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USING MITTAG–LEFFLER FUNCTIONS TO IMPROVE SUFFICIENT CONDITIONS FOR THE UNIQUENESS OF SOLUTIONS TO NABLA BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, the uniqueness of solutions for two prominent classes to nabla fractional boundary value problems are investigated. First, the associated Green's functions are constructed with their equivalent representations and inherent properties are proven. Secondly, the application of the Banach fixed point theorem with sufficient conditions is used to establish the uniqueness and existence of solutions to the considered problems on well-defined spaces with respect to weighted supremum norms. To illustrate the merit, novelty, and applicability of the established results, two examples are presented.

1. Introduction

Nabla fractional calculus is an integrated theory of arbitrary order sums and differences in the backward sense. The concept of nabla fractional difference traces back to the works of many famous researchers in the last two decades. For a detailed introduction to the evolution of nabla fractional calculus, we refer to a recent monograph [8] and the references therein. During the past decade, there has been an increasing interest in analyzing nabla fractional boundary value problems. To name a few notable works, we refer to [2, 5, 6, 7, 9, 10, 11] and the references therein. In this line, we investigate two simple nabla fractional boundary value problems. Specifically, we shall consider

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t) = f(t,u(t)), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = A, \quad u(b) = B, \end{cases} \tag{1}$$

and

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t) + \lambda u(t) = f(t,u(t)), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = A, \quad u(b) = B. \end{cases} \tag{2}$$

Here a, $b \in \mathbb{R}$ such that $b-a \in \mathbb{N}_3$; $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}$; $1 < \nu < 2$; $-1 < \lambda < 1$; A, B are constants and $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th}-order Riemann–Liouville nabla fractional difference of u based at $\rho(a) = a-1$.

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The present article is organized as follows: Section 2 contains preliminaries on discrete fractional calculus. We construct associated Green's functions and obtain some of their properties in Section 3. Also, we deduce sufficient conditions for the existence of solutions to (1) and (2) by employing the Banach fixed point theorem on well-defined spaces with respect to weighted supremum norms coupled with some suitable constraints on f. We also provide two examples to demonstrate the applicability of the established results in Section 4.

2. Preliminaries

We shall use the following preliminaries [3, 4, 8, 10, 13, 14] throughout the article. Denote $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$.

DEFINITION 1. [4] The backward jump operator $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$ is defined by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

DEFINITION 2. [8] For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $\mu \in \mathbb{R}$ such that $(t + \mu) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{\mu}} = \frac{\Gamma(t+\mu)}{\Gamma(t)}.$$

Also, if $t \in \{\ldots, -2, -1, 0\}$ and $\mu \in \mathbb{R}$ such that $(t + \mu) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then we use the convention that $t^{\overline{\mu}} = 0$. Here $\Gamma(\cdot)$ denotes the Euler Gamma function.

DEFINITION 3. [8] For t, $a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\ldots, -2, -1\}$, the μ^{th}-order nabla fractional Taylor monomial is defined by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

provided the right-hand side exists.

In the subsequent lemma, we collect a few properties of nabla fractional Taylor monomials.

LEMMA 1. [8, 9] *Let $s \in \mathbb{N}_a$ and $\mu > -1$. Then, the following properties hold:*

(a) If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geqslant 0$;

(b) If $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$;

(c) If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geqslant 0$, then $H_\mu(t, \rho(s))$ is a nondecreasing function of t;

(d) If $t \in \mathbb{N}_s$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is an increasing function of t.

DEFINITION 4. [8] Let $u:\mathbb{N}_{a+1}\to\mathbb{R}$ and $\nu>0$. The ν^{th}-order nabla fractional sum of u based at a is given by

$$\left(\nabla_a^{-\nu}u\right)(t)=\sum_{s=a+1}^{t}H_{\nu-1}(t,\rho(s))u(s),\quad t\in\mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-\nu}u\right)(a)=0$.

DEFINITION 5. [8] Let $u:\mathbb{N}_{a+1}\to\mathbb{R}$, $\nu>0$, and choose $N\in\mathbb{N}_1$ such that $N-1<\nu\leqslant N$. The ν^{th}-order Riemann–Liouville nabla fractional difference of u based at a is given by

$$\left(\nabla_a^{\nu}u\right)(t)=\left(\nabla^N\left(\nabla_a^{-(N-\nu)}u\right)\right)(t),\quad t\in\mathbb{N}_{a+N}.$$

We observe the following generalized power rules of nabla fractional sum and differences.

LEMMA 2. [8] *Assume the following generalized rising functions and nabla fractional Taylor monomials are well defined. Let $\nu>0$ and $\mu\in\mathbb{R}$. Then,*

1. $\nabla_a^{-\nu}H_\mu(t,a)=H_{\mu+\nu}(t,a)$, *where $\mu+\nu$ is a non-negative integer;*
2. $\nabla_a^{\nu}H_\mu(t,a)=H_{\mu-\nu}(t,a)$, *where $\mu-\nu$ is a non-negative integer,*

for $t\in\mathbb{N}_a$.

Finally, we present the definitions of one-parameter, two-parameter and three parameter nabla Mittag–Leffler functions and state their important properties.

DEFINITION 6. [12] Let $\alpha>0$ and $-1<\lambda<1$. The one-parameter nabla Mittag–Leffler function is defined by

$$F_{\lambda,\alpha}(t,a)=\sum_{n=0}^{\infty}\lambda^n H_{\alpha n}(t,a),\quad t\in\mathbb{N}_a.$$

DEFINITION 7. [8] Let $\alpha\in\mathbb{R}^+$, $\beta\in\mathbb{R}$ and $-1<\lambda<1$. The two-parameter nabla Mittag–Leffler function is defined by

$$E_{\lambda,\alpha,\beta}(t,a)=\sum_{n=0}^{\infty}\lambda^n H_{\alpha n+\beta}(t,a),\quad t\in\mathbb{N}_a.$$

The further generalisation of the Mittag-Leffler function to three parameters allows further generalisation and scope which aligns with the results in the continuous case such as the summation result.

DEFINITION 8. [1] Let $\alpha > 0$, $\beta \in \mathbb{C}$, $\sigma \in \mathbb{C}$ and $-1 < \lambda < 1$. The three-parameter nabla Mittag–Leffler function is defined by

$$E^{\sigma}_{\lambda,\alpha,\beta}(t,a) = \sum_{n=0}^{\infty} \lambda^n \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} H_{\alpha n+\beta-1}(t,a), \quad t \in \mathbb{N}_a.$$

In particular, there are special cases of the Mittag–Leffler function $E^{\sigma}_{\lambda,\alpha,\beta}(t,a)$ for $\sigma = 0$ and $\sigma = -1$:

$$E^{0}_{\lambda,\alpha,\beta}(t,\rho(a)) = \frac{(t-\rho(a))^{\overline{\beta-1}}}{\Gamma(\beta)} = H_{\beta-1}(t,\rho(a));$$

and

$$E^{-1}_{\lambda,\alpha,\beta}(t,\rho(a)) = \frac{(t-\rho(a))^{\overline{\beta-1}}}{\Gamma(\beta)} - \frac{\lambda(t-\rho(a))^{\overline{\alpha+\beta-1}}}{\Gamma(\alpha+\beta)} = H_{\beta-1}(t,\rho(a)) - \lambda H_{\alpha+\beta-1}(t,\rho(a)).$$

Here are a series of Lemmas and results which are useful in dealing with Mittag–Leffler functions of one, two and three parameters.

THEOREM 1. [12] *Assume the following one-parameter nabla Mittag–Leffler functions are well defined. Let* $0 < \lambda < 1$. *Then,*

1. $F_{\lambda,\alpha}(a,a) = 1$;
2. $F_{\lambda,\alpha}(t,a)$ *is a monotonically increasing function with respect to* t *for* $t \in \mathbb{N}_a$;
3. $F_{\lambda,\alpha}(t,a) \to \infty$ *as* $t \to \infty$;
4. $F_{\lambda,\alpha}(\cdot,a) : \mathbb{N}_a \to [1,\infty)$.
5. $\nabla_a^{-\alpha} F_{\lambda,\alpha}(t,a) = \frac{1}{\lambda}\left[F_{\lambda,\alpha}(t,a) - 1\right], \quad t \in \mathbb{N}_a$.

LEMMA 3. [10] *Assume* $1 < \nu < 2$ *and* $t \in \mathbb{N}_{a+2}$. *For each* $0 \leqslant \lambda < 1$, *denote by*

$$g(\lambda) = H_{\nu-3}(t,\rho(a)) + \sum_{n=1}^{\infty} \lambda^n H_{\nu n+\nu-3}(t,\rho(a)).$$

Then, there exists a unique $\bar{\lambda} = \bar{\lambda}(t) \in (0,1)$ *such that* $g(\bar{\lambda}) = 0$.

In the following, if we let $\lambda^* = \min_{t \in \mathbb{N}_{a+2}^b} \bar{\lambda}(t)$, then, $0 < \lambda^* < 1$.

THEOREM 2. [10] *Let* $1 < \nu < 2$ *and* $0 \leqslant \lambda < 1$. *Assume the following two-parameter nabla Mittag–Leffler functions are well defined. Then,*

1. $E_{\lambda,\nu,\nu-1}(t,\rho(a)) > 0$ *for* $t \in \mathbb{N}_a$;

2. $E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is an increasing function with respect to* t *for* $t \in \mathbb{N}_a$*;*
3. *If* $0 < \lambda \leqslant \lambda^* < 1$*, then* $E_{\lambda,\nu,\nu-2}(t,\rho(a))$ *is a decreasing function with respect to* t *for* $t \in \mathbb{N}_{a+1}$.

LEMMA 4. [1] *Let* $\nu > 0$*. Then,*

$$\nabla_a^{\nu} E_{\lambda,\alpha,\beta}^{\sigma}(t,a) = E_{\lambda,\alpha,\beta+\nu}^{\sigma}(t,a), \quad t \in \mathbb{N}_{a+1}.$$

LEMMA 5. [1] *The nabla fractional summation identity:*

$$\sum_{s=a+1}^{t} E_{\lambda,\alpha,\beta}^{\sigma}(t,\rho(s)) E_{\lambda,\alpha,\nu}^{\gamma}(s,\rho(a)) = E_{\lambda,\alpha,\beta+\nu}^{\sigma+\gamma}(t,\rho(a)), \quad t \in \mathbb{N}_a.$$

THEOREM 3. [10] *Assume* $1 < \nu < 2$, $-1 < \lambda < 1$*. The general solution of the homogeneous nabla fractional difference equation*

$$-\left(\nabla_{\rho(a)}^{\nu} u\right)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+2}, \tag{3}$$

is given by

$$u(t) = C_1 E_{\lambda,\nu,\nu-1}(t,\rho(a)) + C_2 E_{\lambda,\nu,\nu-2}(t,\rho(a)), \tag{4}$$

for $t \in \mathbb{N}_a$*. Here* C_1 *and* C_2 *are arbitrary constants.*

THEOREM 4. [10] *Assume* $1 < \nu < 2$, $-1 < \lambda < 1$ *and* $h : \mathbb{N}_{a+2} \to \mathbb{R}$*. The general solution of the non-homogeneous nabla fractional difference equation*

$$-\left(\nabla_{\rho(a)}^{\nu} u\right)(t) + \lambda u(t) = h(t), \quad t \in \mathbb{N}_{a+2}, \tag{5}$$

is given by

$$u(t) = C_1 E_{\lambda,\nu,\nu-1}(t,\rho(a)) + C_2 E_{\lambda,\nu,\nu-2}(t,\rho(a)) - \sum_{s=a+2}^{t} E_{\lambda,\nu,\nu-1}(t,\rho(s)) h(s), \tag{6}$$

for $t \in \mathbb{N}_a$*. Here* C_1 *and* C_2 *are arbitrary constants.*

3. Existence of solutions

In this section, we establish sufficient conditions on the existence of solutions for the nabla fractional boundary value problems (1) and (2) using the Banach fixed point theorem.

THEOREM 5. (Banach fixed point theorem) [15] *Let* S *be a closed subset of a Banach space* X*. Assume* $T : S \to S$ *is a contraction mapping. Then,* T *has a unique fixed point in* S.

THEOREM 6. [11] *A solution* $u : \mathbb{N}_a^b \to \mathbb{R}$ *of the nabla fractional boundary value problem* (1) *is if, and only if,* u *is a solution of the Fredholm summation equation*

$$u(t) = w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b, \tag{7}$$

where

$$G(t,s) = \begin{cases} \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} H_{\nu-1}(b,\rho(s)) - H_{\nu-1}(t,\rho(s)), & t \in \mathbb{N}_a^{\rho(s)}, \\ \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} H_{\nu-1}(b,\rho(s)), & t \in \mathbb{N}_s^b, \end{cases} \tag{8}$$

and

$$w(t) = \frac{1}{H_{\nu-1}(b,a)} \Bigg[B H_{\nu-1}(t,a) + A \Bigg(H_{\nu-2}(t,\rho(a)) H_{\nu-1}(b,\rho(a)) - H_{\nu-1}(t,\rho(a)) H_{\nu-2}(b,\rho(a)) \Bigg) \Bigg], \quad t \in \mathbb{N}_a^b. \tag{9}$$

The next two definitions are the weighted functions for the weighted normed spaces, and their associated properties.

DEFINITION 9. Let $1 < \nu < 2$. Denote by

$$\omega_1(t) = H_{2-\nu}(t,\rho(a)), \quad t \in \mathbb{N}_a.$$

It follows from Lemmas 1 and 2 that

1. $\omega_1(a) = 1$;
2. ω_1 is monotonically increasing with respect to t for $t \in \mathbb{N}_a$;
3. $\omega_1(t) \to \infty$ as $t \to \infty$;
4. $\omega_1 : \mathbb{N}_a \to [1,\infty)$.

DEFINITION 10. Let $0 < \lambda < 1$ and $1 < \nu < 2$. Denote by

$$\omega_2(t) = F_{\lambda,\nu}(t,a), \quad t \in \mathbb{N}_a.$$

It follows from Theorem 1 that

1. $\omega_2(a) = 1$;
2. ω_2 is a monotonically increasing function with respect to t for $t \in \mathbb{N}_a$;
3. $\omega_2(t) \to \infty$ as $t \to \infty$;

4. $\omega_2 : \mathbb{N}_a \to [1, \infty)$;

5. $\left(\nabla_a^{-\nu} \omega_2\right)(t) = \frac{1}{\lambda}\left[\omega_2(t) - 1\right], \quad t \in \mathbb{N}_a$;

We state some key properties for the Green's function in (8) which connects with our weighted norms and upcoming results for uniqueness.

THEOREM 7. *The Green's function $G(t,s)$ defined in (8) satisfies the following properties:*

1. $G(a,s) = G(b,s) = 0$ *for all* $s \in \mathbb{N}_{a+2}^b$;
2. $G(t, a+1) = 0$ *for all* $t \in \mathbb{N}_a^b$;
3. $G(t,s) \geqslant 0$ *for all* $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$;
4. $\sum_{s=a+2}^{b} G(t,s)\omega_1(s) \leqslant H_2(b, \rho(a)) - 1$ *for all* $t \in \mathbb{N}_a^b$;
5. $\sum_{s=a+2}^{b} G(t,s)\omega_2(s) \leqslant \frac{\omega_2(b) - 1}{\lambda}$ *for all* $t \in \mathbb{N}_a^b$.

Proof. We refer to [11] for the proofs of (1), (2) and (3). To prove (4), consider

$$
\begin{aligned}
\sum_{s=a+2}^{b} G(t,s)\omega_1(s) \leqslant & \sum_{s=a+1}^{b} G(t,s)\omega_1(s) \\
&= \sum_{s=a+1}^{b} \left[\frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} H_{\nu-1}(b,\rho(s))\right]\omega_1(s) - \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))\omega_1(s) \\
&= \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} \sum_{s=a+1}^{b} H_{\nu-1}(b,\rho(s)) H_{2-\nu}(s,\rho(a)) \\
&\quad - \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s)) H_{2-\nu}(s,\rho(a)) \\
&= \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} \nabla_a^{-\nu} H_{2-\nu}(b,\rho(a)) - \nabla_a^{-\nu} H_{2-\nu}(t,\rho(a)) \\
&\leqslant \left[H_2(b,\rho(a)) - H_2(t,\rho(a))\right] \text{ (By Lemma 1)} \\
&\leqslant H_2(b,\rho(a)) - 1 \text{ (By Lemma 1)}
\end{aligned}
$$

The proof of (4) is complete. To prove (5), for $t \in \mathbb{N}_a^b$, consider

$$\begin{aligned}
\sum_{s=a+2}^{b} G(t,s)\omega_2(s) &= \sum_{s=a+1}^{b} G(t,s)\omega_2(s) \\
&= \sum_{s=a+1}^{b} \left[\frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} H_{\nu-1}(b,\rho(s))\right]\omega_2(s) - \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))\omega_2(s) \\
&= \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} \sum_{s=a+1}^{b} H_{\nu-1}(b,\rho(s))\omega_2(s) - \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))\omega_2(s) \\
&= \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} \left(\nabla_a^{-\nu}\omega_2\right)(b) - \left(\nabla_a^{-\nu}\omega_2\right)(t) \text{ (By Definition 5)} \\
&= \frac{H_{\nu-1}(t,a)}{H_{\nu-1}(b,a)} \frac{1}{\lambda}\left[\omega_2(b) - 1\right] - \frac{1}{\lambda}\left[\omega_2(t) - 1\right] \text{ (By (5))} \\
&\leqslant \frac{1}{\lambda}\left[\omega_2(b) - 1\right] - \frac{1}{\lambda}\left[\omega_2(t) - 1\right] \text{ (By Lemma 1)} \\
&= \frac{1}{\lambda}\left[\omega_2(b) - \omega_2(t)\right] \\
&\leqslant \frac{1}{\lambda}\left[\omega_2(b) - 1\right]. \text{ (By (2))}
\end{aligned}$$

The proof of (5) is complete. □

Let $\mathscr{B}$ be the set of all real-valued functions defined on $\mathbb{N}_a^b$. Define $T : \mathscr{B} \to \mathscr{B}$ by

$$(Tu)(t) = w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{10}$$

Clearly, u is a fixed point of T if and only if u is a solution of (1). Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1) can be viewed as a real $(b-a+1)$-tuple vector. So, $\mathscr{B}$ is equivalent to $\mathbb{R}^{b-a+1}$.

Finally, we arrive at the first two main uniqueness theorems using the weighted norms under sufficient conditions for the nabla fractional boundary problem (1).

THEOREM 8. *Assume that*

(H 1) f satisfies a uniform Lipschitz condition with respect to its second variable, with Lipschitz constant K;

(H 2) $K(H_2(b,\rho(a)) - 1) < 1$.

Then, the boundary value problem (1) has a unique solution.

Proof. We use the fact that $\mathscr{B}$ is a Banach space equipped with the weighted maximum norm defined by

$$\|u\| = \max_{t \in \mathbb{N}_a^b} \frac{|u(t)|}{\omega_1(t)}.$$

We claim that T is a contraction mapping. To see this, let $u,\ v \in \mathscr{B},\ t \in \mathbb{N}_a^b$, and consider

$$\begin{aligned}
\frac{|(Tu)(t)-(Tv)(t)|}{\omega_1(t)} &= \frac{1}{\omega_1(t)}\left|\sum_{s=a+2}^{b} G(t,s)[f(s,u(s))-f(s,v(s))]\right| \\
&\leqslant \frac{1}{\omega_1(t)}\sum_{s=a+2}^{b} G(t,s)\,|f(s,u(s))-f(s,v(s))| \\
&\leqslant \frac{K}{\omega_1(t)}\sum_{s=a+2}^{b} G(t,s)\,|u(s)-v(s)| \\
&= \frac{K}{\omega_1(t)}\sum_{s=a+2}^{b} G(t,s)\omega_1(s)\frac{|u(s)-v(s)|}{\omega_1(s)} \\
&\leqslant \frac{K\|u-v\|}{\omega_1(t)}\sum_{s=a+2}^{b} G(t,s)\omega_1(s) \\
&\leqslant \frac{K\|u-v\|}{\omega_1(t)} H_2(b,\rho(a)) \text{ (By Lemma 7)} \\
&\leqslant K(H_2(b,\rho(a))-1)\|u-v\|, \text{ (By Definition 9)}
\end{aligned}$$

implying that

$$\|(Tu)-(Tv)\| \leqslant K(H_2(b,\rho(a))-1)\|u-v\|. \tag{11}$$

Since $K(H_2(b,\rho(a))-1)<1$, T is a contraction mapping. Therefore, by Theorem 5, the boundary value problem (1) has a unique solution. The proof is complete. □

THEOREM 9. *Assume that (H 1) and*

(H 3) $\frac{K(\omega_2(b)-1)}{\lambda} < 1$,

hold. Then, the boundary value problem (1) *has a unique solution.*

Proof. We use the fact that $\mathscr{B}$ is a Banach space equipped with the weighted maximum norm defined by

$$\|u\| = \max_{t\in\mathbb{N}_a^b} \frac{|u(t)|}{\omega_2(t)}.$$

We claim that T is a contraction mapping. To see this, let u, $v \in \mathscr{B}$, $t \in \mathbb{N}_a^b$, and consider

$$\begin{aligned}
\frac{|(Tu)(t)-(Tv)(t)|}{\omega_2(t)} &= \frac{1}{\omega_2(t)}\left|\sum_{s=a+2}^{b} G(t,s)[f(s,u(s))-f(s,v(s))]\right| \\
&\leqslant \frac{1}{\omega_2(t)}\sum_{s=a+2}^{b} G(t,s)\,|f(s,u(s))-f(s,v(s))| \\
&\leqslant \frac{K}{\omega_2(t)}\sum_{s=a+2}^{b} G(t,s)\,|u(s)-v(s)| \\
&= \frac{K}{\omega_2(t)}\sum_{s=a+2}^{b} G(t,s)\omega_2(s)\frac{|u(s)-v(s)|}{\omega_2(s)} \\
&\leqslant \frac{K\|u-v\|}{\omega_2(t)}\sum_{s=a+2}^{b} G(t,s)\omega_2(s) \\
&\leqslant \frac{K\|u-v\|}{\omega_2(t)}\frac{\omega_2(b)}{\lambda} \text{ (By Lemma 7)} \\
&\leqslant \frac{K(\omega_2(b)-1)}{\lambda}\|u-v\|, \text{ (By Definition 10)}
\end{aligned}$$

implying that

$$\|(Tu)-(Tv)\| \leqslant \frac{K\omega_2(b)}{\lambda}\|u-v\|. \tag{12}$$

Since $\frac{K(\omega_2(b)-1)}{\lambda} < 1$, T is a contraction mapping. Therefore, by Theorem 5, the boundary value problem (1) has a unique solution. The proof is complete. □

Now, we move our attention to the general nabla fractional boundary value problem (2) where $-1 < \lambda < 1$ is introduced. The equivalent representation is presented and the associated Green's function.

THEOREM 10. [10] *Assume* $1 < \nu < 2$, $-1 < \lambda < 1$ *and* $h : \mathbb{N}_{a+2} \to \mathbb{R}$. *The unique solution of the nabla fractional boundary value problem*

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t) + \lambda u(t) = h(t), & t \in \mathbb{N}^b_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{13}$$

is given by

$$u(t) = \sum_{s=a+2}^{b} H(t,s)h(s), \quad t \in \mathbb{N}^b_a, \tag{14}$$

where

$$H(t,s)=\begin{cases}\dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)}E_{\lambda,\nu,\nu-1}(b,\rho(s))-E_{\lambda,\nu,\nu-1}(t,\rho(s)), & s\in\mathbb{N}_{a+2}^{t},\\ \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)}E_{\lambda,\nu,\nu-1}(b,\rho(s)), & s\in\mathbb{N}_{t+1}^{b}.\end{cases} \tag{15}$$

THEOREM 11. [10] *Assume* $1<\nu<2$ *and* $0<\lambda\leqslant\lambda^*<1$. *The Green's function* $H(t,s)$ *defined in* (15) *satisfies* $H(t,s)\geqslant 0$ *for each* $(t,s)\in\mathbb{N}_a^b\times\mathbb{N}_{a+2}^b$.

Now, we wish to obtain an expression for the unique solution of the following two-point nabla fractional boundary value problem associated with non-homogeneous boundary conditions:

$$\begin{cases}-\left(\nabla_{\rho(a)}^{\nu}u\right)(t)+\lambda u(t)=h(t), & t\in\mathbb{N}_{a+2}^{b},\\ u(a)=A, \quad u(b)=B,\end{cases} \tag{16}$$

where $1<\nu<2$, $-1<\lambda<1$, $h:\mathbb{N}_{a+2}\to\mathbb{R}$ and A, B are constants. For this purpose, we need the following Lemma.

LEMMA 6. *Assume* $1<\nu<2$, $-1<\lambda<1$ *and* A, B *are constants. The unique solution of the nabla fractional boundary value problem*

$$\begin{cases}-\left(\nabla_{\rho(a)}^{\nu}y\right)(t)+\lambda y(t)=0, & t\in\mathbb{N}_{a+2}^{b},\\ y(a)=A, \quad y(b)=B,\end{cases} \tag{17}$$

is

$$y(t)=\frac{1}{E_{\lambda,\nu,\nu-1}(b,a)}\Bigg[BE_{\lambda,\nu,\nu-1}(t,a)+A(1-\lambda)\Bigg(E_{\lambda,\nu,\nu-2}(t,\rho(a))E_{\lambda,\nu,\nu-1}(b,\rho(a)) - E_{\lambda,\nu,\nu-1}(t,\rho(a))E_{\lambda,\nu,\nu-2}(b,\rho(a))\Bigg)\Bigg], \quad t\in\mathbb{N}_a^b. \tag{18}$$

Proof. From Theorem 4, the general solution of the non-homogeneous nabla fractional difference equation in (17) is given by

$$y(t)=C_1E_{\lambda,\nu,\nu-1}(t,\rho(a))+C_2E_{\lambda,\nu,\nu-2}(t,\rho(a)), \quad t\in\mathbb{N}_a^b. \tag{19}$$

Using $y(a)=A$ and $y(b)=B$ in (19), we have

$$A(1-\lambda)=C_1+C_2, \tag{20}$$

$$B=C_1E_{\lambda,\nu,\nu-1}(b,\rho(a))+C_2E_{\lambda,\nu,\nu-2}(b,\rho(a)), \tag{21}$$

respectively. Note that

$$\begin{aligned} &E_{\lambda,\nu,\nu-1}(b,\rho(a)) - E_{\lambda,\nu,\nu-2}(b,\rho(a)) \\ &= \sum_{n=0}^{\infty} \lambda^n \left[H_{\nu n+\nu-1}(b,\rho(a)) - H_{\nu n+\nu-2}(b,\rho(a))\right] \\ &= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(b,a) = E_{\lambda,\nu,\nu-1}(b,a). \end{aligned} \tag{22}$$

Similarly, we have

$$E_{\lambda,\nu,\nu-1}(t,\rho(a)) - E_{\lambda,\nu,\nu-2}(t,\rho(a)) = E_{\lambda,\nu,\nu-1}(t,a). \tag{23}$$

Solving (20) and (21) for C_1 and C_2 and using (22), we obtain

$$C_1 = \frac{B - A(1-\lambda)E_{\lambda,\nu,\nu-2}(b,\rho(a))}{E_{\lambda,\nu,\nu-1}(b,a)}, \tag{24}$$

$$C_2 = \frac{A(1-\lambda)E_{\lambda,\nu,\nu-2}(b,\rho(a)) - B}{E_{\lambda,\nu,\nu-1}(b,a)}. \tag{25}$$

Substituting the expressions of C_1 and C_2 from (24) and (25), respectively, in (19) and rearranging the terms using (23), we obtain (18). The proof is complete. □

It follows from these results now, a unique solution to the non-homogeneous boundary problem has the following representation.

THEOREM 12. *Assume* $1 < \nu < 2$, $-1 < \lambda < 1$, $h : \mathbb{N}_{a+2} \to \mathbb{R}$ *and* A, B *are constants. The unique solution of the nabla fractional boundary value problem* (16) *is given by*

$$u(t) = y(t) + \sum_{s=a+2}^{b} H(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{26}$$

where the Green's function $H(t,s)$ *is as in* (15) *and* y *is given by* (18).

Furthermore, Theorem 12 implies the next crucial equivalent representation result.

THEOREM 13. [10] *A solution* $u : \mathbb{N}_a^b \to \mathbb{R}$ *of the nabla fractional boundary value problem* (2) *is if, and only if,* u *is a solution of the Fredholm summation equation*

$$u(t) = y(t) + \sum_{s=a+2}^{b} H(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b, \tag{27}$$

where the Green's function $H(t,s)$ *is as in* (15) *and* y *is given by* (18).

We now define $S : \mathscr{B} \to \mathscr{B}$ by

$$(Su)(t) = y(t) + \sum_{s=a+2}^{b} H(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{28}$$

Notice that u is a fixed point of S if and only if u is a solution of (2).

Lemma 5 and the special cases of the generalised three-parameter Mittag–Leffler function plays the next key role in the uniqueness results using weighted norms. Equipping the Banach Space $\mathscr{B}$ with an appropriate weighted norm, consider the following weighted norm with an appropriate choice of β:

$$\|x\|_E = \max_{t\in\mathbb{N}_a^b} \frac{\|x(t)\|}{E^{-1}_{\lambda,\nu,\beta}(t,\rho(a))} \geqslant 0.$$

Let $\mathscr{B}_E$ be the Banach space with the weighted norm $\|\cdot\|_E$ and we consider the mapping of $S:\mathscr{B}_E \to \mathscr{B}_E$. This leads to the next main result with a weighted norm to achieve the uniqueness of solutions to (2) under a sufficient condition.

THEOREM 14. *Suppose* $1<\nu<2$, $-1\leqslant\lambda\leqslant 1$, f *satisfies a uniform Lipschitz condition with respect to its second variable with Lipschitz constant* K, *and there exists* β *such that*

$$K+\lambda < \frac{H_{\beta-1}(b,\rho(a))}{H_{\nu+\beta-1}(b,\rho(a))} \tag{29}$$

then it follows that the nabla fractional boundary value problem (2) *has a unique solution.*

Proof. We use the fact that $\mathscr{B}_E$ is a Banach space equipped with the weighted maximum norm defined by

$$\|x\|_E = \max_{t\in\mathbb{N}_a^b} \frac{\|x(t)\|}{E^{-1}_{\lambda,\nu,\beta}(t,\rho(a))}$$

where $E^{-1}_{\lambda,\nu,\beta}(t,\rho(a))>0$ for $t\in\mathbb{N}_a^b$. We claim that S is a contraction mapping. To see this, let u, $v\in\mathscr{B}_E$, $t\in\mathbb{N}_a^b$, and consider

$$\begin{aligned}
|(Su)(t)-(Sv)(t)| &= \left|\sum_{s=a+2}^{b} H(t,s)[f(s,u(s))-f(s,v(s))]\right| \\
&\leqslant \sum_{s=a+2}^{b} H(t,s)\,|f(s,u(s))-f(s,v(s))| \\
&\leqslant K\sum_{s=a+2}^{b} H(t,s)\,|u(s)-v(s)| \\
&\leqslant K\frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} \sum_{s=a+2}^{b} E_{\lambda,\nu,\nu-1}(b,\rho(s)),E^{-1}_{\lambda,\nu,\beta}(s,\rho(a))\,\|u-v\|_E \\
&\leqslant K\sum_{s=a+2}^{b} E^{1}_{\lambda,\nu,\nu}(b,\rho(s)),E^{-1}_{\lambda,\nu,\beta}(s,\rho(a))\,\|u-v\|_E \\
&\leqslant KE^{0}_{\lambda,\nu,\nu+\beta}(b,\rho(a))\,\|u-v\|_E
\end{aligned}$$

$$\leqslant K\|u-v\|_E \frac{(t-\rho(a))^{\overline{\nu+\beta-1}}}{\Gamma(\nu+\beta)} \text{ (By Lemma 7)}$$
$$\leqslant K\|u-v\|_E H_{\nu+\beta-1}(t,\rho(a)).$$

Furthermore, see that

$$\begin{aligned}
\|(Su)(t)-(Sv)(t)\|_E &\leqslant K\|u-v\|_E \frac{H_{\nu+\beta-1}(t,\rho(a))}{E^{-1}_{\lambda,\nu,\beta}(t,\rho(a))} \\
&= \max_{t\in\mathbb{N}_a^b} K\|u-v\|_E \frac{H_{\nu+\beta-1}(t,\rho(a))}{H_{\beta-1}(t,\rho(a))-\lambda H_{\nu+\beta-1}(t,\rho(a))} \\
&\leqslant \max_{t\in\mathbb{N}_a^b} K\|u-v\|_E \frac{H_{\nu+\beta-1}(t,\rho(a))}{H_{\beta-1}(t,\rho(a))-\lambda H_{\nu+\beta-1}(t,\rho(a))} \\
&\leqslant \|u-v\|_E \max_{t\in\mathbb{N}_a^b} \frac{K}{\frac{H_{\beta-1}(t,\rho(a))}{H_{\nu+\beta-1}(t,\rho(a))}-\lambda} \\
&\leqslant \|u-v\|_E \frac{K}{\min_{t\in\mathbb{N}_a^b}\frac{H_{\beta-1}(t,\rho(a))}{H_{\nu+\beta-1}(t,\rho(a))}-\lambda} \\
&\leqslant \|u-v\|_E \frac{K}{\frac{H_{\beta-1}(b,\rho(a))}{H_{\nu+\beta-1}(b,\rho(a))}-\lambda}
\end{aligned}$$

implying that

$$\|(Su)-(Sv)\|_E < \|u-v\|_E. \tag{30}$$

By condition (29), thus S is a contraction mapping. Therefore, by Theorem 5, the boundary value problem (1) has a unique solution. The proof is complete. □

As a remark, the uniqueness result relies on the weighted norm and note that we have not put a restriction on the constant β except for the condition that the weighted norm is well-defined. This is to allow applicability and the potential of future work for new necessary uniqueness conditions.

The last two main theorems of this paper use a weighted norm but formed from the homogeneous BVP with the non-homogeneous boundary conditions under sufficient conditions related to the λ and the Lipschitz constant K of the function f.

THEOREM 15. *Assume $1<\nu<2$, $0<\lambda\leqslant\lambda^*<1$, f satisfies a uniform Lipschitz condition with respect to its second variable, with Lipschitz constant K, and the equation*

$$-\left(\nabla^{\nu}_{\rho(a)}u\right)(t)+\lambda u(t)-Ku(t)=0, \quad t\in\mathbb{N}^b_{a+2}, \tag{31}$$

has a positive solution ω on $\mathbb{N}^b_a$. Then, it follows that the nabla fractional boundary value problem (2) has a unique solution.

Proof. Since the equation (31) has a positive solution ω on $\mathbb{N}_a^b$, it follows that ω is a solution of the nabla fractional boundary value problem

$$\begin{cases} -\left(\nabla_{\rho(a)}^{\nu} u\right)(t) + \lambda u(t) = K\omega(t), & t \in \mathbb{N}_{a+2}^b, \\ u(a) = C, \quad u(b) = D, \end{cases} \tag{32}$$

where $C = \omega(a) > 0$ and $D = \omega(b) > 0$. By using the conclusion of Theorem 12 we have that

$$\omega(t) = x(t) + K \sum_{s=a+2}^{b} H(t,s)\omega(s), \quad t \in \mathbb{N}_a^b, \tag{33}$$

where the Green's function $H(t,s)$ is as in (15) and x is the unique solution of the nabla fractional boundary value problem

$$\begin{cases} -\left(\nabla_{\rho(a)}^{\nu} x\right)(t) + \lambda x(t) = 0, & t \in \mathbb{N}_{a+2}^b, \\ x(a) = C, \quad x(b) = D. \end{cases} \tag{34}$$

Again, by using Lemma 6, x is given by

$$x(t) = \frac{1}{E_{\lambda,\nu,\nu-1}(b,a)} \Bigg[D E_{\lambda,\nu,\nu-1}(t,a) + C(1-\lambda) \bigg(E_{\lambda,\nu,\nu-2}(t,\rho(a)) E_{\lambda,\nu,\nu-1}(b,\rho(a)) - E_{\lambda,\nu,\nu-1}(t,\rho(a)) E_{\lambda,\nu,\nu-2}(b,\rho(a)) \bigg) \Bigg], \quad t \in \mathbb{N}_a^b. \tag{35}$$

We now show that $x(t) > 0$ on $\mathbb{N}_a^b$. It follows from Theorem 2 that $E_{\lambda,\nu,\nu-1}(b,a) > 0$,

$$E_{\lambda,\nu,\nu-1}(t,a) \geqslant 0, \quad t \in \mathbb{N}_a^b,$$

$$E_{\lambda,\nu,\nu-1}(b,\rho(a)) \geqslant E_{\lambda,\nu,\nu-1}(t,\rho(a)), \quad t \in \mathbb{N}_a^b,$$
$$E_{\lambda,\nu,\nu-2}(t,\rho(a)) \geqslant E_{\lambda,\nu,\nu-2}(b,\rho(a)), \quad t \in \mathbb{N}_a^b,$$

implying that

$$\begin{aligned} x(t) &= \frac{1}{E_{\lambda,\nu,\nu-1}(b,a)} \Bigg[D E_{\lambda,\nu,\nu-1}(t,a) + C(1-\lambda) \bigg(E_{\lambda,\nu,\nu-2}(t,\rho(a)) E_{\lambda,\nu,\nu-1}(b,\rho(a)) \\ &\quad - E_{\lambda,\nu,\nu-1}(t,\rho(a)) E_{\lambda,\nu,\nu-2}(b,\rho(a)) \bigg) \Bigg] \\ &\geqslant \frac{1}{E_{\lambda,\nu,\nu-1}(b,a)} \Bigg[D E_{\lambda,\nu,\nu-1}(t,a) + C(1-\lambda) E_{\lambda,\nu,\nu-1}(b,\rho(a)) \bigg(E_{\lambda,\nu,\nu-2}(t,\rho(a)) \\ &\quad - E_{\lambda,\nu,\nu-2}(b,\rho(a)) \bigg) \Bigg] \\ &> 0. \end{aligned}$$

Thus, by (33), we have that

$$\omega(t) > K \sum_{s=a+2}^{b} H(t,s)\omega(s), \quad t \in \mathbb{N}_a^b.$$

Define

$$\alpha := \max_{t\in\mathbb{N}_a^b} \frac{K}{\omega(t)} \sum_{s=a+2}^{b} H(t,s)\omega(s) < 1.$$

We use the fact that $\mathscr{B}$ is a Banach space equipped with the weighted maximum norm defined by

$$\|u\| = \max_{t\in\mathbb{N}_a^b} \frac{|u(t)|}{\omega(t)}.$$

We claim that S is a contraction mapping. To see this, let $u,\ v \in \mathscr{B}$, $t \in \mathbb{N}_a^b$, and consider

$$\begin{aligned}
\frac{|(Su)(t)-(Sv)(t)|}{\omega(t)} &= \frac{1}{\omega(t)} \left| \sum_{s=a+2}^{b} H(t,s)[f(s,u(s)) - f(s,v(s))] \right| \\
&\leqslant \frac{1}{\omega(t)} \sum_{s=a+2}^{b} H(t,s)\,|f(s,u(s)) - f(s,v(s))| \\
&\leqslant \frac{K}{\omega(t)} \sum_{s=a+2}^{b} H(t,s)\,|u(s)-v(s)| \\
&= \frac{K}{\omega(t)} \sum_{s=a+2}^{b} H(t,s)\omega(s) \frac{|u(s)-v(s)|}{\omega(s)} \\
&\leqslant \frac{K\|u-v\|}{\omega(t)} \sum_{s=a+2}^{b} H(t,s)\omega(s) \\
&\leqslant \alpha\|u-v\|,
\end{aligned}$$

implying that

$$\|(Su)-(Sv)\| \leqslant \alpha\|u-v\|.$$

Since $\alpha < 1$, S is a contraction mapping. Therefore, by Theorem 5, the boundary value problem (2) has a unique solution. The proof is complete. □

The final result of this paper removes the restriction in Theorem 15 that the homogeneous equation (31) requiring a positive solution, however, it must be contracting, and in turn provides the uniqueness of solutions for the nabla fractional boundary value problem (2).

THEOREM 16. *If* $1 < \nu < 2,\ 0 < \lambda \leqslant \lambda^* < 1,\ f$ *satisfies a uniform Lipschitz condition with respect to its second variable, with Lipschitz constant* K*, then the nabla fractional boundary value problem* (2) *has a unique solution.*

Proof. The main component of this proof relies on Theorem 15. Consider an alternative boundary value problem

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t)+\lambda u(t)=f(t,u(t)-C)+\lambda C+\nabla^{\nu}_{\rho(a)}C=:g(t,u(t)),\ t\in\mathbb{N}^{b}_{a+2},\\ u(a)=A+C=:A_C,\quad u(b)=B+C=:B_C, \end{cases} \tag{36}$$

where $C\geqslant 0$ such that $A_C:=A+C>0$ and $B_C:=B+C>0$. Then, $u:\mathbb{N}^b_a\to\mathbb{R}$ is a solution of the nabla fractional boundary value problem (36) if, and only if, u is a solution of the Fredholm summation equation

$$u(t)=y(t)+\sum_{s=a+2}^{b}H(t,s)g(s,u(s)),\quad t\in\mathbb{N}^b_a, \tag{37}$$

where the Green's function $H(t,s)$ is as in (15) and y is given by

$$y(t)=\frac{1}{E_{\lambda,\nu,\nu-1}(b,a)}\Bigg[B_CE_{\lambda,\nu,\nu-1}(t,a)+A_C(1-\lambda)\Bigg(E_{\lambda,\nu,\nu-2}(t,\rho(a))E_{\lambda,\nu,\nu-1}(b,\rho(a)) - E_{\lambda,\nu,\nu-1}(t,\rho(a))E_{\lambda,\nu,\nu-2}(b,\rho(a))\Bigg)\Bigg],\quad t\in\mathbb{N}^b_a. \tag{38}$$

Since $E_{\lambda,\nu,\nu-1}(b,a)>0$,

$$E_{\lambda,\nu,\nu-1}(t,a)\geqslant 0,\quad t\in\mathbb{N}^b_a,$$

$$E_{\lambda,\nu,\nu-1}(b,\rho(a))\geqslant E_{\lambda,\nu,\nu-1}(t,\rho(a)),\quad t\in\mathbb{N}^b_a,$$

$$E_{\lambda,\nu,\nu-2}(t,\rho(a))\geqslant E_{\lambda,\nu,\nu-2}(b,\rho(a)),\quad t\in\mathbb{N}^b_a.$$

See that y is a positive function with a suitable choice of C. Since f satisfies a uniform Lipschitz condition with respect to its second variable with Lipschitz constant K, it follows that g is also Lipschitz with respect to its second variable with the same Lipschitz constant. Therefore, the boundary value problem (36) has a unique solution $u(t)$ for $t\in\mathbb{N}^b_a$. However, this consequently implies that $u^*(t):=u(t)-C$ is a unique solution to (2). The proof is complete. □

4. Examples

In this section, we provide two examples to illustrate the applicability of Theorems 8 and 15.

EXAMPLE 1. Consider (1) with $\nu=1.5$, $a=0$, $b=5$, $f(t,u)=t+(0.04)\tan^{-1}u$, $A=1$ and $B=2$. Clearly, f satisfies a uniform Lipschitz condition with respect to its second variable, with Lipschitz constant $K=0.04$. Also,

$$H_2(b,\rho(a))=\frac{(b-a+1)(b-a+2)}{2}=21.$$

Since $KH_2(b,\rho(a)) < 1$, by Theorem 8, the boundary value problem (1) has a unique solution.

EXAMPLE 2. Consider (1) with $\nu = 1.5$, $a = 0$, $b = 10$, $f(t,u) = \frac{\cos u}{200+t}$, $A = 1$ and $B = 2$. In this case, computation by Mathematica yields $\lambda^* = 0.00753$. We choose $\lambda = 0.007$ so that $0 < \lambda \leqslant \lambda^* < 1$. Clearly, f satisfies a uniform Lipschitz condition with respect to its second variable, with Lipschitz constant $K = 0.005$. We have $\lambda - K = 0.002$. Now, consider the equation

$$-\left(\nabla^{1.5}_{\rho(0)}u\right)(t) + (0.02)u(t) = 0, \quad t \in \mathbb{N}_2^{10}. \tag{39}$$

It follows from Theorem 3 and Lemma 1 that the above equation has a positive solution

$$\omega(t) = E_{0.02,1.5,0.5}(t,\rho(0)), \quad t \in \mathbb{N}_0^{10}.$$

Thus, by Theorem 15, the boundary value problem (2) has a unique solution.

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