho+\nu_1',\gamma),(\rho+\eta-\sigma_1-\sigma_1',\gamma),(\rho+\eta-\sigma_1'-\nu_1,\gamma);\end{matrix}\ \left(\frac{x}{s}\right)^{\gamma}\right].$$

THEOREM 10. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(-\nu_1),Re(\sigma_1+\sigma_1'-\eta),Re(\sigma_1+\nu_1'-\eta)\}$ *with* $|1/t|<1$*. Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$*, the following Laplace-transform formula holds true:*

$$L\left\{z^{l-1}\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{z}{t}\right)^{\gamma},s,a;p,q\right)\right\}\right)(x)\right\}$$
$$=\frac{x^{\rho+\eta-\sigma_1-\sigma_1'-1}}{s^{l}}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{1}{xs}\right)^{\gamma},s,a;p,q\right)$$
$$*\ {}_5\Psi_3\left[\begin{matrix}(1,1),(l,\gamma),(1-\rho-\nu_1,\gamma),(1-\rho-\eta+\sigma_1+\sigma_1',\gamma),\\(1-\rho,\gamma),(1-\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho-\eta+\sigma_1+\nu_1',\gamma);\\(1-\rho+\sigma_1-\nu_1,\gamma);\end{matrix}\ \left(\frac{1}{xs}\right)^{\gamma}\right].$$

THEOREM 11. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\eta-\sigma_1-\sigma_1'-\nu_1'),Re(\nu_1-\sigma_1)\}$ *with* $|t|<1$ *. Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$*, the following formula Laplace-transform* *holds true:*

$$L\left\{z^{l-1}\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}((tz)^{\gamma},s,a;p,q)\right\}\right)(x):\right\}$$
$$=\frac{x^{\rho-\eta+\sigma_1+\sigma_1'-1}}{s^l}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{x}{s}\right)^{\gamma},s,a;p,q\right)$$
$$*\,{}_5\Psi_3\left[\begin{matrix}(1,1),(l,\gamma),(\rho,\gamma),(\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(\rho-\nu_1+\sigma_1,\gamma);\\(\rho-\nu_1,\gamma),(\rho-\eta+\sigma_1+\sigma_1',\gamma),(\rho-\eta+\sigma_1+\nu_1',\gamma);\end{matrix}\left(\frac{x}{s}\right)^{\gamma}\right].$$

THEOREM 12. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(\nu_1'),Re(\eta-\sigma_1-\sigma_1',Re(\eta-\sigma_1'-\nu_1)\}$ *with* $|1/t|<1$*. Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$*, the following Laplace-transform formula holds true:*

$$L\left\{z^{l-1}\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{z}{t}\right)^{\gamma},s,a;p,q\right)\right\}\right)(x)\right\}$$
$$=\frac{x^{\rho-\eta+\sigma_1+\sigma_1'-1}}{s^l}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{1}{xs}\right)^{\gamma},s,a;p,q\right)$$
$$*\,{}_5\Psi_3\left[\begin{matrix}(1,1),(l,\gamma),(1-\rho-\nu_1',\gamma),\\(1-\rho,\gamma),(1-\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho+\eta-\sigma_1-\sigma_1',\gamma),(1-\rho+\eta-\sigma_1'-\nu_1,\gamma);\\(1-\rho-\sigma_1'-\nu_1',\gamma);\end{matrix}\left(\frac{1}{xs}\right)^{\gamma}\right].$$

THEOREM 13. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\sigma_1+\sigma_1'+\nu_1-\eta),Re(\sigma_1'-\nu_1')\}$ *with* $|t|<1$ *. Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$*, the following integral formula holds true:*

$$\int_0^{\infty}z^{l-1}e^{-\frac{1}{2}\delta z}W_{\tau,\varsigma}(\delta z)\left\{\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}((wtz)^{\gamma}),s,a;p,q\right\}\right)(x)\right\}\mathrm{d}z$$
$$=\frac{x^{\rho+\eta-\sigma_1-\sigma_1'-1}}{\delta^l}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{wx}{\delta}\right)^{\gamma},s,a;p,q\right)$$
$$*\,{}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\zeta+l,\gamma),(\frac{1}{2}-\zeta+l,\gamma),\\(\frac{1}{2}-\tau+l,\gamma),(\rho+\nu_1',\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(\rho,\gamma),(\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),(\rho+\nu_1'-\sigma_1',\gamma);\\(\rho+\eta-\sigma_1-\sigma_1',\gamma),(\rho+\eta-\sigma_1'-\nu_1,\gamma);\end{matrix}\left(\frac{wx}{\delta}\right)^{\gamma}\right].$$

THEOREM 14. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(-\nu_1),Re(\sigma_1+\sigma_1'-\eta),Re(\sigma_1+\nu_1'-\eta)\}$ *with* $|1/t|<1$*. Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$*, the following integral for-*

mula holds true:

$$\int_0^\infty z^{l-1}e^{-\frac{1}{2}\delta z}\,W_{\tau,\varsigma}(\delta z)\left\{\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{wz}{t}\right)^\gamma,s,a;p,q\right)\right\}\right)(x)\right\}\mathrm{d}z$$
$$=\frac{x^{\rho+\eta-\sigma_1-\sigma_1'-1}}{\delta^l}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{w}{x\delta}\right)^\gamma,s,a;p,q\right)$$
$$*\,{}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\zeta+l,\gamma),(\frac{1}{2}-\zeta+l,\gamma),(1-\rho-\nu_1,\gamma),\\ (\frac{1}{2}-\tau+l,\gamma),(1-\rho,\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho-\eta+\sigma_1+\sigma_1',\gamma),(1-\rho-\eta+\sigma_1+\nu_1',\gamma);\\ (1-\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(1-\rho+\sigma_1-\nu_1,\gamma);\end{matrix}\left(\frac{w}{x\delta}\right)^\gamma\right].$$

THEOREM 15. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)>\max\{0,Re(\eta-\sigma_1-\sigma_1'-\nu_1'),Re(\nu_1-\sigma_1)\}$ *with* $|t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following integral formula holds true:*

$$\int_0^\infty z^{l-1}e^{-\frac{1}{2}\delta z}\,W_{\tau,\varsigma}(\delta z)\left\{\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}((wtz)^\gamma),s,a;p,q\right\}\right)(x)\right\}$$
$$=\frac{x^{\rho-\eta+\sigma_1+\sigma_1'-1}}{\delta^l}\Phi_{\lambda,\mu;\nu}(\left(\frac{wx}{\delta}\right)^\gamma,s,a;p,q)$$
$$*\,{}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\zeta+l,\gamma),(\frac{1}{2}-\zeta+l,\gamma)\\ (\frac{1}{2}-\tau+l,\gamma),(\rho-\nu_1,\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(\rho+\gamma,\gamma),(\rho-\eta+\sigma_1+\sigma_1'+\nu_1',\gamma),(\rho-\nu_1+\sigma_1,\gamma);\\ (\rho-\eta+\sigma_1+\sigma_1',\gamma),(\rho-\eta+\sigma_1+\nu_1',\gamma)\end{matrix}\left(\frac{wx}{\delta}\right)^\gamma\right].$$

THEOREM 16. *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\rho,p,q,\lambda,\mu,s\in\mathbb{C}$ *with* $\gamma\in\mathbb{R}^+$ *and* $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ *such that* $Re(\eta)>0$ *and* $Re(\rho)<1+\min\{Re(\nu_1'),Re(\eta-\sigma_1-\sigma_1',Re(\eta-\sigma_1'-\nu_1)\}$ *with* $|1/t|<1$. *Then for* $\min\{\Re(p),\Re(q)\}\geqslant 0$, *the following integral formula holds true:*

$$\int_0^\infty z^{l-1}e^{-\frac{1}{2}\delta z}\,W_{\tau,\varsigma}(\delta z)\left\{\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\rho-1}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{wz}{t}\right)^\gamma,s,a;p,q\right)\right\}\right)(x)\right\}$$
$$=\frac{x^{\rho-\eta+\sigma_1+\sigma_1'-1}}{\delta^l}\Phi_{\lambda,\mu;\nu}\left(\left(\frac{w}{x\delta}\right)^\gamma,s,a;p,q\right)$$
$$*\,{}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\zeta+l,\gamma),(\frac{1}{2}-\zeta+l,\gamma),(1-\rho-\nu_1',\gamma),\\ (\frac{1}{2}-\tau+l,\gamma),(1-\rho,\gamma),\end{matrix}\right.$$
$$\left.\begin{matrix}(1-\rho+\eta-\sigma_1-\sigma_1',\gamma),(1-\rho+\eta-\sigma_1'-\nu_1,\gamma)p;\\ (1-\rho+\eta-\sigma_1-\sigma_1'-\nu_1,\gamma),(1-\rho-\sigma_1'-\nu_1',\gamma);\end{matrix}\left(\frac{w}{x\delta}\right)^\gamma\right].$$

The above Theorem 5 to Theorem 16 includes various special cases for which particular choice of parameters reduces to Saigo hypergeometric fractional calculus operators, Riemann-Liouville and Erdélyi-Kober fractional calculus operators. These special cases can be easily evaluated and are left for interesting readers.

4. Concluding remarks and observations

Recently, various developments have been surveyed and studied generalized fractional integrals and differential operators for certain extensions and generalizations of higher transcendental functions, incomplete Hurwitz-Lerch zeta function, general families of double and multiple Hurwitz-Lerch zeta function and other related functions of analytic number theory with applications [28, 31, 33, 34]. In our present investigation, with the help of the concept of the Hadamard product (or the convolution) of two analytic functions, we have obtained the composition formulas of the generalized Marichev-Saigo-Maeda fractional integrals and differential operators (1), (2), (3) and (4) involving the (p,q)-extended Hurwitz-Lerch zeta function $\Phi_{\lambda,\mu;\nu}(z,s,a;p,q)$ in terms of the Hadamard product (8) of the (p,q)-extended Hurwitz-Lerch zeta function $\Phi_{\lambda,\mu;\nu}(z,s,a;p,q)(z,s,a)$ and the Fox–Wright function ${}_r\Psi_s(z)$. Further, we have also deduced the certain image formulas associated with the integral transforms such as Euler-Beta transform, Laplace transform and Whittaker transform. Next, as special cases, we can deduce the certain image formulas for the Saigo's, Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators as Corollaries, which we have left as exercise for readers. The results obtained in this paper are assumed to be have applications in various field of Physical and Engineering Sciences of extended version of generalized fractional integrals and differential operators and integral transforms related papers [4, 5, 6, 7, 8, 19, 21, 22, 23, 29, 30, 32, 35, 36].

Acknowledgements. The authors thanks to the anonymous referee for valuable comments and useful suggestions to improve the quality of the paper in present form.

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(Received September 3, 2023)

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