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GLOBAL SOLUTIONS OF ANOMALOUS DIFFUSION SYSTEMS 3×3

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Abstract. In this article, we establish a global existence result for a nonlinear reaction-diffusion system in the case of 3 components with fractional Laplacians. Our proof method is based on the well-known regularizing effect.

1. Introduction

In this article, we consider the fractional reaction system

$$\begin{cases} u_t + d_1(-\Delta)^{\alpha} u = f_1(u,v,w), & t>0,\ x\in\Omega, \\ v_t + d_2(-\Delta)^{\beta} v = f_2(u,v,w), & t>0,\ x\in\Omega, \\ w_t + d_3(-\Delta)^{\gamma} w = f_3(u,v,w), & t>0,\ x\in\Omega, \end{cases} \tag{1}$$

subject to the boundary and initial conditions

$$u(t,x) = v(t,x) = w(t,x) = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{2}$$

$$u(0,x) = u_0(x) \geqslant 0,\ \ v(0,x) = v_0(x) \geqslant 0,\ \ w(0,x) = w_0(x) \geqslant 0, \qquad x\in\Omega, \tag{3}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, initials values $u_0(x)$, $v_0(x)$, $w_0(x)$ are given nonnegative bounded functions, and the constants d_1, d_2, and d_3 are positive.

Here, the functions u, v, and w represent densities of susceptible, infected, and removed individuals; concentrations of some chemical species; electrical charges, ...; the anomalous diffusion is explained by the nonlocal operators $(-\Delta)^s$ $(0<s<1$, $s=\alpha,\beta,\gamma)$ ([12, 13]), this means that the sub-populations or concentrations face some obstacles that slow their movement.

The reaction terms $f_1(u,v,w)$, $f_2(u,v,w)$, $f_3(u,v,w)$ are locally Lipschitzian satisfy the so-called "quasi-postivity" property, namely:

$$f_1(0,v,w)\geqslant 0, f_2(u,0,w)\geqslant 0 \quad \text{and} \quad f_3(u,v,0)\geqslant 0, \qquad \forall u,v,w\geqslant 0, \tag{4}$$

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which implies, that the solution is positive on its interval of existence via the maximum principle (see Smoller [16]). Moreover, we assume that the functions f_1, f_2, f_3 are of polynomial growth , *i.e* for all $r_1, r_2, r_3 \in [0, \infty[$ and a real $m > 1$, we have

$$|f_1(r_1,r_2,r_3)|, |f_2(r_1,r_2,r_3)|, |f_3(r_1,r_2,r_3)| \leqslant C_1(r_1,r_2,r_3)(1+\sum_{i=1}^{3} r_i)^m \text{ on } (0,\infty)^3, \tag{5}$$

and satisfy

$$Af_1(u,v,w) + Bf_2(u,v,w) + f_3(u,v,w) \leqslant C_2(r_1,r_2,r_3)(u+v+w+1), \tag{6}$$

$$\left(\frac{1}{A}(Bf_2+f_3)\right) \geqslant 0, \quad \left(\frac{1}{B}(Af_1+f_3)\right) \geqslant 0, \quad (Af_1+Bf_2) \geqslant 0, \tag{7}$$

where C_1, C_2 are positive and uniformly bounded functions defined on $(0,\infty)^3$ and A, B are positive constants.

The above nonlinearities can be found in the model of a classical irreversible autocatalytic reaction involving chemical species U, V and W :

$$lU + qV \leftrightarrows rW;$$

in this case, if $u = [U]$, $v = [V]$, $w = [W]$, then

$$f_1(u,v,w) = f_2(u,v,w) = -hu^l v^q + kw^r \text{ and } f_3(u,v,w) = hu^l v^q - kw^r,$$

where $[\cdot]$ means the concentration of chemical species.

The reader is referred to ([1, 2, 3, 7, 11, 14, 15, 17]) for results on global existence, asymptotic behaviour and blow-up in classical reaction system (i.e. $\alpha = \beta = \gamma = 1$) or fractional reaction system (i.e. $0 < \alpha, \beta, \gamma < 1$). Daoud et *al.* ([7], Theorem 4.1) studied problem (1)–(2)–(3) under assumptions (4), (6), $\alpha = \beta = \gamma = s$, and

$$f_1 = \alpha_1 g, f_2 = \alpha_2 g, f_3 = -\alpha_3 g, \tag{8}$$

with

$$g(u_1,u_2,u_3) = u_3^{\alpha_3} - u_1^{\alpha_1} u_2^{\alpha_2}, \tag{9}$$

and derived conditions on the data α_1, α_2 and α_3, which imply the global existence of solutions. Here we obtain the global existence of solution for (1)–(2)–(3) in general source terms f_1, f_2, f_3 and the fractions α, β, γ are different from each other.

2. Preliminaries

Let us recall a few preliminaries about the nonlocal operators $(-\Delta)^s$. The nonlocal operators $(-\Delta)^s$ $(0 < s < 1,\ s = \alpha \text{ or } \beta \text{ or } \gamma)$ stands for anomalous diffusion and is defined by its Riesz representation

$$(-\Delta)^s u(x) = C_N P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \tag{10}$$

where P.V. stands for the Cauchy principal value and C_N is a normalizing constant. If we consider the case $s\in(\frac{1}{2},1)$ fractional Laplacian has close properties to classical Laplacian, or

$$(-\Delta)^s=-\Delta \text{ as } s\to 1^- \quad \text{and} \quad (-\Delta)^s=Id \text{ as } s\to 0^+, \tag{11}$$

where Id is an identity operator. Readers unfamiliar with fractional laplacians are referred to ([6, 8, 9]) and the associated references.

The following important Stroock and Varopoulos inequalities will be utilized (see, for instance, ([4], Formula (B7)), Theorem 1)

$$\int_\Omega u(x)(-\Delta)^s u(x)dx \geqslant 0, \tag{12}$$

$$\int_\Omega u^{p-1}(-\Delta)^s u dx \geqslant \frac{4(p-1)}{p^2}\int_\Omega \left|(-\Delta)^{\frac{s}{2}}u^{\frac{p}{2}}\right|^2 dx \geqslant 0, \quad p>1. \tag{13}$$

NOTATION 1. *For $p\in(1,\infty)$, we denote by A_1,A_2,A_3 the realization of $(-\Delta)^\alpha$, $(-\Delta)^\beta$, $(-\Delta)^\gamma$ respectively with homogeneous Dirichlet boundary condition in $L^p(\Omega)$.*

It is well known that $-A_1,-A_2,-A_3$ are a sectorial operator (see [10]); so that $-A_i$ generates an analytic semigroups $S_{A_i}(t)=\left\{e^{-tA_i}\right\}_{t\geqslant 0}$, $i=1,2,3$.

The lemma on local existence is classical. Let us recall its statement here. Since the proof is simple, we omit it.

LEMMA 1. *Let $(u_0,v_0,w_0)\in(L^\infty(\Omega))^3$. Assume that the f_i's $(i=1,2,3)$ are locally Lipschitz continuous. Then, problem (1)–(2)–(3) has a unique local solution (u,v,w) on $[0,T_{\max}[\times\Omega$, satisfying*

$$\begin{cases} u(t)=S_{A_1}(t)u_0+\int_0^t S_{A_1}(t-s)f_1(u(s),v(s),w(s))ds,\\ v(t)=S_{A_2}(t)v_0+\int_0^t S_{A_2}(t-s)f_2(u(s),v(s),w(s))ds,\\ w(t)=S_{A_3}(t)w_0+\int_0^t S_{A_3}(t-s)f_3(u(s),v(s),w(s))ds,\end{cases}$$

Moreover, the following alternatives hold

i) $T_{\max}=+\infty$ or,

ii) $T_{\max}<+\infty$ and $\lim_{t\to T_{\max}}(\|u(t,.)\|_\infty+\|v(t,.)\|_\infty+\|w(t,.)\|_\infty)=+\infty$.

DEFINITION 1. Let $u(t,.),v(t,.),w(t,.)$ be solutions of problem (1). We define the maximal existence time $T_{\max}$ of $u(t,.),v(t,.),w(t,.)$ as follows

(i) If $u(t,.),v(t,.),w(t,.)$ exist for $0\leqslant t<\infty$, then $T_{\max}=+\infty$,

(ii) If there exist a $t_0\in(0,\infty)$ such that $u(t,.),v(t,.),w(t,.)$ exist for $0\leqslant t<t_0$, but do not exist at $t=t_0$, then $T_{\max}=t_0$.

— If (i) are satisfied, we say that the solutions $u(t,.),v(t,.),w(t,.)$ are global.

To study the problem (1)–(2)–(3) and to show our main result, we need the following inequalities:

LEMMA 2. (Young's inequality [5]) *Let* $p,q \in]1,\infty[$, $s \geqslant 1$ *such that* $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}$. *Then, for all* $a,b \geqslant 0$*; we have*

$$\frac{(ab)^s}{s} \leqslant \frac{a^p}{p}+\frac{b^q}{q}.$$

LEMMA 3. (Hölder's inequality [5]) *Let* $p,q \in]1,\infty[$*; such that* $\frac{1}{p}+\frac{1}{q}=1$. *If* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$, *then* $fg \in L^1(\Omega)$, *with*

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q .$$

3. Global existence

According to the regularizing effect method (see Henry [10], pp. 35-62), in order to prove global existence of solution to (1)–(2)–(3), it is sufficient to derive a uniform estimate of $\|f_1(u,v,w)\|_P, \|f_2(u,v,w)\|_P$ and $\|f_3(u,v,w)\|_P$ on $[0,T_{\max}[$ in the space $L^P(\Omega)$ for some $P>\frac{n}{2}, (n=\dim\Omega)$.

We first start with the following Theorem, which we will use in the proof of Corollary 1. This Theorem will play an essential role in proving our main result.

THEOREM 1. *Suppose that the conditions on* $f_1(u,v,w)$, $f_2(u,v,w)$, *and* $f_3(u,v,w)$, *given in section 1, hold. Then all solutions of* (1)–(2)–(3) *with positive initial data in* $L^\infty(\Omega)$ *can be estimated in the form*

$$\left\|u^{p+1}+v^{p+1}+w^{p+1}\right\|_{L^1(\Omega)} \leqslant KL(0)\exp\left(p+1\right)\xi t,\ \ \forall t\in[0,T^*], T^*\leqslant T_{\max}, \quad (14)$$

where ξ *and* K *are positive constants,* $p \geqslant 1$ *is a positive integer and the constant* $L(0)=\left\|u_0^{p+1}+v_0^{p+1}+w_0^{p+1}\right\|_{L^1(\Omega)}$.

Proof. Multiplying the first differential equations in (1) by $(u(t,x))^p$, the second one by $(v(t,x))^p$, the third one by $(w(t,x))^p$, integrating the three equations over Ω, adding the three results,

$$\int_\Omega (u^p u_t + v^p v_t + w^p w_t)\,dx + d_1\int_\Omega u^p(-\Delta)^\alpha u dx + d_2\int_\Omega \left(v^p(-\Delta)^\beta v\right)dx$$

$$+d_3\int_\Omega (w^p(-\Delta)^\gamma w)\,dx = \int_\Omega (u^p f_1(u,v,w)+v^p f_2(u,v,w)+w^p f_3(u,v,w))\,dx. \quad (15)$$

By using the so-called Stroock and Varopoulos inequality (13), we have

$$\begin{cases} \int_\Omega u^p(-\Delta)^\alpha u dx \geqslant \frac{4p}{(p+1)^2}\int_\Omega \left|(-\Delta)^{\frac{\alpha}{2}} u^{\frac{p+1}{2}}\right|^2 dx \geqslant 0, \\ \int_\Omega v^p(-\Delta)^\beta v dx \geqslant \frac{4p}{(p+1)^2}\int_\Omega \left|(-\Delta)^{\frac{\beta}{2}} v^{\frac{p+1}{2}}\right|^2 dx \geqslant 0, \\ \int_\Omega w^p(-\Delta)^\gamma w dx \geqslant \frac{4p}{(p+1)^2}\int_\Omega \left|(-\Delta)^{\frac{\gamma}{2}} w^{\frac{p+1}{2}}\right|^2 dx \geqslant 0. \end{cases} \tag{16}$$

By (15) and (16), we see

$$\int_\Omega (u^p u_t + v^p v_t + w^p w_t)\, dx \leqslant \int_\Omega (u^p f_1(u,v,w) + v^p f_2(u,v,w) + w^p f_3(u,v,w))\, dx. \tag{17}$$

This implies that

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx \leqslant I, \tag{18}$$

where

$$I = \int_\Omega (u^p f_1(u,v,w) + v^p f_2(u,v,w) + w^p f_3(u,v,w))\, dx. \tag{19}$$

We can write formula (19) as follows

$$I = \int_\Omega \left(\frac{1}{A}u^p + \frac{1}{B}v^p + w^p\right)(Af_1 + Bf_2 + f_3)\, dx - \int_\Omega u^p\left(\frac{1}{A}(Bf_2+f_3)\right) dx \tag{20}$$
$$-\int_\Omega v^p\left(\frac{1}{B}(Af_1+f_3)\right) dx - \int_\Omega w^p (Af_1+Bf_2)\, dx.$$

From condition (6) on f_1, f_2 and f_3, it follows that

$$\int_\Omega \left(\frac{1}{A}u^p + \frac{1}{B}v^p + w^p\right)(Af_1 + Bf_2 + f_3)\, dx \leqslant C_3 \int_\Omega (u^p+v^p+w^p)(1+u+v+w)\, dx, \tag{21}$$

where $C_3 = \max(\frac{1}{A}, \frac{1}{B}, 1, \sup C_2(u,v)) > 0$.

By inserting (21) into (20), we obtain

$$I \leqslant C_3 \int_\Omega \left(\left(u^{p+1}+v^{p+1}+w^{p+1}\right) + (u^p+v^p+w^p)\right) dx$$
$$+\int_\Omega u^p\left(C_3(v+w) - \left(\frac{1}{A}(Bf_2+f_3)\right)\right) dx$$
$$+\int_\Omega v^p\left(C_3(u+w) - \frac{1}{B}(Af_1+f_3)\right) dx + \int_\Omega w^p (C_3(u+v) - (Af_1+Bf_2))\, dx.$$

Taking into account condition (7) on f_1, f_2 and f_3, we find

$$I \leqslant C_3 \int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx + C_3\int_\Omega u^p(1+v+w)\, dx$$
$$+C_3\int_\Omega v^p(1+u+w)\, dx + C_3\int_\Omega w^p(1+u+v)\, dx. \tag{22}$$

Since $v+w,\ u+w,\ u+v<u+v+w,$ one easily sees that

$$I \leqslant C_3 \int_\Omega \left[\left(u^{p+1}+v^{p+1}+w^{p+1}\right)+\left(u^p+v^p+w^p\right)(1+u+v+w)\right] dx$$

$$= C_3 \int_\Omega \left[\left(u^{p+1}+v^{p+1}+w^{p+1}\right)+R_{p+1}(u,v,w)+\left(u^p+v^p+w^p\right)\right] dx, \quad (23)$$

where

$$R_{p+1}(u,v,w) = u^{p+1}+v^{p+1}+w^{p+1}+u^p(v+w)+v^p(u+w)+w^p(u+v)$$

is a homogeneous polynomial of degrees $p+1$. First, using the fact that

$$(U+V)^{p+1} \leqslant 2^p(U^{p+1}+V^{p+1}), \text{ for all } U,V \geqslant 0, \text{ and } p>0,$$

we get

$$\int_\Omega R_{p+1}(u,v,w)dx \leqslant \int_\Omega (u+v+w)^{p+1}\, dx \leqslant C_4 \int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx, \quad (24)$$

where $C_4 = 2^{2p}$, then applying Hölder's inequality to the last term of (23), one gets

$$\int_\Omega \left(u^p+v^p+w^p\right) dx \leqslant (meas\Omega)^{\frac{1}{p+1}} \left[\left(\int_\Omega (u^p)^{\frac{p+1}{p}}\, dx\right)^{\frac{p}{p+1}} + \left(\int_\Omega (v^p)^{\frac{p+1}{p}}\, dx\right)^{\frac{p}{p+1}} + \left(\int_\Omega (w^p)^{\frac{p+1}{p}}\, dx\right)^{\frac{p}{p+1}}\right]. \quad (25)$$

By inserting (24) and (25) into (23), estimate (23) becomes

$$I \leqslant C_5 \int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx$$

$$+C_6\left[\left(\int_\Omega u^{p+1}dx\right)^{\frac{p}{p+1}} + \left(\int_\Omega v^{p+1}dx\right)^{\frac{p}{p+1}} + \left(\int_\Omega w^{p+1}dx\right)^{\frac{p}{p+1}}\right].$$

Therefore, we arrive at

$$I \leqslant C_5 \int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx + C_7\left[\int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx\right]^{\frac{p}{p+1}}. \quad (26)$$

If we insert (26) into (18), we then obtain

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx \leqslant C_5 \int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx$$

$$+C_7\left[\int_\Omega \left(u^{p+1}+v^{p+1}+w^{p+1}\right) dx\right]^{\frac{p}{p+1}}. \quad (27)$$

Now, put $L(t)=\int_\Omega\left(u^{p+1}+v^{p+1}+w^{p+1}\right)dx$, one can obtain the differential inequality

$$L^{'}\leqslant C_5L+C_7L^{\frac{p}{p+1}}, \tag{28}$$

which for $Z=L^{\frac{1}{1+p}}$ can be written as

$$(1+p)Z^{'}\leqslant C_5Z+C_7. \tag{29}$$

Integrating (29) from 0 to t, we get

$$z(t)\leqslant C_6Z(0)\exp\xi t+C_8,$$

then

$$L(t)\leqslant KL(0)\exp(p+1)\xi t, \qquad t\in[0,T^*],$$

where K,ξ are positive constants and $L(0)=\left\|u_0^{p+1}+v_0^{p+1}+w_0^{p+1}\right\|_{L^1(\Omega)}$. This completes the proof. □

COROLLARY 1. *Assume that conditions* (4)*,* (6)*,* (7) *on* $f_1(u,v,w), f_2(u,v,w)$*, and* $f_3(u,v,w)$ *hold. Then all solutions of* (1)–(2)–(3) *with positive initial data in* $L^\infty(\Omega)$ *are in* $L^\infty(0,T^*;L^{p+1}(\Omega))$.

Proof. We want to show that $\sup_{t\in[0,T^*]}\|U\|_{p+1}<\infty$ for $U=u,v,w$. For this, we set $z=u+v$ and by using the fact that $(z+w)^{p+1}\leqslant 2^p(z^{p+1}+w^{p+1})$ for all $z,w\geqslant 0$ and $p>1$, we get

$$\begin{aligned}\int_\Omega (z+w)^{p+1}dx &\leqslant 2^p\int_\Omega\left(z^{p+1}+w^{p+1}\right)dx\\ &\leqslant 2^{2p}\int_\Omega\left(u^{p+1}+v^{p+1}+w^{p+1}\right)dx.\end{aligned}$$

Now, we use theorem 1, we get

$$\int_\Omega (u+v+w)^{p+1}dx\leqslant 2^{2p}L(t)\leqslant 2^{2p}KL(0)\exp p\xi t, \quad \text{on } [0,T^*].$$

Together with positivity, this implies a uniform $L^\infty(0,T^*;L^{p+1}(\Omega))$ bound on u,v,w. Hence, the proof is completed. □

Now, let us prove the global existence to (1)–(2)–(3).

PROPOSITION 1. *Suppose that the conditions on* f_1*,* f_2*, and* f_3*, given in section 1, hold. Furthermore assume that* $\frac{p+1}{m}>\frac{n}{2}$ $(n=\dim\Omega)$*, then all solutions of* (1)–(2)–(3) *with positive initial data in* $L^\infty(\Omega)$ *are global.*

Proof. Our goal is to derive a uniform estimate of $\|f_1(u,v,w)\|_P$, $\|f_2(u,v,w)\|_P$ and $\|f_3(u,v,w)\|_P$ on $[0,T_{\max}[$ in the space $L^P(\Omega)$ for some $P > \frac{n}{2}$ $(n = \dim\Omega)$ which leads to global existence (see Henry [10], pp. 35–62).

From Corollary 1, there exists a positive constant C_9 such that

$$\int_\Omega (1+u+v+w)^{p+1}\,dx \leqslant C_9, \qquad \text{on} \quad [0,T_{\max}[, \tag{30}$$

for all $p > 1$. Making use (5), we get

$$|f_1(u,v,w)|^{\frac{p+1}{m}},\ |f_2(u,v,w)|^{\frac{p+1}{m}},\ |f_3(u,v,w)|^{\frac{p+1}{m}} \leqslant C_{10}(1+u+v+w)^{p+1}. \tag{31}$$

It follows from (30) and (31) that $f_1, f_2, f_3 \in L^\infty(0,T_{\max}; L^P(\Omega))$ with $P = \frac{p+1}{m}$.

Consequently, the solution to the problem (1) is global for all $P \geqslant 1$. Hence, the proof is completed. □

REMARK 1. It is actually the case in the system (1)–(2)–(3), when the condition $f_1+f_2+f_3 \leqslant 0$ hold, if $d_1 = d_2 = d_3 = d$ and the fractions α, β and γ are equal $\alpha = \beta = \gamma = s$, then the solutions u, v, w are uniformly bounded on $[0,T)$. Thus, a priori L^∞-bounds imply global existence.

Indeed, in this case

$$\partial_t(u+v+w) + d(-\Delta)^s(u+v+w) \leqslant 0.$$

By maximum principle

$$\forall t \in [0,T), \|(u+v+w)\|_\infty \leqslant \|(u_0+v_0+w_0)\|_\infty.$$

Together with positivity, this implies a uniform $L^\infty(\Omega)$ bound on u, v, w, hence $T = \infty$.

4. Application

Many chemical reactions, when modelled through the mass action law, lead to reaction-diffusion. We consider the reversible reaction

$$lU + qV \leftrightarrows rW. \tag{32}$$

Then, according to the mass action law and with a Fickian diffusion, the evolution of the concentrations u, v, w of U, V, W respectivly is governed by the following reaction-diffusion system:

$$u_t + d_1(-\Delta)^\alpha u = -\lambda u^l v^q + \mu w^r, \qquad t > 0,\ x \in \Omega, \tag{33}$$

$$v_t + d_2(-\Delta)^\beta v = -\lambda u^l v^q + \mu w^r, \qquad t > 0,\ x \in \Omega, \tag{34}$$

$$w_t + d_3(-\Delta)^\gamma w = \lambda u^l v^q - \mu w^r, \qquad t > 0,\ x \in \Omega, \tag{35}$$

with boundary conditions (2), initial conditions (3), and λ, μ, l, q and r are positive constants, and $0 < \alpha, \beta, \gamma < 1$.

THEOREM 2. *Assume that* $(u_0, v_0, w_0) \in (L^\infty(\Omega)^+)^3$. *System* (33)–(34)–(35) *with boundary conditions* (2) *admits a non-negative global solution in the following cases:*

1. $r \leqslant 1$ *whatever* l *and* q,
2. $l+q \leqslant 1$ *whatever* r,
3. $d_1 = d_3$ *or* $d_2 = d_3$ *whatever are* l, q *and* r.

Proof. **1.** *The case* $r \leqslant 1$.

According to (Smoller [16]), the positivity of the solutions is preserved for all time if and only if $f = (f_1, f_2, f_3)$ is quasi-positive.

If we denote

$$f_1(u,v,w) = f_2(u,v,w) = -f_3(u,v,w) = -\lambda u^l v^q + \mu w^r,$$

then for all $u, v, w \geqslant 0$,

$$f_1(0,v,w) = \mu w^r \geqslant 0, \;\; f_2(u,0,w) = \mu w^r \geqslant 0, \;\; f_3(u,v,0) \geqslant \lambda u^l v^q \geqslant 0,$$

so u, v, w are positive.

In order to prove the global existence, it is sufficient to prove that (6)–(7) are satisfied. By choosing $A+B>1$, we can easily see that

$$Af_1(u,v,w) + Bf_2(u,v,w) + f_3(u,v,w) \leqslant \mu w^r < C(1+u+v+w).$$

Moreover, f_3 and f_1 (resp. f_2) satisfy (7). In fact, $\frac{1}{B}(Af_1+f_3) \geqslant 0$ and $\frac{1}{A}(Bf_2+f_3) \geqslant 0$ by choosing $A = B = 1$. Also Condition (7) is satisfied for f_1 and f_2 while choosing $\mu w^r \geqslant \lambda u^l v^q$.

So that (6)–(7) holds for the system (33)–(34)–(35) when $r<1$. Then corollary 1 implies that all components of the solution are in $L^\infty(0,T;L^{p+1}(\Omega))$ for all $p \geqslant 1$. Since the reaction terms are of polynomial growth, then $T_{\max} = +\infty$.

2. *The case* $l+q \leqslant 1$.

The conditions (6) is obviously satisfied in the case $l+q \leqslant 1$ for the system in the order (35)–(34)–(33) by choosing $A > B+1$ and applying the Young inequality to the term $u^l v^q$. Condition (7) is satisfied while choosing $\mu w^r \geqslant \lambda u^l v^q$. Then Corollary 1 implies that $u, v, w \in L^\infty(0,T;L^{p+1}(\Omega))$ for all $p \geqslant 1$, then proposition applied to (33)–(34)–(35) permits us to give the global existence.

The situation is quite more complicated if $l+q>1$ because the condition (6)–(7) becomes difficult to verify.

3. *The case* $d_1 = d_3 = d$ *or* $d_2 = d_3 = d$.

We set $Z = u+w$, then

$$\partial Z_t - d(-\Delta)^\alpha Z \leqslant 0.$$

The point is that thanks to the nonnegativity of u, v, we have

$$\|Z\|_\infty = \|u+w\|_\infty \leqslant \|u_0 + w_0\|_\infty,$$

this implies a uniform $L^\infty(\Omega)$ bound on u, w.

Since u, w are uniform bounded ($u \leqslant M_1$ and $w \leqslant M_2$), the conditions (6)–(7) are obviously satisfied for the system (33)–(34)–(35) by choosing $A < B+1$ whatever are l, q and r, which implies that $T = \infty$. □

- Another example is an SIR-type epidemiological model:

$$S_t + d_1(-\Delta)^\alpha S = -\lambda SI \qquad t > 0, x \in \Omega, \tag{36}$$

$$I_t + d_2(-\Delta)^\beta I = \lambda SI - \mu I \quad t > 0, x \in \Omega, \tag{37}$$

$$R_t + d_3(-\Delta)^\gamma R = \mu I \qquad t > 0, x \in \Omega. \tag{38}$$

The spread of epidemics within a restricted population is described by this system. Densities of susceptible and infected individuals are represented by the functions $S(t,x)$, $I(t,x)$ and $R(t,x)$. The infection rate and removal rate are denoted by the positive constants λ and μ, respectively (see [11]).

System (36)–(37)–(38) with boundary conditions (2) and positive initial data in $L^\infty(\Omega)$ admits a non negative global solution.

Indeed, the positivity of the solutions is preserved for all time because of

$$f_1(0,I,R), f_2(S,0,R), f_3(I,S,0) \geqslant 0, \quad \text{for all } S, I, R \geqslant 0.$$

By using maximum principle to equation (36), one easily sees that

$$\forall t \in [0,T), \|S(t)\|_\infty \leqslant \|S_0(t)\|_\infty = M,$$

this implies a uniform $L^\infty(\Omega)$ bound on S.

The condition (6) is obviously satisfied by choosing $B < A$ and $B < 1$. The condition (7) is satisfied for f_3 and f_1 by choosing $A \leqslant \frac{\mu}{M\lambda}$; for f_3 and f_2 by choosing $B < 1$; for f_1 and f_2 by choosing $(A-B)\lambda M \geqslant \mu B$, where $M = \|S_0(t)\|_\infty = \sup |S_0(t)|$. Then, from Corollary 1, $u, v, w \in L^\infty(0,T; L^{p+1}(\Omega))$ for all $p \geqslant 1$. Whence, by using proposition 1, $T = \infty$.

The proof of the global existence to the problem (36)–(37)–(38) in the case $d_1 = d_2 = d_3$ and $\alpha = \beta = \gamma$ is an immediate consequence of remark 1.

- Finally we illustrate our results with the system

$$\begin{cases} u_t + d_1(-\Delta)^{\alpha_1} u = -u^{\theta_1} v^{\theta_2} - u^{\theta_3} w^\rho + \mu_1 v + \mu_2 w & t > 0, \ x \in \Omega, \\ v_t + d_2(-\Delta)^{\alpha_2} v = -u^{\theta_1} v^{\theta_2} + u^{\theta_3} w^\rho & t > 0, \ x \in \Omega, \\ w_t + d_3(-\Delta)^{\alpha_3} w = u^{\theta_1} v^{\theta_2} + u^{\theta_3} w^\rho & t > 0, \ x \in \Omega, \end{cases} \tag{39}$$

where θ_1, θ_2, θ_3, ρ, μ_1 and μ_2 are positive constants.

If we denote

$$\begin{cases} f_1(u,v,w) = -u^{\theta_1} v^{\theta_2} - u^{\theta_3} w^\rho + \mu_1 v + \mu_2 w, \\ f_2(u,v,w) = -u^{\theta_1} v^{\theta_2} + u^{\theta_3} w^\rho, \\ f_3(u,v,w) = u^{\theta_1} v^{\theta_2} + u^{\theta_3} w^\rho, \end{cases}$$

then for all $u,v,w \geqslant 0$,

$$f_1(0,v,w) = \mu_1 v + \mu_2 w \geqslant 0,\ f_2(u,0,w) = u^{\theta_3} w^{\rho} \geqslant 0,\ f_3(u,v,0) \geqslant u^{\theta_1} v^{\theta_2} \geqslant 0.$$

Again, existence of a global solution to (39) follows from conditions (6) and (7). By choosing $A > 1$ and $B > 0$ we can easily see that

$$Af_1(u,v,w) + Bf_2(u,v,w) + f_3(u,v,w) \leqslant A(\mu_1 v + \mu_2 w) < C(1+u+v+w);$$

moreover, f_3 and f_1 (resp. f_2) satisfy (7). In fact, $\frac{1}{B}(Af_1 + f_3) \geqslant 0$ by choosing $A > 1$ and $\frac{1}{A}(Bf_2 + f_3) \geqslant 0$ by choosing $B = 1$. Also (7) is satisfied for f_1 and f_2 while choosing $(A+B)w^r < A(\mu_1 v + \mu_2 w)$.

So that (6)–(7) holds for the system (39). The result of the Corollary applied to this system is summarized in the following proposition

PROPOSITION 2. *Assume that $(u_0, v_0, w_0) \in \left(L^\infty(\Omega)^+\right)^3$. System (39) with boundary conditions (2) admits a non negative global solution for all positive constants θ_1, θ_2, θ_3, ρ, μ_1 and μ_2.*

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