Fractional
Differential
Calculus
Volume 15, Number 1 (2025), 35–52
doi:10.7153/fdc-2025-15-03

FRACTIONAL TELEGRAPH EQUATION WITH THE SEQUENTIAL RIEMANN–LIOUVILLE DERIVATIVE

RAJAPBOY SAPARBAYEV

(*Communicated by A. Pskhu*)

Abstract. In recent years, the telegraph equation has attracted much attention from researchers due to its practical importance. In this paper, we discuss the telegraph equation

$$(\partial_t^{\rho})^2 u(x,t) + 2\alpha \partial_t^{\rho} u(x,t) - u_{xx}(x,t) = f(x,t),$$

where $0 < t \leqslant T$ and $0 < \rho < 1$, with the Riemann-Liouville derivative. The boundary value problem is investigated. Using the Fourier method, conditions are found for the initial functions and the right-hand side of the equation that guarantee both the existence and uniqueness of the solution of the boundary value problem. Stability inequalities are obtained. An explicit form of the solution to the Cauchy problem for the corresponding ordinary differential equation was found. Note that for such a Cauchy problem, only the existence of a solution was previously known.

1. Introduction

The integral of the Riemann-Liouville order $\rho > 0$ of the function $g(t)$ in the interval $[0, +\infty)$ is defined by the following formula (see, e.g. [18], p. 69):

$$D_t^{-\rho} g(t) \equiv I_t^{\rho} g(t) = \frac{1}{\Gamma(\rho)} \int_0^t \frac{g(\xi)}{(t-\xi)^{1-\rho}} \, d\xi, \quad t > 0$$

provided the right-hand side exists. Here $\Gamma(\rho)$ is Euler's gamma function. Using this definition, one can define the Riemann-Liouville fractional derivative of order $0 < \rho < 1$:

$$\partial_t^{\rho} g(t) = \frac{d}{dt} I_t^{1-\rho} g(t).$$

If we swap the integral and derivative in this equality, we obtain the definition of a derivative in the sense of Caputo of order ρ:

$$D_t^{\rho} g(t) = I_t^{1-\rho} \frac{d}{dt} g(t)$$

Mathematics subject classification (2020): Primary 35R11; Secondary 74S25.
Keywords and phrases: Telegraph equation, Laplace transform, the Riemann-Liouville derivatives, Fourier method.

Sequential derivatives were introduced by Miller and Ross (see, [27], p. 209). The Miller-Ross sequential derivatives are defined as follows:

$$(\partial_t^\rho \circ \partial_t^\rho)u(t) = (\frac{d}{dt} \circ I_t^{1-\rho} \circ \frac{d}{dt} \circ I_t^{1-\rho})u(t),$$

whenever the right-hand side exists for all $t \in (0,T]$. Simple calculations show that

$$(\partial_t^\rho)^2 u(t) := (\partial_t^\rho \circ \partial_t^\rho)u(t) \neq \partial_t^{2\rho} u(t).$$

For functions in the class $C^1[0,T]$ we can write the operator in the following form [19]

$$(\partial_t^\rho)^2 u(t) = \begin{cases} \left(\partial_t^{2\rho} + \frac{t^{1-2\rho}}{\Gamma(2-2\rho)}\delta \circ \frac{d}{dt} + \frac{\Gamma(\rho)}{\Gamma(1-\rho)}\delta\right)u(t), & 0<\rho<\frac{1}{2}, \\ \left(\frac{d}{dt}+\delta\right)u(t), & \rho=\frac{1}{2}, \\ \left(\partial_t^{2\rho} + \frac{\Gamma(\rho)}{\Gamma(1-\rho)}\delta\right)u(t), & \frac{1}{2}<\rho\leqslant 1, \end{cases}$$

where δ denotes the Dirac delta functional.

Let $\rho \in (0,1)$ be a fixed number and $\Omega = (0,\pi)\times(0,T]$. Consider the following boundary value problem for the telegraph equation:

$$\begin{cases} (\partial_t^\rho)^2 u(x,t) + 2\alpha \partial_t^\rho u(x,t) - u_{xx}(x,t) = f(x,t), & (x,t)\in\Omega, \\ u(0,t) = u(\pi,t) = 0, & 0\leqslant t\leqslant T, \\ \lim\limits_{t\to 0} I_t^{1-\rho}(\partial_t^\rho u(x,t)) = \varphi_0(x), & 0\leqslant x\leqslant \pi, \\ \lim\limits_{t\to 0} I_t^{1-\rho} u(x,t) = \varphi_1(x), & 0\leqslant x\leqslant\pi, \end{cases} \tag{1}$$

where $t^{1-\rho}f(x,t)\in C(\overline{\Omega})$ and $\varphi_0(x), \varphi_1(x)$ are continuous functions in the segment $[0,\pi]$.

DEFINITION 1. If a function $u(x,t)$ with the properties $(\partial_t^\rho)^2 u(x,t)$, $\partial_t^\rho u(x,t)$, $u_{xx}(x,t)\in C(\Omega)$ and $t^{1-\rho}\partial_t^\rho u(x,t)$, $t^{1-\rho}u(x,t)\in C(\overline{\Omega})$ satisfies condition (1), then it is called the solution of problem (1).

Taking into account the boundary conditions in problem (1), it is convenient for us to introduce the Hölder classes as follows. Let $\omega_g(\delta)$ be the modulus of continuity of a function $g(x)\in C[0,\pi]$, i.e.

$$\omega_g(\delta) = \sup_{|x_1-x_2|\leqslant\delta} |g(x_1)-g(x_2)|, \quad x_1,x_2\in[0,\pi].$$

If $\omega_g(\delta)\leqslant C\delta^a$ is true for some $a>0$, where C does not depend on δ and $g(0) = g(\pi) = 0$, then $g(x)$ is said to belong to the Hölder class $C^a[0,\pi]$. Let us denote the smallest of all such constants C by $||g||_{C^a[0,\pi]}$. Similarly, if the continuous function $h(x,t)$ is defined on $[0,\pi]\times[0,T]$, then the value

$$\omega_h(\delta;t) = \sup_{|x_1-x_2|\leqslant\delta} |h(x_1,t)-h(x_2,t)|, \quad x_1,x_2\in[0,\pi]$$

is the modulus of continuity of function $h(x,t)$ with respect to the variable x. In case when $\omega_h(\delta;t) \leqslant C\delta^a$, where C does not depend on t and δ and $h(0,t) = h(\pi,t) = 0$, $t \in [0,T]$, then we say that $h(x,t)$ belongs to the Hölder class $C_x^a(\overline{\Omega})$. Similarly, we denote the smallest constant C by $||h||_{C_x^a(\overline{\Omega})}$.

Let $C_{1,x}^a(\overline{\Omega})$ denote the class of functions $h(x,t)$ such that $h_x(x,t) \in C_x^a(\overline{\Omega})$ and $h(0,t) = h(\pi,t) = 0$, $t \in [0,T]$.

THEOREM 1. *Let $\alpha > 0$, $a > \frac{1}{2}$, $t^{1-\rho} f(x,t) \in C_x^a(\overline{\Omega})$ and $\varphi_0(x), \varphi_1(x) \in C_1^a[0,\pi]$. Then problem* (1) *has a unique solution.*

Furthermore, there is a constant $C > 0$ such that the following stability estimate

$$||t^{1-\rho}(\partial_t^\rho)^2 u(x,t)||_{C(\overline{\Omega})} + ||t^{1-\rho}\partial_t^\rho u(x,t)||_{C(\overline{\Omega})} + ||t^{1-\rho} u_{xx}(x,t)||_{C(\overline{\Omega})}$$

$$\leqslant C\left[||\varphi_0(x)||_{C_1^a[0,\pi]} + ||\varphi_1||_{C_1^a[0,\pi]} + ||t^{1-\rho} f(x,t)||_{C_x^a(\overline{\Omega})}\right],$$

holds.

The telegraph equation first appeared in the work of Oliver Heaviside in 1876, when the author modeled the passage of electrical signals in marine telegraph cables. This equation is represented as:

$$u_{tt} + au_t + bu - cu_{xx} = 0,$$

where a and b are non-negative constants and c is a positive constant (see, for example, [1,20]). Specialists subsequently utilized this equation to model various physical, medical, and biological processes (a review of the various applications of the telegraph equation is provided in [16]). Some applications of the telegraph equation to the theory of random walks are contained in [6].

The problem (1) with Caputo derivatives was studied in the works [8] and [2]. The authors of [8] investigated several aspects of the fractional telegraph equations to better understand the anomalous diffusion processes observed in blood flow experiments. The paper [2] investigates the existence of a solution to an initial boundary value problem and the Sobolev regularity of the solution.

For the first time, in the fundamental work of R. Cascaval et al. [8], the highest derivative with respect to time in the telegraph equation is taken in the form $(D_t^\rho)^2$. Usually, authors have studied the telegraph equation with the highest derivative $D_t^{2\rho}$ (see, for example, [3,7,9,12,13,21,23–25,28]), or the second-order classical derivatives [4,5]. The rationale for this choice is given.

As for our choice of the sequential Riemann-Liouville derivative, it is primarily justified by the fact that in the fundamental monograph by Kilbas et al. [18] (see also [11]), the telegraph equation was studied precisely with such a derivative. It should also be noted that the Riemann-Liouville derivative is widely used and applied in various fields of science and technology, as well as in numerical methods for solving fractional differential equations [14,26,31].

2. Preliminaries

DEFINITION 2. (Mittag-Leffler function) The function

$$E_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)}$$

is called the Mittag-Leffler function with two parameters (see, [15], p. 64) where $\rho>0$, $\mu\in\mathbb{C}$ and $z\in\mathbb{C}$.

If the parameter $\mu=1$, then we have the classical Mittag-Leffler function: $E_\rho(z)=E_{\rho,1}(z)$. Prabhakar (see, [29]) introduced the function $E^{\gamma}_{\rho,\mu}(z)$ of the form

$$E^{\gamma}_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\rho k+\mu)}\cdot\frac{z^k}{k!},$$

where $z\in C$, ρ, μ and γ are arbitrary positive constants, and $(\gamma)_k$ is the Pochhammer symbol. When $\gamma=1$, one has $E^1_{\rho,\mu}(z)=E_{\rho,\mu}(z)$. We also have [29]

$$E^{2}_{\rho,\mu}(z)=\frac{1}{\rho}\left[E_{\rho,\mu-1}(z)+(1+\rho-\mu)E_{\rho,\mu}(z)\right]. \tag{2}$$

LEMMA 1. (see [10], p. 136) *Let μ be an arbitrary complex number. Further, let β be a fixed number, such that $\frac{\pi}{2}\rho<\beta<\pi\rho$ and $\beta\leqslant|\arg z|\leqslant\pi$. Then, the following estimate:*

$$|E_{\rho,\mu}(z)|\leqslant\frac{C}{1+|z|},$$

holds, where C is a constant, independent of z.

COROLLARY 1. (see, [2]) *Let $\lambda>0$, $\alpha>0$, and $0<\varepsilon<1$. Then*

$$|t^{\rho-1}E_{\rho,\mu}(-(\alpha\pm\sqrt{\alpha^2-\lambda})t^{\rho})|\leqslant C\lambda^{\varepsilon-\frac{1}{2}}t^{2\varepsilon\rho-1},\quad t>0.$$

Everywhere below, we will denote constants by C , which are not necessarily the same. If a constant depends on, say, ρ, we will denote it by C_ρ.

LEMMA 2. (see, [17]) *If $\rho>0$ and $\lambda\in\mathbb{C}$, then*

$$\partial_t^{\rho}\left(t^{\rho-1}E_{\rho,\rho}(\lambda t^{\rho})\right)=t^{\rho-1}\lambda E_{\rho,\rho}(\lambda t^{\rho}),\quad t>0,$$

$$\partial_t^{\rho}\left(t^{2\rho-1}E^{2}_{\rho,2\rho}(\lambda t^{\rho})\right)=t^{\rho-1}E^{2}_{\rho,\rho}(\lambda t^{\rho}),\quad t>0.$$

If $g(t)\in C[0,T]$, then

$$\partial_t^{\rho}\left(\int_0^t(t-\tau)^{2\rho-1}E^{2}_{\rho,2\rho}(\lambda(t-\tau)^{\rho})g(\tau)d\tau\right)=\int_0^t(t-\tau)^{\rho-1}E^{2}_{\rho,\rho}(\lambda(t-\tau)^{\rho})g(\tau)d\tau.$$

LEMMA 3. *The solution to the Cauchy problem*

$$\begin{cases} \partial_t^\rho u(t) - \lambda u(t) = f(t), \quad 0 < t \leqslant T; \\ \lim\limits_{t\to 0} I_t^{1-\rho} u(t) = 0, \end{cases}$$

with $0 < \rho < 1$ *and* $\lambda \in \mathbb{C}$ *has the form*

$$u(t) = \int_0^t (t-\tau)^{\rho-1} E_{\rho,\rho}(\lambda(t-\tau)^\rho) f(\tau) d\tau.$$

The proof of this Lemma for $\lambda \in \mathbb{R}$ can be found in [18], p. 224. In the complex case, similar ideas will lead us to the same conclusion.

LEMMA 4. *Let* $\alpha > 0$, $\lambda > 0$, $t^{1-\rho} g(t) \in C[0,T]$ *and* φ_0, φ_1 *be known numbers. Then the unique solution to the Cauchy problem*

$$\begin{cases} (\partial_t^\rho)^2 y(t) + 2\alpha \partial_t^\rho y(t) + \lambda y(t) = g(t), \quad 0 < t \leqslant T, \\ \lim\limits_{t\to 0} I_t^{1-\rho} \partial_t^\rho y(t) = \varphi_0, \\ \lim\limits_{t\to 0} I_t^{1-\rho} y(t) = \varphi_1, \end{cases} \tag{3}$$

has the form

$$y(t) = \begin{cases} y_1(t,\lambda), \quad \alpha^2 \neq \lambda, \\ y_2(t,\lambda), \quad \alpha^2 = \lambda. \end{cases} \tag{4}$$

Here

$$y_1(t,\lambda) = \frac{\alpha_2}{\alpha_2-\alpha_1} t^{\rho-1} E_{\rho,\rho}(-\alpha_1 t^\rho)\varphi_1 - \frac{\alpha_1}{\alpha_2-\alpha_1} t^{\rho-1} E_{\rho,\rho}(-\alpha_2 t^\rho)\varphi_1 \tag{5}$$

$$+\frac{t^{\rho-1}}{\alpha_2-\alpha_1}\left(E_{\rho,\rho}(-\alpha_1 t^\rho) - E_{\rho,\rho}(-\alpha_2 t^\rho)\right)\varphi_0$$

$$+\frac{1}{\alpha_2-\alpha_1}\int_0^t (t-\tau)^{\rho-1}\Big(E_{\rho,\rho}(-\alpha_1(t-\tau)^\rho) - E_{\rho,\rho}(-\alpha_2(t-\tau)^\rho)\Big) g(\tau) d\tau,$$

$$y_2(t,\lambda) = t^{\rho-1}E_{\rho,\rho}(-\alpha t^\rho)\varphi_1 + \alpha t^{2\rho-1} E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_1 + t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_0 \tag{6}$$

$$+\int_0^t (t-\tau)^{2\rho-1} E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho) g(\tau) d\tau,$$

where

$$\alpha_1 = \begin{cases} \alpha - \sqrt{\alpha^2-\lambda}, \quad \alpha^2 > \lambda, \\ \alpha - i\sqrt{\lambda-\alpha^2}, \quad \alpha^2 < \lambda, \end{cases} \qquad \alpha_2 = \begin{cases} \alpha + \sqrt{\alpha^2-\lambda}, \quad \alpha^2 > \lambda, \\ \alpha + i\sqrt{\lambda-\alpha^2}, \quad \alpha^2 < \lambda. \end{cases}$$

Here, it is important to note that $i^2 = -1$.

It should be especially emphasized that problem (3) also has an independent interest, as proved by the fact that in the fundamental monograph [18], p. 396 (see also [11]), several theorems are devoted to the study of this problem. The equation discussed in this monograph has variable coefficients, and the authors were able to prove only the existence and uniqueness of the solution using the method of compressed mappings. We have obtained an explicit solution where the coefficients are constant and the initial conditions are arbitrary real numbers.

Proof. To find a representation of the solution to the problem (3), we use the fact that the problem can be considered on the semi-real line $(0,\infty)$. Let $g(t)$ be a continuous and bounded extension of the given function $g(t)\in C(0,T]$ to the region $[T,\infty)$. Now consider the following problem

$$\begin{cases} (\partial_t^\rho)^2 y(t) + 2\alpha\partial_t^\rho y(t) + \lambda y(t) = g(t), \quad t>0, \\ \lim\limits_{t\to 0} I_t^{1-\rho}\partial_t^\rho y(t) = \varphi_0, \\ \lim\limits_{t\to 0} I_t^{1-\rho} y(t) = \varphi_1. \end{cases} \tag{7}$$

Using the Laplace transform we can find the solution to the above problem. Recall that the Laplace transform of $f(t)$ is defined as (see [18], p. 18)

$$L[f](p) = \hat{f}(p) = \int_0^\infty e^{-pt} f(t)dt.$$

The inverse Laplace transform has the form

$$L^{-1}[\hat{f}](t) = \frac{1}{2\pi i}\int_C e^{pt}\hat{f}(p)dp,$$

where C is a contour parallel to the imaginary axis and the right of the singularities of $\hat{f}$. Note, that the convolution theorem for the Laplace transform can be written as (see [18], p. 19)

$$L^{-1}\left[\hat{f}(p)\hat{g}(p)\right] = L^{-1}[\hat{f}(p)] * L^{-1}[\hat{g}(p)](t) = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{8}$$

Let us apply the Laplace transform to the equation in (7). Then

$$p^{2\rho}\hat{y}(p) + 2\alpha p^\rho\hat{y}(p) + \lambda\hat{y}(p) - p^\rho\lim_{t\to 0} I_t^{1-\rho} y(t) - \lim_{t\to 0} J_t^{\rho-1}\partial_t^\rho y(t)$$

$$-2\alpha\lim_{t\to 0} I_t^{1-\rho} y(t) = \hat{g}(p).$$

Therefore

$$\hat{y}(p) = \frac{\hat{g}(p) + p^\rho\lim\limits_{t\to 0} I_t^{1-\rho} y(t) + \lim\limits_{t\to 0} I_t^{1-\rho}\partial_t^\rho y(t)}{p^{2\rho}+2\alpha p^\rho+\lambda} + \frac{2\alpha\lim\limits_{t\to 0} I_t^{1-\rho} y(t)}{p^{2\rho}+2\alpha p^\rho+\lambda}. \tag{9}$$

Case 1. Let $\alpha^2 \neq \lambda$. Write $\hat{y}_1(p) = \hat{l}_0(p) + \hat{l}_1(p)$, where

$$\hat{l}_0(p) = \frac{p^\rho \varphi_1 + \varphi_0 + 2\alpha\varphi_1}{p^{2\rho} + 2\alpha p^\rho + \lambda}, \quad \hat{l}_1(p) = \frac{\hat{g}(p)}{p^{2\rho} + 2\alpha p^\rho + \lambda},$$

furthermore

$$y_1(t) = L^{-1}[\hat{l}_0(p)] + L^{-1}[\hat{l}_1(p)],$$

and in fact we need to find $l_0(p)$ and $l_1(p)$.

For the first term of $y_1(t)$, the inverse Laplace transform can be obtained by splitting the function $\hat{l}_0(p)$ into simpler functions:

$$L^{-1}[\hat{l}_0(p)] = L^{-1}\left[\frac{p^\rho}{p^{2\rho} + 2\alpha p^\rho + \lambda}\right]\varphi_1 + L^{-1}\left[\frac{1}{p^{2\rho} + 2\alpha p^\rho + \lambda}\right](\varphi_0 + 2\alpha\varphi_1). \quad (10)$$

It is easy to see that

$$\frac{1}{p^{2\rho} + 2\alpha p^\rho + \lambda} = \frac{1}{\alpha_2 - \alpha_1}\left(\frac{1}{p^\rho + \alpha_1} - \frac{1}{p^\rho + \alpha_2}\right), \quad (11)$$

where

$$\alpha_1 = \begin{cases} \alpha - \sqrt{\alpha^2 - \lambda}, & \alpha^2 > \lambda, \\ \alpha - i\sqrt{\lambda - \alpha^2}, & \alpha^2 < \lambda, \end{cases} \qquad \alpha_2 = \begin{cases} \alpha + \sqrt{\alpha^2 - \lambda}, & \alpha^2 > \lambda, \\ \alpha + i\sqrt{\lambda - \alpha^2}, & \alpha^2 < \lambda. \end{cases}$$

Therefore equality (10) can be rewritten as

$$\begin{aligned} L^{-1}\left[\hat{l}_0(p)\right] &= \frac{\varphi_1}{\alpha_2 - \alpha_1} L^{-1}\left[\frac{p^\rho}{p^\rho + \alpha_1}\right] - \frac{\varphi_1}{\alpha_2 - \alpha_1} L^{-1}\left[\frac{p^\rho}{p^\rho + \alpha_2}\right] \\ &+ \frac{(\varphi_0 + 2\alpha\varphi_1)}{\alpha_2 - \alpha_1}\left(L^{-1}\left[\frac{1}{p^\rho + \alpha_1}\right] - L^{-1}\left[\frac{1}{p^\rho + \alpha_2}\right]\right). \end{aligned}$$

The result of the inverse Laplace transform has the form (see [22], p. 223, E53):

$$\begin{aligned} l_0(t) &= \frac{\alpha_2}{\alpha_2 - \alpha_1} t^{\rho-1} E_{\rho,\rho}(-\alpha_1 t^\rho)\varphi_1 - \frac{\alpha_1}{\alpha_2 - \alpha_1} t^{\rho-1} E_{\rho,\rho}(-\alpha_2 t^\rho)\varphi_1 \\ &+ \frac{t^{\rho-1}}{\alpha_2 - \alpha_1}\left(E_{\rho,\rho}(-\alpha_1 t^\rho) - E_{\rho,\rho}(-\alpha_2 t^\rho)\right)\varphi_0 \end{aligned}$$

For the second term of $y_1(t)$ one can obtain the inverse Laplace transform by splitting the function $\hat{l}_1(p)$ into the simpler functions using equality (11).

According to (8) we have

$$L^{-1}[\hat{l}_1(p)] = L^{-1}\left[\frac{\hat{g}(p)}{p^{2\rho} + 2\alpha p^\rho + \lambda}\right] = L^{-1}\left[\frac{1}{p^{2\rho} + 2\alpha p^\rho + \lambda}\right] * L^{-1}[\hat{g}(p)].$$

On the other hand one has (see [22], p. 223, E53)

$$\frac{1}{\alpha_2 - \alpha_1} L^{-1}\left[\frac{1}{p^\rho + \alpha_1}\right] = \frac{t^{\rho-1}}{\alpha_2 - \alpha_1} E_{\rho,\rho}(-\alpha_1 t^\rho), \quad (12)$$

$$\frac{1}{\alpha_2-\alpha_1}L^{-1}\left[\frac{1}{p^\rho+\alpha_2}\right]=\frac{t^{\rho-1}}{\alpha_2-\alpha_1}E_{\rho,\rho}\left(-\alpha_2 t^\rho\right). \tag{13}$$

Applying first (11) and then (12), (13) and (8), we obtain

$$y_1(t,\lambda)=\frac{\alpha_2}{\alpha_2-\alpha_1}t^{\rho-1}E_{\rho,\rho}(-\alpha_1 t^\rho)\varphi_1-\frac{\alpha_1}{\alpha_2-\alpha_1}t^{\rho-1}E_{\rho,\rho}(-\alpha_2 t^\rho)\varphi_1$$

$$+\frac{t^{\rho-1}}{\alpha_2-\alpha_1}\left(E_{\rho,\rho}(-\alpha_1 t^\rho)-E_{\rho,\rho}(-\alpha_2 t^\rho)\right)\varphi_0$$

$$+\frac{1}{\alpha_2-\alpha_1}\int_0^t(t-\tau)^{\rho-1}\Big(E_{\rho,\rho}(-\alpha_1(t-\tau)^\rho)-E_{\rho,\rho}(-\alpha_2(t-\tau)^\rho)\Big)g(\tau)d\tau,$$

Case 2. Let $\alpha^2=\lambda$. In this case (9) has the following form

$$\hat{y}_2(p)=\frac{\hat{g}(p)+p^\rho\lim\limits_{t\to0}I_t^{1-\rho}y(t)+\lim\limits_{t\to0}I_t^{1-\rho}\partial_t^\rho y(t)+2\alpha\lim\limits_{t\to0}I_t^{1-\rho}y(t)}{(p^\rho+\alpha)^2}.$$

Passing to the inverse Laplace transform (see [22], p. 226, E67):

$$y_2(t,\lambda)=L^{-1}\left[\frac{1}{p^\rho+\alpha}\right]\lim_{t\to0}I_t^{1-\rho}y(t)-L^{-1}\left[\frac{\alpha}{(p^\rho+\alpha)^2}\right]\lim_{t\to0}I_t^{1-\rho}y(t)$$
$$+L^{-1}\left[\frac{1}{(p^\rho+\alpha)^2}\right]\lim_{t\to0}I_t^{1-\rho}\partial_t^\rho y(t)+L^{-1}\left[\frac{2\alpha}{(p^\rho+\alpha)^2}\right]\lim_{t\to0}I_t^{1-\rho}y(t)$$
$$+L^{-1}\left[\frac{1}{(p^\rho+\alpha)^2}\right]*L^{-1}[\hat{g}(p)]$$

one has

$$y_2(t,\lambda)=t^{\rho-1}E_{\rho,\rho}(-\alpha t^\rho)\varphi_1+\alpha t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_1+t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_0$$

$$+\int_0^t(t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho)g(\tau)d\tau. \quad \square$$

Let H be a separable Hilbert space and $A:H\to H$ be an arbitrary self-adjoint unbounded positive operator with a domain of definition $D(A)$. Suppose that A has a complete set of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\{\lambda_k\}$.

Let $f(\lambda)$ be an arbitrary continuous function defined in $[0,+\infty)$. Then we can define the operator $f(A)$ by the von Neumann theorem as (see [30], p. 263)

$$f(A)h=\sum_{k=1}^{\infty}f(\lambda_k)h_k v_k,\ h\in D(f(A))\subset H, \tag{14}$$

here and everywhere below, by h_k we will denote the Fourier coefficients of a vector $h\in H$: $h_k=(h,v_k)$, and

$$D(f(A))=\{h\in H:\sum_{k=1}^{\infty}|f(\lambda_k)|^2|h_k|^2<\infty\}.$$

In particular, if τ is an arbitrary real number and $f(\lambda)=\lambda^{\tau}$, then one can define the power of operator A, acting in H according to the rule

$$A^{\tau}h=\sum_{k=1}^{\infty}\lambda_k^{\tau}h_kv_k. \tag{15}$$

Next, we take operator A as the operator $-d^2/dx^2$ with the domain $D(A)=\{v(x)\in W_2^2(0,\pi):v(0)=v(\pi)=0\}$, where $W_2^2(0,\pi)$ is the standard Sobolev space. Operator A is self-adjoint in $H=L_2(0,\pi)$ and has the complete set of eigenfunctions $\{v_k(x)=\sin kx\}$ and eigenvalues $\lambda_k=k^2$, $k=1,2,\dots$.

Let us introduce the operator $E_{\rho,\mu}(tA)$. From the definition (14) it follows that

$$E_{\rho,\mu}(tA)h(x,t)=\sum_{n=1}^{\infty}E_{\rho,\mu}(t\lambda_n)h_n(t)v_n(x),\ h(\cdot,t)\in D(E_{\rho,\mu}(tA)). \tag{16}$$

By $h_k(t)$, we will denote the Fourier coefficients of a function $h(x,t)$: $h_k(t)=(h(\cdot,t),v_k(\cdot))$.

LEMMA 5. *Let $g(x)\in C^a[0,\pi]$. Then for any $\sigma\in[0,a-1/2)$ one has*

$$\sum_{k=1}^{\infty}k^{\sigma}|g_k|<\infty.$$

For $\sigma=0$ this assertion coincides with the well-known theorem of S. N. Bernshtein on the absolute convergence of trigonometric series and is proved in the same way as this theorem. For the convenience of readers, we recall the main points of the proof (see, e.g. [32], p. 384).

Proof. In theorem (3.1) of A. Zygmund [32], p. 384, it is proved that for an arbitrary function $g(x)\in C[0,\pi]$, with the properties $g(0)=g(\pi)=0$, one has the estimate

$$\sum_{k=2^{n-1}+1}^{2^n}|g_k|^2\leqslant\omega_g^2\left(\frac{1}{2^{n+1}}\right).$$

Therefore, if $\sigma\geqslant 0$, then by the Cauchy-Bunyakovsky inequality

$$\sum_{k=2^{n-1}+1}^{2^n}k^{\sigma}|g_k|\leqslant\left(\sum_{k=2^{n-1}+1}^{2^n}|g_k|^2\right)^{\frac12}\left(\sum_{k=2^{n-1}+1}^{2^n}k^{2\sigma}\right)^{\frac12}\leqslant C2^{n(\frac12+\sigma)}\omega_g\left(\frac{1}{2^{n+1}}\right),$$

and finally

$$\sum_{k=2}^{\infty}k^{\sigma}|g_k|=\sum_{n=1}^{\infty}\sum_{k=2^{n-1}+1}^{2^n}k^{\sigma}|g_k|\leqslant C\sum_{n=1}^{\infty}2^{n(\frac12+\sigma)}\omega_g\left(\frac{1}{2^{n+1}}\right).$$

Obviously, if $\omega_g(\delta)\leqslant C\delta^a$, $a>1/2$ and $0<\sigma<a-1/2$, then the last series converges:

$$\sum_{k=2}^{\infty}k^{\sigma}|g_k|\leqslant C||g||_{C^a[0,\pi]}.\quad\square$$

Let I be the identity operator in $L_2(0,\pi)$. One can define the operator $(\alpha^2 I - A)^{\frac{1}{2}}$ using (15). We denote by $S^- = \alpha I - (\alpha^2 I - A)^{\frac{1}{2}}$ and $S^+ = \alpha I + (\alpha^2 I - A)^{\frac{1}{2}}$.

LEMMA 6. *Let* $\alpha > 0$, $a > \frac{1}{2}$, *then for any* $g(x,t) \in C_x^a(\overline{\Omega})$ *one has* $E_{\rho,\mu}(-St^\rho)g(x,t) \in C(\overline{\Omega})$ *and* $SE_{\rho,\mu}(-St^\rho)g(x,t) \in C([0,\pi]\times(0,T])$. *Moreover, the following estimates hold:*

$$||E_{\rho,\mu}(-St^\rho)g(x,t)||_{C(\overline{\Omega})} \leqslant C||g||_{C_x^a(\overline{\Omega})}, \tag{17}$$

$$\left|\left|SE_{\rho,\mu}(-St^\rho)g(x,t)\right|\right|_{C([0,\pi]\times(0,T])} \leqslant Ct^{-\rho}||g||_{C_x^a(\overline{\Omega})}, \quad t>0. \tag{18}$$

If $g(x,t) \in C_{1,x}^a(\overline{\Omega})$, *then*

$$\left|\left|SE_{\rho,\mu}(-St^\rho)g(x,t)\right|\right|_{C(\overline{\Omega})} \leqslant C||g_x||_{C_x^a(\overline{\Omega})}, \tag{19}$$

$$\left|\left|\frac{\partial^2}{\partial x^2}E_{\rho,\mu}(-St^\rho)g(x,t)\right|\right|_{C([0,\pi]\times(0,T])} \leqslant Ct^{-\rho}||g_x||_{C_x^a(\overline{\Omega})}. \tag{20}$$

Here, S *has two states:* S^- *and* S^+.

Proof. By definition one has

$$|E_{\rho,\mu}(-S^-t^\rho)g(x,t)| = \left|\sum_{k=1}^\infty E_{\rho,\mu}\left(-\alpha_{1k}t^\rho\right)g_k(t)v_k(x)\right| \leqslant \sum_{k=1}^\infty \left|E_{\rho,\mu}\left(-\alpha_{1k}t^\rho\right)g_k(t)\right|,$$

where

$$\alpha_{1k} = \begin{cases} \alpha - \sqrt{\alpha^2-\lambda_k}, & \alpha^2 > \lambda_k, \\ \alpha - i\sqrt{\lambda_k-\alpha^2}, & \alpha^2 < \lambda_k, \end{cases} \quad \alpha_{2k} = \begin{cases} \alpha + \sqrt{\alpha^2-\lambda_k}, & \alpha^2 > \lambda_k, \\ \alpha + i\sqrt{\lambda_k-\alpha^2}, & \alpha^2 < \lambda_k. \end{cases}$$

Lemmas 1 and 5 imply that

$$|E_{\rho,\mu}(-S^-t^\rho)g(x,t)| \leqslant C\sum_{k=1}^\infty \frac{|g_k(t)|}{1+t^\rho|\alpha_{1k}|} \leqslant C||g||_{C_x^a(\overline{\Omega})}.$$

On the other hand,

$$|S^-E_{\rho,\mu}(-S^-t^\rho)g(x,t)| \leqslant C\sum_{k=1}^\infty \frac{|\alpha_{1k}||g_k(t)|}{1+t^\rho|\alpha_{1k}|} \leqslant Ct^{-\rho}||g||_{C_x^a(\overline{\Omega})}, \quad t>0.$$

If $g(x,t) \in C_{1,x}^a(\overline{\Omega})$, then $g_k(t) = \lambda_k^{-\frac{1}{2}}(g_x(t,x),\cos kx)$. Therefore,

$$|S^-E_{\rho,\mu}(-S^-t^\rho)g(x,t)| \leqslant C\sum_{k=1}^\infty \frac{|\alpha_{1k}||g_k(t)|}{1+t^\rho|\alpha_{1k}|}. \tag{21}$$

We analyze the convergence of the series on the right-hand side of inequality (21). Using the definition of α_{1k}, it is known that

$$\sum_{k=1}^{\infty} u_{\lambda_k}(t) = \sum_{k=1}^{j} \frac{\left|\alpha - \sqrt{\alpha^2-\lambda_k}\right| |g_k(t)|}{1+t^{\rho}\left|\alpha - \sqrt{\alpha^2-\lambda_k}\right|} + \sum_{k=j+1}^{\infty} v_{\lambda_k}(t). \tag{22}$$

For the first term on the right-hand side of the above series, we obtain the following estimate

$$\sum_{k=1}^{j} \frac{\left|\alpha - \sqrt{\alpha^2-\lambda_k}\right| |g_k(t)|}{1+t^{\rho}\left|\alpha - \sqrt{\alpha^2-\lambda_k}\right|} \leqslant \sum_{k=1}^{j} \left|\alpha - \sqrt{\alpha^2-\lambda_k}\right| |g_k(t)|,$$

and we obtain an inequality for the second term

$$\sum_{k=j+1}^{\infty} v_{\lambda_k}(t) = \sum_{k=j+1}^{\infty} \frac{\lambda_k^{\frac{1}{2}} |g_k(t)|}{1+t^{\rho}\sqrt{\lambda_k}} \leqslant \sum_{k=j+1}^{\infty} \lambda_k^{\frac{1}{2}} |g_k(t)|.$$

We rewrite equality (22) and, taking into account the inequalities derived for the first and second terms along with Lemma 5, we obtain the desired result.

$$\begin{aligned}
\sum_{k=1}^{\infty} u_{\lambda_k}(t) &= \sum_{k=1}^{j} \frac{\left|\alpha - \sqrt{\alpha^2-\lambda_k}\right| |g_k(t)|}{1+t^{\rho}\left|\alpha - \sqrt{\alpha^2-\lambda_k}\right|} + \sum_{k=j+1}^{\infty} v_{\lambda_k}(t) \\
&\leqslant \sum_{k=1}^{j} \left|\alpha - \sqrt{\alpha^2-\lambda_k}\right| |g_k(t)| + \sum_{k=j+1}^{\infty} \lambda_k^{\frac{1}{2}} |g_k(t)| \leqslant C \sum_{k=j+1}^{\infty} \lambda_k^{\frac{1}{2}} |g_k(t)| \\
&\leqslant C\|g_x\|_{C_x^a(\overline{\Omega})}.
\end{aligned}$$

Consequently,

$$\|S^- E_{\rho,\mu}(-t^{\rho}S^-)g(x,t)\|_{C(\overline{\Omega})} \leqslant C\|g_x\|_{C_x^a(\overline{\Omega})}.$$

Similarly, we have

$$\left\|\frac{\partial^2}{\partial x^2} E_{\rho,\mu}(-S^- t^{\rho})g(x,t)\right\|_{C(\overline{\Omega})} \leqslant C\sum_{k=1}^{\infty}\left|\frac{\lambda_k|g_k(t)|}{1+t^{\rho}\sqrt{\lambda_k}}\right| \leqslant Ct^{-\rho}\|g_x\|_{C_x^a(\overline{\Omega})}, \quad t>0.$$

A similar estimate is proved in exactly the same way with the operator S^- replaced by the operator S^+. □

Operator $R^{-1} = (\alpha^2 I - A)^{-\frac{1}{2}}$ is defined using (15) on the domain

$$D(R^{-1}) = \left\{h \in H \ : \ \sum|\alpha^2-\lambda_k|^{-1}|h_k|^2 < \infty\right\}.$$

LEMMA 7. *Let* $\alpha>0$, $a>\frac{1}{2}$ *and* $\lambda_k\neq\alpha^2$, *for all* k. *Then for any* $g(x,t)\in C_x^a(\overline{\Omega})$ *one has* $R^{-1}E_{\rho,\mu}(-St^\rho)g(x,t)$, $SR^{-1}E_{\rho,\mu}(-St^\rho)g(x,t)\in C(\overline{\Omega})$ *and* $\frac{\partial^2}{\partial x^2}R^{-1}E_{\rho,\mu}(-t^\rho S)$ $g(x,t)\ \in C([0,\pi]\times(0,T])$. *Moreover, the following estimates hold:*

$$||R^{-1}E_{\rho,\mu}(-t^\rho S)g(x,t)||_{C(\overline{\Omega})}\leqslant C||g||_{C_x^a(\overline{\Omega})},\tag{23}$$

$$||SR^{-1}E_{\rho,\mu}(-t^\rho S)g(x,t)||_{C(\overline{\Omega})}\leqslant C||g||_{C_x^a(\overline{\Omega})},\tag{24}$$

$$||\frac{\partial^2}{\partial x^2}R^{-1}E_{\rho,\mu}(-t^\rho S)g(x,t)||_{C([0,\pi]\times(0,T])}\leqslant Ct^{-\rho}||g||_{C_x^a(\overline{\Omega})},\quad t>0.\tag{25}$$

Proof. The proof follows ideas similar to the proof of Lemma 6. □

LEMMA 8. *Let* $\alpha>0$ *and* $t^{1-\rho}g(x,t)\in C_x^a(\overline{\Omega})$. *Then*

$$\left|\left|t^{1-\rho}\int_0^t(t-\tau)^{\rho-1}E_{\rho,\rho}(-S(t-\tau)^\rho)g(x,\tau)d\tau\right|\right|\leqslant C_\rho||t^{1-\rho}g||_{C_x^a(\overline{\Omega})},\tag{26}$$

$$\left|\left|t^{1-\rho}\int_0^t(t-\tau)^{2\rho-1}E^2_{\rho,2\rho}(-\alpha(t-\tau)^\rho)g(x,\tau)d\tau\right|\right|\leqslant C_\rho||t^{1-\rho}g||_{C_x^a(\overline{\Omega})}\tag{27}$$

$$\left|\left|\int_0^t(t-\tau)^{\rho-1}E^2_{\rho,\rho}(-\alpha(t-\tau)^\rho)g(x,\tau)d\tau\right|\right|\leqslant C_\rho||t^{1-\rho}g||_{C_x^a(\overline{\Omega})}.\tag{28}$$

Proof. Apply estimate (17) to get

$$\left|\left|t^{1-\rho}\int_0^t(t-\tau)^{\rho-1}E_{\rho,\rho}(-S(t-\tau)^\rho)g(x,\tau)d\tau\right|\right|$$
$$\leqslant Ct^{1-\rho}\int_0^t(t-\tau)^{\rho-1}\tau^{\rho-1}d\tau\cdot||t^{1-\rho}g||_{C_x^a(\overline{\Omega})}.$$

For the integral one has

$$\int_0^t(t-\tau)^{\rho-1}\tau^{\rho-1}d\tau=t^{2\rho-1}\frac{\Gamma^2(\rho)}{\Gamma(2\rho)}.$$

this implies the assertion of the (26). (27) and (28) are obtained in the same way as in the proof of (26) combining. □

LEMMA 9. *Let* $\alpha>0$, $a>\frac{1}{2}$ *and* $\lambda_k\neq\alpha^2$, *for all* k. *Then for any* $t^{1-\rho}g(x,t)\in C(\overline{\Omega})$, *we have*

$$\left|\left|\int_0^t(t-\tau)^{\rho-1}\frac{\partial^2}{\partial x^2}R^{-1}E_{\rho,\rho}(-S(t-\tau)^\rho)g(x,\tau)d\tau\right|\right|_{C(\overline{\Omega})}\leqslant C||t^{1-\rho}g||_{C_x^a(\overline{\Omega})},\tag{29}$$

$$\left|\left|\int_0^t (t-\tau)^{\rho-1} SR^{-1} E_{\rho,\rho}(-S(t-\tau)^\rho) g(x,\tau) d\tau\right|\right|_{C(\overline{\Omega})} \leqslant C||t^{1-\rho} g||_{C_x^a(\overline{\Omega})}, \tag{30}$$

$$\left|\left|\int_0^t (t-\tau)^{\rho-1} R^{-1} E_{\rho,\rho}(-S(t-\tau)^\rho) g(x,\tau) d\tau\right|\right|_{C(\overline{\Omega})} \leqslant C||t^{1-\rho} g||_{C_x^a(\overline{\Omega})}. \tag{31}$$

Proof. Let

$$S_j(x,t) = \sum_{k=1}^{j} \left[\int_0^t (t-\tau)^{\rho-1} \alpha_{1k} E_{\rho,\rho}(-\alpha_{1k}(t-\tau)^\rho) g_k(\tau) d\tau\right] \lambda_k v_k(x).$$

Choose ε so that $0<\varepsilon<a-1/2$ and apply Corollary 1 to get

$$|S_j(t)| \leqslant C \sum_{k=1}^{j} \int_0^t (t-\tau)^{\varepsilon\rho-1} \tau^{\rho-1} \lambda_k^{\varepsilon} |\tau^{1-\rho} g_k(\tau)| ds.$$

By Lemma 5 we have

$$|S_j(t)| \leqslant C||t^{1-\rho} g||_{C_x^a(\overline{\Omega})}$$

and since

$$\int_0^t (t-\tau)^{\rho-1} \frac{\partial^2}{\partial x^2} R^{-1} E_{\rho,\rho}(-S^-(t-\tau)^\rho) g(x,\tau) d\tau = \lim_{j\to\infty} S_j(t),$$

This implies the assertion of (29). The results (30) and (31) are obtained in the same way as in the proof of (29), by combining the relevant arguments. □

3. Proof of the theorem on the forward problem

Before formulating the main results, we note that in solving the problems posed, the location of the spectrum of operator $-d^2/dx^2$ and the parameter α of the equation plays a significant role. In other words, the representation of the solution depends on whether λ_k for some k coincides with the parameter α^2. Obviously, there can only be at most one such k. For the sake of completeness, we assume there is such k for which $\alpha^2=\lambda_k$ we will denote it by k_0.

According to the Fourier method, we will seek the solution to this problem in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) v_k(x),$$

where $T_k(t)$ is a solution to the problem

$$\begin{cases} (\partial_t^\rho)^2 T_k(t) + 2\alpha \partial_t^\rho T_k(t) + k^2 T_k(t) = f_k(t), \quad 0<t\leqslant T, \\ \lim\limits_{t\to 0} I_t^{1-\rho}(\partial_t^\rho T_k(t)) = \varphi_{0k}, \\ \lim\limits_{t\to 0} I_t^{1-\rho} T_k(t) = \varphi_{1k}, \end{cases} \tag{32}$$

Lemma 4 implies

$$T_k(t) = \begin{cases} y_{1k}(t,\lambda_k), & k \neq k_0, \\ y_{2k}(t,\lambda_k), & k = k_0, \end{cases}$$

where $y_{1k}(t,\lambda_k)$ and $y_{2k}(t,\lambda_k)$ are the solution of (32) given as (5) and (6). Hence, we have a formal solution to the problem (1), which is an element of the Hilbert space $L_2(0,\pi)$ and has the following form.

$$u(x,t) = \sum_{k=1}^{\infty} y_{1k}(t,\lambda_k)v_k(x) + y_{2k_0}(t,\lambda_{k_0})v_{k_0}(x). \tag{33}$$

According to (16) then we may rewrite (33):

$$\begin{aligned} u(x,t) = &\frac{1}{2}\left(\tilde{E}_{\rho,\rho}(-S^- t^\rho) + \tilde{E}_{\rho,\rho}(-S^+ t^\rho)\right) t^{\rho-1}\varphi_1(x) \\ &+\frac{\alpha}{2}\left(R^{-1}\tilde{E}_{\rho,\rho}(-S^- t^\rho) - R^{-1}\tilde{E}_{\rho,\rho}(-S^+ t^\rho)\right) t^{\rho-1}\varphi_1(x) \\ &+\frac{1}{2}\left(R^{-1}\tilde{E}_{\rho,\rho}(-S^- t^\rho) - R^{-1}\tilde{E}_{\rho,\rho}(-S^+ t^\rho)\right) t^{\rho-1}\varphi_0(x) \\ &+t^{\rho-1}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}(x) + \alpha t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_{1k_0}v_{k_0}(x) \\ &+t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_{0k_0}v_{k_0}(x) + \int_0^t (t-\tau)^{2\rho-1}E^2_{\rho,2\rho}\left(-\alpha(t-\tau)^\rho\right) f_{k_0}(\tau)v_{k_0}(x)d\tau \\ &+\frac{1}{2}\left(\int_0^t (t-\tau)^{\rho-1}\left(R^{-1}\tilde{E}_{\rho,\rho}\left(-S^-(t-\tau)^\rho\right) - R^{-1}\tilde{E}_{\rho,\rho}\left(-S^-(t-\tau)^\rho\right)\right) f(x,\tau)d\tau\right), \end{aligned} \tag{34}$$

where we denote by

$$\tilde{E}_{\rho,\mu}(-St^\rho)f(x,t) = \sum_{k\neq k_0} E_{\rho,\mu}(-(\alpha \pm \sqrt{\alpha^2-\lambda_k})t^\rho)f_k(t)v_k(x).$$

Let us prove the convergence of series (34). Using Lemma 1, (2), (17), (23), (31):

$$\begin{aligned} ||u(x,t)||||_{C(\overline{\Omega})} \leqslant &C||\varphi_1||_{C^a[0,\pi]} + C||\varphi_0||_{C^a[0,\pi]} + C_\rho|\varphi_{1k_0}| \\ &+C_\rho|\varphi_{0k_0}| + C_\rho||t^{1-\rho}f_{k_0}(t)||_{C[0,T]} + C_\rho||t^{1-\rho}f(x,t)||_{C^a_x(\overline{\Omega})}. \end{aligned}$$

Thus, if $\varphi_0(x), \varphi_1(x) \in C^a[0,\pi]$ and $t^{1-\rho}f(x,t) \in C^a_x(\overline{\Omega})$ we obtain $u(x,t) \in C(\overline{\Omega})$.

We show that $u(x,t)$ is a solution of the problem (1) in the above cases according to Definition 1.

We estimate $||t^{1-\rho}u(x,t)||_{C(\overline{\Omega})}$ using (17), (23), Lemma 1, (2), (26), (27):

$$\begin{aligned} ||t^{1-\rho}u(x,t)||_{C(\overline{\Omega})} \leqslant &C||\varphi_1||_{C^a[0,\pi]} + C||\varphi_0||_{C^a[0,\pi]} + C_\rho|\varphi_{1k_0}| \\ &+C_\rho|\varphi_{0k_0}| + C_\rho||t^{1-\rho}f_{k_0}(t)||_{C[0,T]} + C_\rho||t^{1-\rho}f(x,t)||_{C^a_x(\overline{\Omega})}. \end{aligned}$$

Hence, it is sufficient to have $\varphi_0(x), \varphi_1(x) \in C^a[0,\pi]$ and $t^{1-\rho}f(x,t) \in C^a_x(\overline{\Omega})$ for having $t^{1-\rho}u(x,t) \in C(\overline{\Omega})$.

Next we prove that (16) series converges after applying operator $\frac{\partial^2}{\partial x^2}$ and the derivatives $(\partial_t^\rho)^2$, ∂_t^ρ.

If $S_j(x,t)$ is a partial sum of (34), then we may write

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}S_j(x,t)\\
&=\frac{1}{2}\sum_{\substack{k=1\\k\neq k_0}}^{j}\Big[t^{\rho-1}E_{\rho,\rho}\left(-\alpha_{1k}t^\rho\right)\varphi_{1k}+t^{\rho-1}E_{\rho,\rho}\left(-\alpha_{2k}t^\rho\right)\varphi_{1k}\\
&\quad+\frac{\alpha}{\alpha_{2k}-\alpha_{1k}}t^{\rho-1}E_{\rho,\rho}\left(-\alpha_{1k}t^\rho\right)\varphi_{1k}-\frac{\alpha}{\alpha_{2k}-\alpha_{1k}}t^{\rho-1}E_{\rho,\rho}\left(-\alpha_{2k}t^\rho\right)\varphi_{1k}\\
&\quad+\frac{2}{\alpha_{2k}-\alpha_{1k}}t^{\rho-1}E_{\rho,\rho}\left(-\alpha_{1k}t^\rho\right)\varphi_{0k}-\frac{2}{\alpha_{2k}-\alpha_{1k}}t^{\rho-1}E_{\rho,\rho}\left(-\alpha_{2k}t^\rho\right)\varphi_{0k}\\
&\quad+\frac{2}{\alpha_{2k}-\alpha_{1k}}\int_0^t(t-\tau)^{\rho-1}\Big(E_{\rho,\rho}\left(-\alpha_{1k}(t-\tau)^\rho\right)-E_{\rho,\rho}\left(-\alpha_{2k}(t-\tau)^\rho\right)\Big)f_k(\tau)d\tau\\
&\quad+t^{\rho-1}E_{\rho,\rho}(-\alpha t^\rho)\varphi_{1k_0}\lambda_{k_0}v_{k_0}(x)+\alpha t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_{1k_0}\lambda_{k_0}v_{k_0}(x)\\
&\quad+t^{2\rho-1}E^2_{\rho,2\rho}(-\alpha t^\rho)\varphi_{0k_0}\lambda_{k_0}v_{k_0}(x)\\
&\quad+\int_0^t(t-\tau)^{2\rho-1}E^2_{\rho,2\rho}\left(-\alpha(t-\tau)^\rho\right)f_{k_0}(\tau)\lambda_{k_0}v_{k_0}(x)d\tau.
\end{aligned}
$$

Using estimates (20), (25) and (2), Lemma 1, (29) consequently for above given expression we get

$$
\begin{aligned}
\|\frac{\partial^2}{\partial x^2}S_j(x,t)\|_{C(\Omega)}\leqslant\ & t^{\rho-1}(C+\alpha t^{-\rho}C)||\varphi_{1x}||_{C^a[0,\pi]}+Ct^{-1}||\varphi_{0x}||_{C^a[0,\pi]}\\
&+Ct^{\rho-1}\alpha^2|\varphi_{1k_0}|+t^{2\rho-1}C_\rho|\varphi_{1k_0}|+t^{2\rho-1}C_\rho|\varphi_{0k_0}|\\
&+C\|t^{1-\rho}f(x,t)\|_{C^a_x(\overline{\Omega})},\quad t>0.
\end{aligned}
$$

If $\varphi_0(x),\varphi_1(x)\in C_1^a[0,\pi]$ and $t^{1-\rho}f(x,t)\in C_x^a(\overline{\Omega})$, then we have $\frac{\partial}{\partial x^2}u(x,t)\in C(\Omega)$.

Let us now estimate $\partial_t^\rho u(x,t)$. If $S_j(x,t)$ is a partial sum of (34), then by Lemmas 2–3 we see that

$$
\begin{aligned}
\partial_t^\rho S_j(x,t)=\frac{1}{2}\sum_{\substack{k=1\\k\neq k_0}}^{j}\Big[&t^{\rho-1}\varphi_{1k}\Big(-\alpha_{1k}E_{\rho,\rho}\left(-\alpha_{1k}t^\rho\right)-\alpha_{2k}E_{\rho,\rho}\left(-\alpha_{2k}t^\rho\right)\Big)\\
&-\frac{2\alpha\varphi_{1k}t^{\rho-1}}{\alpha_{2k}-\alpha_{1k}}\Big(\alpha_{1k}E_{\rho,\rho}\left(-\alpha_{1k}t^\rho\right)-\alpha_{2k}E_{\rho,\rho}\left(-\alpha_{2k}t^\rho\right)\Big)\\
&-\frac{2\varphi_{0k}t^{\rho-1}}{\alpha_{2k}-\alpha_{1k}}\Big(\alpha_{1k}E_{\rho,\rho}\left(-\alpha_{1k}t^\rho\right)+\alpha_{2k}E_{\rho,\rho}\left(-\alpha_{2k}t^\rho\right)\Big)\\
&-\frac{2}{\alpha_{2k}-\alpha_{1k}}\int_0^t(t-\tau)^{\rho-1}\Big(\alpha_{1k}E_{\rho,\rho}\left(-\alpha_{1k}(t-\tau)^\rho\right)
\end{aligned}
$$

$$-\alpha_{2k}E_{\rho,\rho}\left(-\alpha_{2k}(t-\tau)^{\rho}\right)\Big)f_k(\tau)d\tau\Big]v_k(x)$$
$$-t^{\rho-1}\alpha E_{\rho,\rho}(-\alpha t^{\rho})\varphi_{1k_0}v_{k_0}(x)+\alpha t^{\rho-1}E^2_{\rho,\rho}(-\alpha t^{\rho})\varphi_{1k_0}v_{k_0}(x)$$
$$+t^{\rho-1}E^2_{\rho,\rho}(-\alpha t^{\rho})\varphi_{0k_0}v_{k_0}(x)$$
$$+\int_0^t(t-\tau)^{\rho-1}E^2_{\rho,\rho}\left(-\alpha(t-\tau)^{\rho}\right)f_{k_0}(\tau)v_{k_0}(x)d\tau.$$

Applying the estimates (18), (24), Lemma 1, (2) and (28), (30) for corresponding terms of the above expression we have

$$\begin{aligned}\|\partial_t^{\rho}S_j(x,t)\|_{C(\Omega)} \leqslant{}& (t^{-1}C+\alpha t^{\rho-1}C)||\varphi_1||_{C^a[0,\pi]}+Ct^{\rho-1}||\varphi_0||_{C^a[0,\pi]}\\ &+Ct^{\rho-1}\alpha|\varphi_{1k_0}|+\frac{2t^{\rho-1}C\alpha}{\rho}|\varphi_{1k_0}|+\frac{2t^{\rho-1}C}{\rho}|\varphi_{0k_0}|\\ &+t^{3\rho-1}C_{\rho}||t^{1-\rho}f_{k_0}||_{C[0,T]}+C\|t^{1-\rho}f(x,t)\|_{C_x^a(\overline{\Omega})},\quad t>0.\end{aligned}$$

Therefore, if $\varphi_0(x),\varphi_1(x)\in C^a[0,\pi]$ and $t^{1-\rho}f(x,t)\in C_x^a(\overline{\Omega})$, then we have $\partial_t^{\rho}u(x,t)\in C(\Omega)$. Further, equation (1) implies $(\partial_t^{\rho})^2u(x,t)=-2\alpha\partial_t^{\rho}u(x,t)+u_{xx}(x,t)+f(x,t)$. Therefore, we find that $(\partial_t^{\rho})^2u(x,t)\in C(\Omega)$.

Using estimates (19) and similar ideas as in the proof of the above estimate we have

$$\begin{aligned}\|t^{1-\rho}\partial_t^{\rho}S_j(x,t)\|_{C(\overline{\Omega})} \leqslant{}& C||\varphi_{1x}||_{C^a[0,\pi]}+C||\varphi_1||_{C^a[0,\pi]}+C||\varphi_0||_{C^a[0,\pi]}\\ &+C\alpha|\varphi_{1k_0}|+\frac{2C\alpha}{\rho}|\varphi_{1k_0}|+\frac{2C}{\rho}|\varphi_{0k_0}|+C_{\rho}||t^{1-\rho}f_{k_0}||_{C[0,T]}\\ &+C\|t^{1-\rho}f(x,t)\|_{C_x^a(\overline{\Omega})}.\end{aligned}$$

Finally, if $\varphi_0(x),\varphi_1(x)\in C^a[0,\pi]$ and $t^{1-\rho}f(x,t)\in C_x^a(\overline{\Omega})$ then $t^{1-\rho}\partial_t^{\rho}u(x,t)\in C(\overline{\Omega})$.

REMARK 1. Note that we are treating all the terms $k\neq k_0$ similarly and the term corresponding to k_0 will be treated independently. Therefore, the case where there is no k_0 is can be shown with similar reasoning.

Let us prove the uniqueness of the solution. We use a standard technique based on the completeness of the set of eigenfunctions $\{v_k(x)\}$ in $L_2(0,\pi)$.

Let $u(x,t)$ be a solution to the problem

$$\begin{cases}(\partial_t^{\rho})^2u(x,t)+2\alpha\partial_t^{\rho}u(x,t)-u_{xx}(x,t)=0, & (x,t)\in\Omega,\\ u(0,t)=u(\pi,t)=0, & 0\leqslant t\leqslant T,\\ \lim\limits_{t\to0}I_t^{1-\rho}(\partial_t^{\rho}u(x,t))=0, & 0\leqslant x\leqslant\pi,\\ \lim\limits_{t\to0}I_t^{1-\rho}u(x,t)=0, & 0\leqslant x\leqslant\pi.\end{cases}$$

Consider the function

$$T_k(t)=\int_0^{\pi}u(x,t)v_k(x)dx.$$

By definition of the solution we may write

$$
\begin{aligned}
(\partial_t^\rho)^2 T_k(t) &= \int_0^\pi (\partial_t^\rho)^2 u(x,t) v_k(x) dx = \int_0^\pi \left(-2\alpha \partial_t^\rho u(x,t) + u_{xx}(x,t)\right) v_k(x) dx \\
&= -2\alpha \int_0^\pi \partial_t^\rho u(x,t) v_k(x) dx + \int_0^\pi u(x,t) v_{kxx}(x) dx \\
&= -2\alpha \partial_t^\rho T_k(t) - \lambda_k T_k(t).
\end{aligned}
$$

Hence, we have the following problem for $T_k(t)$:

$$
\begin{cases}
(\partial_t^\rho)^2 T_k(t) + 2\alpha \partial_t^\rho T_k(t) + \lambda_k T_k(t) = 0, \quad 0 < t \leqslant T, \\
\lim\limits_{t \to 0} I_t^{1-\rho} (\partial_t^\rho T_k(t)) = 0, \\
\lim\limits_{t \to 0} I_t^{1-\rho} T_k(t) = 0.
\end{cases}
$$

Lemma 4 implies that $T_k(t) \equiv 0$ for all k . Consequently, due to the completeness of the system of eigenfunctions $v_k(x)$, we have $u(x,t) \equiv 0$, as required.

Acknowledgement. The author is grateful to A. O. Ashyralyev for posing the problem and I convey thanks to R. R. Ashurov for discussions of these results. The author acknowledges financial support from the Innovative Development Agency under the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan, Grant No F-FA-2021-424.

We would also like to express special thanks to the anonymous reviewer of the journal for his/her comments and suggestions that led to the improvement of the original manuscript.

REFERENCES

[1] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Math., Birkhäuser, Germany, 2001.

[2] R. Ashurov, R. Saparbayev, *Fractional telegraph equation with the Caputo derivative*, Fractal Fract. **7** (6), (2023), 1–17.

[3] R. Ashurov, R. Saparbayev, *Time-dependent identification problem for a fractional telegraph equation with the Caputo derivative*, Fract. Calc. Appl. Anal. **27**, (2024), 652–676.

[4] A. Ashyralyev, A. Al-Hammouri, Ch. Ashyralyev, *On the absolute stable difference scheme for the space-wise dependent source identification problem for elliptic-telegraph equation*, Numerical Methods for Partial Differential Equations **37** (2), (2021), 962–986.

[5] A. Ashyralyev, H. Al-Hazaimeh, *Numerical solution of the time-dependent source of identification problem for the telegraph equation with nonlocal condition*, AIP Conference Proceedings **2781**, 020003, (2023).

[6] J. Banasiak, J. R. Mika, *Singularly perturbed telegraph equations with applications in the random walk theory*, J. Appl. Math. Stoch. Anal. **11** (1), (1998), 9–28.

[7] R. F. Camargo, A. O. Chiacchio, E. C. Oliveira, *Differentiation to fractional orders and the fractional telegraph equation*, J. Math. Phys. **49**, 033505, (2008).

[8] R. Cascaval, E. Eckstein, C. Frota, A. Goldstein, *Fractional telegraph equations*, J. Math. Anal. Appl. **276**, (2002), 145–159.

[9] V. B. L. Chaurasia, R. Sh. Dubey, *Analytical solution for the generalized time-fractional telegraph equation*, Fractional Differential Calculus **3** (1), (2013), 21–29.

[10] M. M. DJRBASHIAN, *Integral Transforms and Representation of Functions in the Complex Domain* (in Russian), Nauka, Moscow, 1966.
[11] M. D'OVIDIO, P. LORETI, A. MOMEMZADEH, S. SARV AHRABI, *Determination of order in linear fractional differential equations*, Fract. Calc. Appl. Anal. **21**, (2018), 937–948.
[12] M. D'OVIDIO, F. POLITO, *Fractional diffusion-telegraph equations and their associated stochastic solutions*, Theory of Probability and its Applications **62**, (2017), 692–718.
[13] M. D'OVIDIO, B. TOALDO, E. ORSINGHER, *Time changed processes governed by space-time fractional telegraph equations*, Stochastic Analysis and Applications **32** (6), (2014), 1009–1045.
[14] N. ENGHETA, *On fractional calculus and fractional multipoles in electromagnetism*, IEEE Trans. Antennas Propag. **44**, (1996), 554–566.
[15] R. GORENFLO, A. A. KILBAS, F. MAINARDI, S. V. ROGOZIN, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, 2020.
[16] P. M. JORDAN, A. PURI, *Digital signal propagation in dispersive media*, J. Appl. Phys. **85** (3), (1999), 1273–1282.
[17] A. A. KILBAS, M. SAIGO, R. K. SAXENA, *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transforms and Special Functions **15** (1), (2004), 31–49.
[18] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
[19] T. KLEINER, R. HILFER, *Sequential generalized Riemann–Liouville derivatives based on distributional convolution*, Fract. Calc. Appl. Anal. **25**, (2022), 267–298.
[20] H. M. LIEBERSTEIN, *Theory of Partial Differential Equations*, Academic Press, 1972.
[21] H. LOPUSHANSKA, V. RAPITA, *Inverse coefficient problem for the semi-linear fractional telegraph equation*, Electronic Journal of Differential Equations **2015** (153), (2015), 1–13.
[22] F. MAINARDI, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, 2010.
[23] M. O. MAMCHUEV, *Modified Cauchy problem for a loaded second-order parabolic equation with constant coefficients*, Differential Equations **51** (9), (2015), 1137–1144.
[24] M. O. MAMCHUEV, *Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients*, Differential Equations **52** (6), (2016), 789–797.
[25] M. O. MAMCHUEV, *Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method*, Fract. Calc. Appl. Anal. **20**, (2017), 190–211.
[26] L. MESCIA, P. BIA, D. CARATELLI, *FDTD-based electromagnetic modeling of dielectric materials with fractional dispersive response*, Electronics **11** (10), (2022), 1588.
[27] K. S. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[28] E. ORSINGHER, L. BEGHIN, *Time-fractional telegraph equation and telegraph processes with Brownian time*, Probab. Theory Relat. Fields **128**, (2004), 141–160.
[29] T. R. PRABHAKAR, *A singular integral equation with a generalized Mittag–Leffler function in the kernel*, Yokohama Math. J. **19** (1), (1971), 7–15.
[30] M. REED, B. SIMON, *Methods of Modern Mathematical Physics*, Academic Press, 1980.
[31] R. K. SAXENA, R. GARRA, E. ORSINGHER, *Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives*, arXiv:1506.06645v1 [math.PR], 19 Jun 2015.
[32] A. ZYGMUND, *Trigonometric Series*, Vol. 1, Cambridge University Press, 1959.

(Received April 8, 2024)

Rajapboy Saparbayev
V. I. Romanovskiy Institute of Mathematics
Uzbekistan Academy of Sciences
University street 9, Tashkent 100174, Uzbekistan
and
Tashkent State University of Economics (TSUE)
Karimov street 49, Tashkent 100066, Uzbekistan
e-mail: rajapboy1202@gmail.com
r.saparbayev@tsue.uz