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LYAPUNOV–TYPE INEQUALITIES FOR A RIEMANN–LIOUVILLE FRACTIONAL HYBRID BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we investigate a Riemann–Liouville fractional hybrid boundary value problem to delve into the complexities of fractional calculus. We introduce novel Lyapunov-type inequalities that are specifically adapted to the distinct features of the problem at hand. Our results enhance the theoretical framework and include a comprehensive example that highlights the practical significance and implications of our theoretical advancements.

1. Introduction

Fractional calculus (FC), an extension of classical calculus to non-integer orders of differentiation and integration, plays a pivotal role in the advancement of various scientific and engineering fields. Its significance lies in its ability to model complex phenomena that are not adequately described by traditional integer-order calculus. By incorporating fractional derivatives and integrals, FC provides a more sophisticated understanding of dynamic systems exhibiting anomalous diffusion, memory effects, and hereditary properties. This extended framework enables more accurate and flexible representations of real-world processes, ranging from control structures, chemistry, dynamic procedures, mechanics, viscoelastic materials, etc [3, 7, 14, 15, 16, 17, 19, 22, 28, 29, 30, 31, 32].

Lyapunov-type inequalities (LTIs) are crucial for analyzing fractional boundary value problems (FBVPs), providing essential insights into the existence, uniqueness, and stability of solutions. These inequalities serve as fundamental tools in evaluating the boundedness and growth behavior of solutions to fractional differential equations (FDEqs), thereby shedding light on their qualitative characteristics. In the context of FBVPs, LTIs are particularly valuable for deriving a priori estimates, which are vital for demonstrating the existence of solutions and ensuring their stability across different scenarios. By utilizing these inequalities, researchers can establish more precise bounds and enhance their understanding of the dynamics within fractional systems. This, in turn, improves the robustness and reliability of both theoretical findings and practical applications. Integral inequalities, notably Lyapunov-type, serve as essential in exploring the quantitative aspects of solutions to differential and integral equations.

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This highlights their critical importance in the overall study of DEqs [2, 24]. The famous Lyapunov theorem [21] is as follows:
If the BVP

$$\left.\begin{aligned} \mathtt{u}''(\mathtt{z})+\ell(\mathtt{z})\mathtt{u}(\mathtt{z})=0, \quad \mathtt{z}\in(\mathtt{z}_0,\mathtt{z}_1), \\ \mathtt{u}(\mathtt{z}_0)=0=\mathtt{u}(\mathtt{z}_1), \end{aligned}\right\} \tag{1}$$

has a nontrivial solution, where $\ell \in \mathtt{C}([\mathtt{z}_0,\mathtt{z}_1];\mathbb{R})$, then

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1} \big|\ell(\mathtt{y})\big| d\mathtt{y} > \frac{4}{\mathtt{z}_1-\mathtt{z}_0}. \tag{2}$$

The Lyapunov inequality (LTI) (2) is useful in a variety of DEq-related problems. It has been generalized in various forms due to its significance. Notably, investigators have discovered LTIs for various BVPs. In [9], Ferreira studied an LTI for the FBVP

$$\left.\begin{aligned} \mathfrak{D}^{\sigma}_{\mathtt{z}_0^+}\mathtt{u}(\mathtt{z})+\ell(\mathtt{z})\mathtt{u}(\mathtt{z})=0, \quad \mathtt{z}\in(\mathtt{z}_0,\mathtt{z}_1), \\ \mathtt{u}(\mathtt{z}_0)=0=\mathtt{u}(\mathtt{z}_1), \end{aligned}\right\} \tag{3}$$

has a nontrivial solution, then

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1} \big|\ell(\mathtt{y})\big| d\mathtt{y} > \Gamma(\sigma)\left(\frac{4}{\mathtt{z}_1-\mathtt{z}_0}\right)^{\sigma-1},$$

where ℓ is a real continuous function, $\sigma \in (1,2]$, $\mathfrak{D}^{\sigma}_{\mathtt{z}_0^+}$ is the Riemann-Liouville (RL) derivative. Further, the paper aims to describe an LTI for a Caputo FBVP in [10]. The author has discussed some fascinating applications for identifying the real zeros of specific Mittag-Leffler functions in both of his works.

In [18], the authors investigated the existence results for a fractional system subject to Sturm-Liouville type conditions. The approach used relies on the Guo-Krasnosel'skii and Wardowski fixed point theorems. In 2018, Ntouyas et al. [24] published survey results on LTIs for FDEqs with numerous different BCs. For more details, see BVPs for ODEqs [4, 5, 27], for FDEqs [12, 13, 33]. Nonlinear DEqs with quadratic perturbations produce hybrid DEqs, which are more significant and exist as exceptional situations of dynamical systems. One such type of Eqs includes the arbitrary real order derivative of an unknown function that has been hybridized with nonlinearity. There are some recent results in the literature on FBVPs for hybrid DEqs, see [1, 6, 11, 23]. In [34], the authors considered the following RL-type hybrid DEqs

$$\mathfrak{D}^{\sigma}\left[\frac{\mathtt{u}(\mathtt{z})}{\Psi\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big)}\right] = \mathtt{f}\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big), \quad a.e. \quad \mathtt{z}\in(0,\mathtt{T}),$$

where $0<\sigma<1$. They used the fixed point theorem in a Banach algebra to obtain some existence results. For nonlinear fractional hybrid BVP, Lopez et al. [20] derived an LTI. In [8], Eloe and Krushna established LTIs for a family of two-point $(\mathtt{n},\mathtt{p})$-type FBVPs. To illustrate the applicability of their findings, they provided several examples, including one involving a FDEq with delay. Ntouyas and Ahmad [25] provide a comprehensive survey on LTIs for FDEqs and systems of FDEqs, covering a wide range of fractional BCs. These include sequential, Caputo–Fabrizio, Hadamard, Hilfer, Katugampola, hybrid, and nested derivatives. More recently, Ntouyas et al. [26] presented an extensive review of the latest developments on LTIs for various types of FBVPs.

Based on the latest developments and insights from preceding research, we now devote our attention to the study of nonlinear FBVPs. These issues include a complicated interaction between fractional differential operators and nonlinear BCs, which presents both significant hurdles and great potential for practical applications. The hybrid nature of these BVPs, which combine FC and nonlinear dynamics, provides an ideal platform for the development of novel solution methods. By tackling such a complicated issue, we hope to get a better understanding of FDEqs and their applications, potentially leading to new techniques and solutions. This paper discusses the nonlinear fractional hybrid BVP:

$$ {}^{RL}\mathfrak{D}^{\sigma}_{z_0^+}\left[\frac{u(z)}{\Psi\big(z,u(z)\big)}\right]+\ell(z)Fu(z)=0, \quad z\in(z_0,z_1), \tag{4}$$

$$\left.\begin{aligned} &u^{(i)}(z_0)=0, \quad i=\overline{0,n-2},\\ &u(z_1)=0, \end{aligned}\right\} \tag{5}$$

where $z_1>z_0\geqslant 0, \sigma\in(n-1,n)$, $n\geqslant 2$ and ${}^{RL}\mathfrak{D}^{\bullet}_{z_0^+}$ is the RL derivative.

Three suppositions are taken into account throughout the paper:

($\mathbf{H}_1$) $\Psi:[z_0,z_1]\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ is continuous and bounded function.

($\mathbf{H}_2$) $F:C[z_0,z_1]\to C[z_0,z_1]$ and there exists $\eta>0$ such that if $u(z)\in C[z_0,z_1]$ and $u(z)\geqslant 0$ for $z_0\leqslant z\leqslant z_1$, then

$$\|Fu\|_\infty\leqslant\eta\|u\|_\infty.$$

($\mathbf{H}_3$) $\ell\in C\big([z_0,z_1],\mathbb{R}\big)$.

This article is structured as follows. Some auxiliary results are included in Sect. 2. Sect. 3 presents the key theorems, and Sect. 4 illustrates how to apply the findings.

2. Preliminary results

Our key findings will be based on a few auxiliary results that we present here.

DEFINITION 1. [14, 28] Let $\varphi\in L^1\big((z_0,z_1);\mathbb{R}\big)$, where $\big(z_0,z_1\big)\in\mathbb{R}^2$, $z_0<z_1$. The RL fractional integral of order $\sigma>0$ of φ is defined by

$$I^{\sigma}_{z_0^+}\varphi(z)=\frac{1}{\Gamma(\sigma)}\int_{z_0}^{z}(z-y)^{\sigma-1}\varphi(y)dy, \quad a.e.\ z\in[z_0,z_1].$$

DEFINITION 2. [14, 28] The RL derivative of order σ of a function $\varphi : [z_0, z_1] \to \mathbb{R}$ is defined by

$$ {}^{RL}\mathfrak{D}^{\sigma}_{z_0^+}\varphi(z) = \frac{1}{\Gamma(m-\sigma)}\left(\frac{d}{dz}\right)^{m}\int_{z_0}^{z}(z-y)^{m-\sigma-1}\varphi(y)dy, \quad a.e.\ z \in [z_0, z_1], $$

where $(z_0, z_1) \in \mathbb{R}^2, z_0 < z_1$, $\sigma > 0$ and $m = [\sigma] + 1$.

LEMMA 1. [14, 28] *Assume that* $\varphi \in C(z_0, z_1) \cap L^1(z_0, z_1)$. *Then*

$$ I^{\sigma}_{z_0^+}\,{}^{RL}\mathfrak{D}^{\sigma}_{z_0^+}\varphi(z) = \varphi(z) + \sum_{k=1}^{n} d_k (z - z_0)^{\sigma-k}, \quad z \in [z_0, z_1], $$

where $d_k \in \mathbb{R}$, $k = \overline{1,n}$ *and* $n = [\sigma] + 1$.

3. Kernel, bounds of kernel, and Lyapunov-type inequalities

LEMMA 2. *Suppose* $(\mathbf{H}_1)$ *holds and let* $h \in C([z_0, z_1], \mathbb{R})$. *Then* $u(z) \in C([z_0, z_1], \mathbb{R})$ *is a solution of the FDEq*

$$ {}^{RL}\mathfrak{D}^{\sigma}_{z_0^+}\left[\frac{u(z)}{\Psi(z, u(z))}\right] + h(z) = 0, \quad z \in (z_0, z_1), \tag{6} $$

with (5) *if and only if*

$$ u(z) = \Psi(z, u(z))\int_{z_0}^{z_1} G_\sigma(z, y)h(y)dy, $$

where

$$ G_\sigma(z, y) = \begin{cases} \dfrac{(z_1 - y)^{\sigma-1}(z - z_0)^{\sigma-1}}{(z_1 - z_0)^{\sigma-1}\Gamma(\sigma)}, & z_0 \leqslant z \leqslant y \leqslant z_1, \\[2ex] \dfrac{(z_1 - y)^{\sigma-1}(z - z_0)^{\sigma-1}}{(z_1 - z_0)^{\sigma-1}\Gamma(\sigma)} - \dfrac{(z - y)^{\sigma-1}}{\Gamma(\sigma)}, & z_0 \leqslant y \leqslant z \leqslant z_1. \end{cases} \tag{7} $$

Proof. Let $u(z) \in C[z_0, z_1]$ be the solution to the FBVP (6), (5). Then we have by Lemma 1,

$$ \frac{u(z)}{\Psi(z, u(z))} = \sum_{k=1}^{n} d_k (z - z_0)^{\sigma-k} - I^{\sigma}_{z_0^+}h(y)dy, $$

where $d_k \in \mathbb{R}$ for $k = \overline{1,n}$. So,

$$ u(z) = \Psi(z, u(z))\left[\sum_{k=1}^{n} d_k (z - z_0)^{\sigma-k} - \frac{1}{\Gamma(\sigma)}\int_{z_0}^{z}(z - y)^{\sigma-1}h(y)dy\right]. $$

Using $u^{(i)}(z_0)=0,\ i=\overline{0,n-2}$, we get $d_n=d_{n-1}=\cdots=d_2=0$. Therefore

$$u(z)=\Psi\big(z,u(z)\big)\left[d_1\big(z-z_0\big)^{\sigma-1}-\frac{1}{\Gamma(\sigma)}\int_{z_0}^{z}(z-y)^{\sigma-1}h(y)dy\right]. \tag{8}$$

Thus, $u(z_1)=0$ implies $d_1=\displaystyle\int_{z_0}^{z_1}\left[\frac{(z_1-y)^{\sigma-1}}{\Gamma(\sigma)\big(z_1-z_0\big)^{\sigma-1}}\right]h(y)dy$ and the unique solution of FBVP (6), (5):

$$\begin{aligned}
u(z)&=\Psi\big(z,u(z)\big)\left\{\begin{array}{l}\displaystyle\int_{z_0}^{z}\left(\frac{(z-z_0)^{\sigma-1}(z_1-y)^{\sigma-1}}{\Gamma(\sigma)(z_1-z_0)^{\sigma-1}}-\frac{(z-y)^{\sigma-1}}{\Gamma(\sigma)}\right)h(y)dy\\ \displaystyle+\int_{z}^{z_1}\left(\frac{(z-z_0)^{\sigma-1}(z_1-y)^{\sigma-1}}{\Gamma(\sigma)(z_1-z_0)^{\sigma-1}}\right)h(y)dy\end{array}\right.\\
&=\Psi\big(z,u(z)\big)\int_{z_0}^{z_1}G_\sigma(z,y)h(y)dy,
\end{aligned}$$

where $G_\sigma(z,y)$ is given in (7). The proof is now complete. □

For $n=2$ and $n\geqslant 3$ cases, we now present estimates of the Kernel G_σ represented by (7). Let us begin with the case $n=2$.

LEMMA 3. [20] *Let* $n=2$. *The Kernel* $G_\sigma(z,y)$ *given by* (7) *has the properties:*

(a) $G_\sigma(z,y)\geqslant 0,\ \forall\, z,y\in[z_0,z_1]$,

(b) $$\max_{z\in[z_0,z_1]}G_\sigma(z,y)=G_\sigma(y,y)=\frac{(y-z_0)^{\sigma-1}(z_1-y)^{\sigma-1}}{\Gamma(\sigma)(z_1-z_0)^{\sigma-1}},\quad \forall\ z_0\leqslant y\leqslant z_1,$$

(c) $$\max_{y\in[z_0,z_1]}G_\sigma(y,y)=\frac{1}{\Gamma(\sigma)}\left(\frac{z_1-z_0}{4}\right)^{\sigma-1}.$$

The following lemma provides the estimates of the Kernel G_σ for Case $n\geqslant 3$.

LEMMA 4. *If* $n\in\mathbb{N}$, $n\geqslant 3$, *then the Kernel* $G_\sigma(z,y)$ *has the properties:*

(a) $G_\sigma(z,y)\geqslant 0,\ \forall\, z,y\in[z_0,z_1]$,

(b) For $z\in[z_0,z_1]$, $y\in(z_0,z_1)$,

$$G_\sigma(z,y)\leqslant G_\sigma(y^*,y)=\frac{(z_1-y)^{\sigma-1}(y-z_0)^{\sigma-1}}{(z_1-z_0)^{\sigma-1}\Gamma(\sigma)\left[1-\left(\dfrac{z_1-y}{z_1-z_0}\right)^{\frac{\sigma-1}{\sigma-2}}\right]^{\sigma-2}},$$

where $$y^*=\frac{y-z_0\left(\dfrac{z_1-y}{z_1-z_0}\right)^{\frac{\sigma-1}{\sigma-2}}}{1-\left(\dfrac{z_1-y}{z_1-z_0}\right)^{\frac{\sigma-1}{\sigma-2}}}.$$

Proof. The Kernel $G_\sigma(z,y)$ provided in (7) is clearly continuous on $[z_0,z_1]\times[z_0,z_1]$. It is evident that $G_\sigma(z,y)\geqslant 0,\ \forall\, z,y\in[z_0,z_1]$. We establish the inequality (b). Let $y\in(z_0,z_1)$ be fixed. For $y\leqslant z\leqslant z_1$, we have

$$\begin{aligned}\frac{\partial}{\partial z}G_\sigma(z,y) &= \frac{(\sigma-1)(z_1-y)^{\sigma-1}(z-z_0)^{\sigma-2}}{(z_1-z_0)^{\sigma-1}\Gamma(\sigma)}-\frac{(\sigma-1)(z-y)^{\sigma-2}}{\Gamma(\sigma)}\\ &=\left[\frac{(z-z_0)^{\sigma-2}}{\Gamma(\sigma-1)}\right]\left\{\left(\frac{z_1-y}{z_1-z_0}\right)^{\sigma-1}-\left(1-\frac{y-z_0}{z-z_0}\right)^{\sigma-2}\right\}.\end{aligned}$$

We notice that $\dfrac{\partial}{\partial z}G_\sigma(z,y)=0\Leftrightarrow z=y^*$. In addition, we also have

$$y^*-z_0=\frac{y-z_0}{1-\left(\dfrac{z_1-y}{z_1-z_0}\right)^{\frac{\sigma-1}{\sigma-2}}}>0,\quad y\in(z_0,z_1)$$

and

$$z_1-y^*=\frac{z_1-y}{1-\left(\dfrac{z_1-y}{z_1-z_0}\right)^{\frac{\sigma-1}{\sigma-2}}}\left[1-\left(\frac{z_1-y}{z_1-z_0}\right)^{\frac{1}{\sigma-2}}\right]>0,\quad y\in(z_0,z_1).$$

As a result, for all $y\in(z_0,z_1)$, we have $y^*\in(z_0,z_1)$. Furthermore, we have $G_\sigma(z,y)$ maximizes at y^* occurs when $y\leqslant z$, for a given value of $y\in(z_0,z_1)$. This leads to the conclusion that (b) is true due to the fact that $G_\sigma(z,y)$ is increasing on $y>z$. $\square$

REMARK 1. Notice that in the case $n=2$, that is, $\sigma\in(1,2)$, we have $y^*<z_0$. It follows that the estimates for $G_\sigma(z,y)$ for $n>3$ presented in Lemma 4 cannot cover those for $n=2$ given in Lemma 3.

REMARK 2. A straightforward calculation yields

$$\lim_{y\to z_0^+}G_\sigma(y^*,y)=\lim_{y\to z_1^-}G_\sigma(y^*,y)=0.$$

LEMMA 5. [13] *If* $n\in\mathbb{N},\ n\geqslant 3$, *then*

$$\max_{y\in[z_0,z_1]}G_\sigma(y^*,y)=\frac{(z_1-z_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}},$$

where ϖ_σ *is the unique zero of* $\varpi^{\frac{2\sigma-3}{\sigma-2}}-2\varpi+1=0$ *in* $\left(0,\left(\dfrac{2\sigma-4}{2\sigma-3}\right)^{\frac{\sigma-2}{\sigma-1}}\right)$.

For RL FBVP (4)–(5), we can now construct an LTI. As we are focused on continuous solutions, we consider the Banach space $\mathtt{E}=\Big\{\mathtt{u}:\ \mathtt{u}\in\mathtt{C}[\mathtt{z}_0,\mathtt{z}_1]\Big\}$ with the norm

$$\|\mathtt{u}\|=\max_{\mathtt{z}\in[\mathtt{z}_0,\mathtt{z}_1]}\big|\mathtt{u}(\mathtt{z})\big|.$$

We distinguish two cases.

3.1. The case $\mathtt{n}=2$

In this case, problem (4)–(5) reduces to the nonlinear fractional hybrid BVP

$$^{RL}\mathfrak{D}^{\sigma}_{\mathtt{z}_0^+}\left[\frac{\mathtt{u}(\mathtt{z})}{\Psi\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big)}\right]+\ell(\mathtt{z})\mathtt{Fu}(\mathtt{z})=0,\quad \mathtt{z}\in(\mathtt{z}_0,\mathtt{z}_1),\tag{9}$$

$$\mathtt{u}(\mathtt{z}_0)=0,\quad \mathtt{u}(\mathtt{z}_1)=0,\tag{10}$$

where $\mathtt{z}_1>\mathtt{z}_0\geqslant 0,\sigma\in(1,2)$.

THEOREM 1. *Suppose* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *hold. If* $\mathtt{u}$ *is a nontrivial solution of the FBVP* (9)–(10)*, then*

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1}\big|\ell(\mathtt{y})\big|d\mathtt{y}\geqslant\frac{\Gamma(\sigma)}{\Re\eta}\left(\frac{4}{\mathtt{z}_1-\mathtt{z}_0}\right)^{\sigma-1}.\tag{11}$$

Proof. Let $\Re=\sup\Big\{\big|\Psi(\mathtt{z},\mathtt{u})\big|:\mathtt{z}\in[\mathtt{z}_0,\mathtt{z}_1],\mathtt{u}\in\mathbb{R}\Big\}$. Let $\mathtt{u}$ be a nontrivial solution to the FBVP (9)–(10). By implementing Lemma 2 and $z(\mathtt{y})=\ell(\mathtt{y})\mathtt{Fu}(\mathtt{y})$, $\mathtt{u}$ can now be expressed as

$$\mathtt{u}(\mathtt{z})=\Psi\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big)\int_{\mathtt{z}_0}^{\mathtt{z}_1}\mathtt{G}_\sigma(\mathtt{z},\mathtt{y})\ell(\mathtt{y})\mathtt{Fu}(\mathtt{y})d\mathtt{y},\quad \mathtt{z}\in[\mathtt{z}_0,\mathtt{z}_1].\tag{12}$$

We can see from $(\mathbf{H}_1)$–$(\mathbf{H}_3)$, for $\mathtt{z}_0\leqslant\mathtt{z}\leqslant\mathtt{z}_1$, we obtain

$$\begin{aligned}\big|\mathtt{u}(\mathtt{z})\big|&\leqslant\big|\Psi\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big)\big|\int_{\mathtt{z}_0}^{\mathtt{z}_1}\big|\mathtt{G}_\sigma(\mathtt{z},\mathtt{y})\big|\ \big|\ell(\mathtt{y})\big|\ \big|\mathtt{Fu}(\mathtt{y})\big|d\mathtt{y}\\&\leqslant\Re\ \big\|\mathtt{Fu}\big\|\int_{\mathtt{z}_0}^{\mathtt{z}_1}\big|\mathtt{G}_\sigma(\mathtt{z},\mathtt{y})\big|\ \big|\ell(\mathtt{y})\big|d\mathtt{y}.\end{aligned}$$

Using Lemma 3 (c), we obtain

$$\big\|\mathtt{u}\big\|\leqslant\Re\ \eta\ \big\|\mathtt{u}\big\|\int_{\mathtt{z}_0}^{\mathtt{z}_1}\frac{1}{\Gamma(\sigma)}\left(\frac{\mathtt{z}_1-\mathtt{z}_0}{4}\right)^{\sigma-1}\big|\ell(\mathtt{y})\big|d\mathtt{y}.$$

As a result,

$$\big\|\mathtt{u}\big\|\leqslant\int_{\mathtt{z}_0}^{\mathtt{z}_1}\big|\ell(\mathtt{y})\big|d\mathtt{y}\,\frac{1}{\Gamma(\sigma)}\left(\frac{\mathtt{z}_1-\mathtt{z}_0}{4}\right)^{\sigma-1}\Re\eta\big\|\mathtt{u}\big\|.$$

Since $\mathtt{u}$ is a nontrivial solution, we have $\|\mathtt{u}\|>0$. Therefore,

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1}\big|\ell(\mathtt{y})\big|d\mathtt{y}\,\frac{1}{\Gamma(\sigma)}\left(\frac{\mathtt{z}_1-\mathtt{z}_0}{4}\right)^{\sigma-1}\Re\eta\geqslant 1,$$

which yields the desired inequality (11). □

3.2. The case $\mathtt{n} \geqslant 3$

THEOREM 2. *Suppose* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *hold. If* $\mathtt{u}$ *is a nontrivial solution of the FBVP* (4)–(5), *then*

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1} |\ell(\mathtt{y})| d\mathtt{y} \geqslant \frac{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}}{\Re\eta(\mathtt{z}_1-\mathtt{z}_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}, \tag{13}$$

where ϖ_σ *is the unique zero of* $\varpi^{\frac{2\sigma-3}{\sigma-2}} - 2\varpi + 1 = 0$ *in* $\left(0, \left(\frac{2\sigma-4}{2\sigma-3}\right)^{\frac{\sigma-2}{\sigma-1}}\right)$.

Proof. Let $\mathtt{u}$ be a nontrivial solution to the FBVP (9)–(10). Following from $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ and using (12), we obtain

$$\begin{aligned} |\mathtt{u}(\mathtt{z})| &\leqslant |\Psi(\mathtt{z},\mathtt{u}(\mathtt{z}))| \int_{\mathtt{z}_0}^{\mathtt{z}_1} |\mathtt{G}_\sigma(\mathtt{z},\mathtt{y})|\, |\ell(\mathtt{y})|\, |\mathtt{Fu}(\mathtt{y})| d\mathtt{y} \\ &\leqslant \Re\, \|\mathtt{Fu}\| \int_{\mathtt{z}_0}^{\mathtt{z}_1} |\mathtt{G}_\sigma(\mathtt{z},\mathtt{y})|\, |\ell(\mathtt{y})| d\mathtt{y}. \end{aligned}$$

Using Lemma 5, we obtain

$$\|\mathtt{u}\| \leqslant \Re\eta\|\mathtt{u}\| \int_{\mathtt{z}_0}^{\mathtt{z}_1} \frac{(\mathtt{z}_1-\mathtt{z}_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}} |\ell(\mathtt{y})| d\mathtt{y}.$$

As a result,

$$\|\mathtt{u}\| \leqslant \int_{\mathtt{z}_0}^{\mathtt{z}_1} \frac{(\mathtt{z}_1-\mathtt{z}_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}} |\ell(\mathtt{y})| d\mathtt{y}\, \Re\eta\|\mathtt{u}\|.$$

Since $\mathtt{u}$ is a nontrivial solution, we have $\|\mathtt{u}\| > 0$. Therefore,

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1} \frac{(\mathtt{z}_1-\mathtt{z}_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}} |\ell(\mathtt{y})| d\mathtt{y}\Re\eta \geqslant 1,$$

which yields the desired inequality (13). □

COROLLARY 1. *Suppose* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *hold. Then the estimate*

$$\int_{\mathtt{z}_0}^{\mathtt{z}_1} |\ell(\mathtt{y})| d\mathtt{y} < \frac{\Gamma(\sigma)}{\Re\eta}\left(\frac{4}{\mathtt{z}_1-\mathtt{z}_0}\right)^{\sigma-1}$$

implies that the FDEq

$$^{RL}\mathfrak{D}^{\sigma}_{\mathtt{z}_0^+}\mathtt{u}(\mathtt{z}) + \ell(\mathtt{z})F\mathtt{u}(\mathtt{z}) = 0, \quad \mathtt{z}_0 < \mathtt{z} < \mathtt{z}_1, \tag{14}$$

satisfying (5), *where* $\mathtt{z}_1 > \mathtt{z}_0 \geqslant 0$, $\sigma \in (1,2)$, *has only the trivial solution* $\mathtt{u}(\mathtt{z}) \equiv 0$.

COROLLARY 2. *Suppose* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *hold. Then the estimate*

$$\int_{z_0}^{z_1} |\ell(y)|dy < \frac{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}}{\Re\eta(z_1-z_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}$$

implies that the FDEq

$$^{RL}\mathfrak{D}^{\sigma}_{z_0^+}u(z)+\ell(z)Fu(z)=0, \quad z_0<z<z_1, \tag{15}$$

satisfying (5)*, where* $z_1>z_0\geqslant 0$, $\sigma\in(n-1,\ n)$, $n\geqslant 3$, ϖ_σ *is the unique zero of* $\varpi^{\frac{2\sigma-3}{\sigma-2}}-2\varpi+1=0$ *in* $\left(0,\left(\frac{2\sigma-4}{2\sigma-3}\right)^{\frac{\sigma-2}{\sigma-1}}\right)$*, has only the trivial solution* $u(z)\equiv 0$.

COROLLARY 3. *Suppose that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_3)$ *hold. Then we assert that any eigenvalue* λ *of the FDEq*

$$^{RL}\mathfrak{D}^{\sigma}_{z_0^+}\left[\frac{u(z)}{\Psi\big(z,u(z)\big)}\right]+\lambda Fu(z)=0, \quad z_0<z<z_1, \tag{16}$$

coupled with BCs (5)*, where* $z_1>z_0\geqslant 0$, $\sigma\in(n-1,n)$, $n\geqslant 3$, ϖ_σ *is the unique zero of* $\varpi^{\frac{2\sigma-3}{\sigma-2}}-2\varpi+1=0$ *in* $\left(0,\left(\frac{2\sigma-4}{2\sigma-3}\right)^{\frac{\sigma-2}{\sigma-1}}\right)$*, satisfies*

$$|\lambda|\geqslant\frac{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}}{\Re\eta(z_1-z_0)^{\sigma}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}.$$

Proof. If λ is an eigenvalue of FBVP (16), (5). Then FBVP (16), (5) admits at least one nontrivial solution u_λ. As a result of Theorem 2, we have

$$\int_{z_0}^{z_1}|\lambda|dy\geqslant\frac{\Gamma(\sigma)\left(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\right)^{\sigma-2}}{\Re\eta(z_1-z_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}$$

and it provides us with the desired outcome. $\square$

4. An Example

EXAMPLE 1. Let $z_1=1$, $z_0=0$, $\sigma=\frac{11}{3}$, $\Psi(z,u)\equiv 1$, $\ell(z)=53\ln(2+3z)$ and $Fu=u$ on $C[0,1]$. Consider a nonlinear FBVP:

$$^{RL}\mathfrak{D}^{\frac{11}{3}}_{0^+}\left[\frac{u(z)}{\Psi(z,u)}\right]+\ell(z)u(z)=0, \quad 0<z<1, \tag{17}$$

$$\mathrm{u}(0)=\mathrm{u}'(0)=\mathrm{u}''(0)=\mathrm{u}(1)=0. \tag{18}$$

In view of the data given, we get $\Re=1$, $\eta=1$ and by straight forward calculation, we get $\varpi_\sigma=0.691912$ is the unique zero of $\varpi^{\frac{13}{5}}-2\varpi+1=0$ in $\big(0,0.745461\big)$. Then

$$\textbf{(i)}\quad \int_{\mathrm{z}_0}^{\mathrm{z}_1}\big|\ell(\mathrm{y})\big|d\mathrm{y}=53\int_0^1\ln(2+3\mathrm{y})d\mathrm{y}=\frac{53}{3}\Big[(2+3\mathrm{y})\ln(2+3\mathrm{y})-(2+3\mathrm{y})\Big]_0^1$$
$$=\frac{265}{3}\ln(5)-\frac{106}{3}\ln(2)-53\approx 64.675815,$$

$$\textbf{(ii)}\quad \frac{\Gamma(\sigma)\Big(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\Big)^{\sigma-2}}{\Re\eta(\mathrm{z}_1-\mathrm{z}_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}=\frac{\Gamma(\frac{11}{3})\Big(1-(0.691912)^{\frac{\frac{11}{3}-1}{\frac{11}{3}-2}}\Big)^{\frac{11}{3}-2}}{(0.691912)^{\frac{11}{3}-1}\big(1-(0.691912)\big)^{\frac{11}{3}-1}}$$
$$\approx 61.163746.$$

From $(\mathbf{i})$ and $(\mathbf{ii})$, we have $\displaystyle\int_{\mathrm{z}_0}^{\mathrm{z}_1}\big|\ell(\mathrm{y})\big|d\mathrm{y}\geqslant\frac{\Gamma(\sigma)\Big(1-\varpi_\sigma^{\frac{\sigma-1}{\sigma-2}}\Big)^{\sigma-2}}{\Re\eta(\mathrm{z}_1-\mathrm{z}_0)^{\sigma-1}\varpi_\sigma^{\sigma-1}(1-\varpi_\sigma)^{\sigma-1}}$. By Theorem 2, the FBVP (17)–(18) has a nontrivial solution.

5. Conclusion

In conclusion, this paper effectively addresses the RL fractional hybrid BVP by establishing sufficient conditions for deriving LTIs. The theoretical results are validated through an illustrative example, which highlights the practical applicability and effectiveness of the proposed method. These findings enhance our understanding of FDEqs and their boundary value problems, opening potential avenues for further research into the analysis and application of fractional systems.

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