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THEORETICAL ANALYSIS OF A CLASS OF NONLOCAL ϕ–CAPUTO FRACTIONAL NONLINEAR EVOLUTION EQUATIONS USING THE MEASURE OF NON–COMPACTNESS IN BANACH SPACES

NAJAT CHEFNAJ, ABDELLAH TAQBIBT*,
KHALID HILAL AND M'HAMED ELOMARI

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Abstract. This study explores the existence of solutions for nonlocal fractional differential evolution equations through the concept of the measure of non-compactness. Our approach incorporates probability density functions, operator semigroup theory, and the Mönch fixed point theorem. To demonstrate the practical significance of our findings, we conclude with an application.

1. Introduction

In recent years, fractional calculus has garnered increasing attention within the scientific community, leading to a surge in applications across various fields such as biophysics, physics, viscoelasticity, electrochemistry, biomedicine, control theory, and signal processing. Fractional differential equations, in particular, have proven to be valuable tools for modeling a wide range of phenomena in science and engineering, reflecting the significant advancements made in this area, see the monographs of Kilbas et al. [6], Podlubny [13], Zhou [15], the papers [4, 9, 10, 14, 16] and the references therein.

Melliani et al. [11] have investigated a comprehensive class of periodic boundary value problems for nonlinear impulsive evolution equations in Banach spaces, described by the system

$$\begin{cases} v'(t) = Av(t) + f\big(t, v(t), \rho(t)\big) + \mathscr{B}(t)c(t), & t \in (s_i, t_{i+1}], \ \ i = 0,1,2,\dots,m, c \in \mathscr{U}_{ad}, \\ v(t) = T(t-t_i)g_i(t, v(t)), & t \in (t_i, s_i], \ \ i = 0,1,2,\dots,m, \\ v(0) = v(b) \in X, \end{cases}$$

where $\{T(t), t \geqslant 0\}$ is a strongly continuous semigroup on X, a Banach space with norm $\|.\|$, and the operator $A : D(A) \to X$ is its infinitesimal generator. The fixed points t_i and s_i, satisfying $0 = s_0 < t_1 \leqslant s_1 \leqslant t_2 < \dots < t_m \leqslant s_m \leqslant t_{m+1} = b$ are pre-specified

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* Corresponding author.

numbers. The function $f : [0,b] \times X \times X \to X$ is continuous, $g_i : [t_i, s_i] \times X \to X$ is continuous for every $i = 1,2,\dots,m$, and $\rho : [0,b] \to [0,b]$ is continuous.

Malar et al. [8], have studied the existence results of abstract impulsive integro-differential problems with noncompactnes smeasure

$$\begin{cases} v'(t) = Av(t) + f\big(t, v(t), \int_0^t \rho(t,s)\sigma(t,s,v(s))ds\big), \ t \in (s_i, t_{i+1}], \ i = 0,1,2,\dots,m, \\ v(t) = g_i(t, v(t)), \quad t \in (t_i, s_i], \ i = 0,1,2,\dots,m, \\ v(0) = v_0 + \Phi(v), \end{cases}$$

where $\{T(t), t \geqslant 0\}$ is a C_0-semigroup of bounded linear operator on X, a Banach space with norm $\|.\|$, and the operator $A : D(A) \to X$ is its infinitesimal generator. The fixed points t_i and s_i satisfying $0 = t_0 = s_0 < t_1 \leqslant s_1 \leqslant t_2 < \dots < t_n \leqslant s_n \leqslant t_{n+1} = b$ are pre-specified numbers, and $v_0 \in X$. The function $g_i : [t_i, s_i] \times X \to X$ is continuous for every $i = 1,2,\dots,m$, $f : [0,b] \times X \times X \to X$, $\Phi : X \to X$, $\sigma \in \mathscr{C}(\Delta, \mathbb{R}^+)$, $\Delta = \big\{(t,s) \ / \ t,s \in [0,b], t \geqslant s\big\}$ and $\rho : [0,b] \to [0,b]$ is continuous.

In this paper, we explore the existence of a solution for a nonlinear fractional evolution problem with nonlocal conditions

$$\begin{cases} {}^C D_{0^+}^{\gamma,\phi} v(t) = Av(t) + f\big(t, v(t), \int_0^t \rho(t,s)\sigma(t,s,v(s))ds\big) + \mathscr{B}(t)c(t), \ t \in \Sigma, c \in U_{ad}, \\ v(0) = v_0 + \Phi(v), \end{cases} \tag{1}$$

where $\Sigma = [0,b]$, ${}^C D_{0^+}^{\gamma,\phi}$ is the ϕ-Caputo fractional derivative operator of order $\gamma \in (0,1)$, $\{T(t), t \geqslant 0\}$ is an analytic semigroup of bounded linear operators on X, a Banach space with norm $\|.\|$, and the operator $A : D(A) \to X$ is its infinitesimal generator. $v_0 \in X$, $\mathscr{B} : [0,b] \to \mathscr{L}(Y,X)$, f is a function $f : [0,b] \times X \times X \to X$, $\Phi : X \to X$, $\sigma \in \mathscr{C}(\Delta, \mathbb{R}^+)$, $\Delta = \{(t,s) \ / \ t,s \in [0,b], \ t \geqslant s\}$ and $\rho : [0,b] \to [0,b]$ is continuous.

This paper is structured as follows: In Section 2, we review some notations and key existing results. Section 3 presents our main findings on the existence of solutions to the aforementioned problem. Finally, in Section 4, we provide an application to illustrate our main results.

2. Preliminaries

In this section, we will present some notations, definitions and we will also state some results that will be used in this work.

Let X be a Banach space equipped with the norm $\|.\|$, and let $\mathscr{C}(\Sigma, X)$ be the Banach space of continuous functions defined on Σ and taking values in X, characterized by the norm

$$\|v\|_\infty = \sup_{t \in \Sigma} \|v(t)\|.$$

Consider a separable reflexive Banach space Y, wherein the controls c are valued. Let $P_f(y)$ represent a collection of nonempty, convex, nd subsets of Y. We assume that the multi-valued mapping $w : [0,T] \to P_f(y)$ is measurable, such that $w(t)$ is a subset of E, a bounded set in Y. The admissible control set is defined as

$$\mathscr{U}_{ad} = \{c \in \mathscr{L}^p(E) : c(s) \in w(s),\ a.e.\},$$

where $p > 1$. Additionally, we denote by $B_r = \{v \in \mathscr{C}(\Sigma, X) : \|v\|_\infty \leqslant r\}$ and $\Delta = \{(t,s) : t,s \in [0,b], t \geqslant s\}$.

DEFINITION 1. [1] Let $\gamma > 0$, $g \in L^1(\Sigma,\mathbb{R})$ and $\phi \in \mathscr{C}^n(\Sigma,\mathbb{R})$ such that $\phi'(t) > 0$ for all $t \in \Sigma$. The fractional integral of order γ in the sense of the ϕ-Riemann-Liouville for the function g is defined by

$$I_{0^+}^{\gamma,\phi} g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\gamma-1} g(s)ds. \tag{2}$$

REMARK 1. Observe that when $\phi(t) = t$ and $\phi(t) = \log(t)$, the equation (2) simplifies to the Riemann-Liouville and Hadamard fractional integrals, respectively.

DEFINITION 2. [1] Let $\gamma > 0$, $g \in C^{n-1}(\Sigma,\mathbb{R})$ and $\phi \in \mathscr{C}^n(\Sigma,\mathbb{R})$ such that $\phi'(t) > 0$ for all $t \in \Sigma$. The fractional derivative of order γ in the sense of the ϕ-Caputo for the function g is defined by

$${}^C D_{0^+}^{\gamma,\phi} g(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{n-\gamma-1} g_\phi^{[n]}(s)ds, \tag{3}$$

where $g_\phi^{[n]}(s) = \left(\frac{1}{\phi'(s)}\frac{d}{ds}\right)^n g(s)$, $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ. In particular, if $\gamma \in]0,1[$, then

$${}^C D_{0^+}^{\gamma,\phi} g(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (\phi(t) - \phi(s))^{-\gamma} g'(s)ds \text{ and } {}^C D_{0^+}^{\gamma,\phi} g(t) = I_{0^+}^{1-\gamma,\phi}\left(\frac{g'(t)}{\phi'(t)}\right).$$

REMARK 2. Specifically, note that when $\phi(t) = t$ and $\phi(t) = \log(t)$, the equation (3) reduces to the Caputo fractional derivative and the Caputo-Hadamard fractional derivative, respectively.

PROPOSITION. [1] *Let γ be a strictly positive real number, and $g \in C^{n-1}(\Sigma,\mathbb{R})$. Then, we have*

1) ${}^C D_{0^+}^{\gamma,\phi} I_{0^+}^{\gamma,\phi} g(t) = g(t)$.

2) $I_{0^+}^{\gamma,\phi}\, {}^C D_{0^+}^{\gamma,\phi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_\phi^{[k]}(0)(\phi(t) - \phi(0))^k}{k!}$.

3) $I_{0^+}^{\gamma,\phi}$ *is linear operator and bounded defined on $\mathscr{C}(\Sigma,\mathbb{R})$ to $\mathscr{C}(\Sigma,\mathbb{R})$.*

PROPOSITION. [1] *Let γ be a strictly positive real number, and $\beta \in \mathbb{R}$ where $\beta > n$, with $n = [\gamma]+1$, then*

1) $I_{0^+}^{\gamma,\phi}(\phi(t)-\phi(0))^{\beta-1} = \dfrac{\Gamma(\beta)(\phi(t)-\phi(0))^{\gamma+\beta-1}}{\Gamma(\gamma+\beta)}$.

2) ${}^{C}D_{0^+}^{\gamma,\phi}(\phi(t)-\phi(0))^{k} = 0, \quad$ *for all* $n > k \in \mathbb{N}$.

3) ${}^{C}D_{0^+}^{\gamma,\phi}(\phi(t)-\phi(0))^{\beta-1} = \dfrac{\Gamma(\beta)(\phi(t)-\phi(0))^{\gamma-\beta-1}}{\Gamma(\beta-\gamma)}$.

DEFINITION 3. [5] The generalized Laplace transform of the function $v : \Sigma \to X$ is defined by

$$\mathscr{L}_{\phi}\{v(t)\}(s) := \widehat{v}(s) = \int_0^{\infty} \phi'(t)e^{-s(\phi(t)-\phi(0))}v(t)dt.$$

DEFINITION 4. [5] Let f and g be two functions that are piecewise continuous with exponential order on Σ. We define the generalized ϕ-convolution of f and f as follows

$$(f \underset{\phi}{*} g)(t) = \int_0^{t} f(s)g[\phi^{-1}(\phi(t)+\phi(0)-\phi(s))]\phi'(s)ds.$$

LEMMA 1. (See [5]) *Let γ be a strictly positive real number, and let v be a piecewise continuous function on the interval $[0,t]$ and $\phi(t)$-exponential order. Then,*

1. $\mathscr{L}_{\phi}\{I_{0^+}^{\gamma,\phi}v(t)\}(s) = \dfrac{\widehat{v}(s)}{s^{\gamma}}$.

2. $\mathscr{L}_{\phi}\{{}^{C}D_{0^+}^{\gamma,\phi}v(t)\}(s) = s^{\gamma}\left[\mathscr{L}_{\phi}\{v(t)\} - \sum_{k=0}^{n-1} s^{-k+1}v^{(k)}(0)\right]$, *such that* $n = [\gamma]+1$.

DEFINITION 5. (See [7]) Let ρ be a positive real number. The one-sided stable probability density is given by

$$\omega_{\gamma}(\rho) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\gamma n+1)}{n!}(\phi(\rho)-\phi(0))^{-\gamma n-1}\sin(n\pi\gamma).$$

LEMMA 2. [7] *The Laplace transform of the function $\omega_{\gamma}(t)$ is given as follows*

$$\int_0^{\infty} e^{-\lambda(\phi(t)-\phi(0))}\omega_{\gamma}(t)\phi'(t)dt = e^{-\lambda^{\gamma}}.$$

LEMMA 3. [2] *Let X be a real Banach space and C, B be bounded subsetsof X, the following properties are satisfied*

1. $\mu(C) = 0$ *if and only if C is pre-compact,*

2. $\mu(C)=\mu(\overline{C})=\mu(convC)$, *where* $\overline{C}$ *and* $convC$ *represent the closure and convex hull of* C, *respectively,*

3. $\mu(C)\leqslant\mu(B)$ *when* C *is subset of* B,

4. $\mu(C+B)\leqslant\mu(C)+\mu(B)$, *where* $C+B=\{z+x,z\in C,x\in B\}$,

5. $\mu(C\cup B)\leqslant\max\{\mu(C),\mu(B)\}$,

6. $\mu(\lambda B)=|\lambda|\mu(B)$ *for any* $\lambda\in\mathbb{R}$,

7. *If the map* $\mathscr{F}:D(\mathscr{F})\subset X\to Y$ *is Lipschitz continuous with constant* k. *Then,* $\mu(\mathscr{F}C)\leqslant k\mu(C)$, *for each bounded subset* $C\subset D(\mathscr{F})$, *where* Y *be a Banach space.*

8. *We have*

$$\begin{aligned}\mu(C)&=\inf\left\{d(C,B),B\subset X \text{ be pre-compact}\right\}\\&=\inf\{d(C,B),B\subset X \text{ be finite valued}\},\end{aligned}$$

where $d(C,B)$ *denote the symmetric (or nonsymmetric) Hausdorff distance between* C *and* B *in* X.

9. *If* $\{B_n\}_n^{\infty}=1$ *is a decreasing sequence of closed, bounded, nonempty subsets of* X *and* $\lim\limits_{n\to\infty}\mu(B_n)=0$, *then* $\cap_{n=1}^{\infty}B_n$ *is nonempty and compact in* X.

LEMMA 4. [12] *If* $\{v_n\}_{n=1}^{\infty}\subset L^1(0,K,X)$ *is uniformly integrable. Then,* $\mu(\{v_n\}_{n=1}^{\infty})$ *is measurable and*

$$\mu\big(\{\int_0^t v_n(s)ds\}_{n=1}^{\infty}\big)\leqslant 2\int_0^t\mu\big(\{v_n(s)\}_{n=1}^{\infty}\big)ds.$$

LEMMA 5. [3] *If* B_r *is bounded, then for any strictly positive real number* ε, *there exists a sequence* $\{v_n\}_{n=1}^{\infty}\subset B_r$ *with*

$$\mu(B_r)\leqslant 2\mu(\{v_n\}_{n=1}^{\infty})+\varepsilon.$$

THEOREM 1. [12] *Let* H *be a convex and closed subset of* X, *a Banach space with* $0\in H$. *Suppose that the map* $\mathscr{P}:H\to X$ *is continuous and satisfies Mönch's condition, meaning that if* $D\subseteq H$ *is countable, then* $D\subseteq\overline{co}(\{0\}\cup\mathscr{P}(D))$ *implies that* $\overline{D}$ *is compact. Under these conditions,* $\mathscr{P}$ *possesses a fixed point in* H.

3. Main results

In this section, we use the ϕ-Laplace transform to construct the integral solution for the fractional evolution equation (1). To achieve this, it is necessary to prove the following lemma.

LEMMA 6. *The problem (1) is equivalent to the equation given below*

$$v(t)=v_0+\Phi(v)+I_{0^+}^{\gamma,\phi}\left[Av(t)+f\big(t,v(t),\int_0^t\rho(t,s)\sigma(t,s,v(s))ds\big)+\mathscr{B}(t)c(t)\right], \quad (4)$$

for all $t\in J$.

Proof. Let v be a solution of the system (1). By applying the ϕ-fractional integral operator $I_{0^+}^{\gamma,\phi}$ to both sides of (1), we can get the following

$$I_{0^+}^{\gamma,\phi}\,{}^{C}D_{0^+}^{\gamma,\phi}v(t)=I_{0^+}^{\gamma,\phi}\left[Av(t)+f\left(t,v(t),\int_0^t\rho(t,s)\sigma(t,s,v(s))ds\right)+\mathscr{B}(t)c(t)\right],$$

and by applying Proposition 2, we derive

$$v(t)-v(0)=I_{0^+}^{\gamma,\phi}[Av(t)+f(t,v(t),\int_0^t\rho(t,s)\sigma(t,s,v(s))ds)+\mathscr{B}(t)c(t)],$$

since $v(0)=v_0+\Phi(v)$, this implies that

$$\begin{aligned}v(t)&=v_0+\Phi(v)\\&\quad+\frac{1}{\Gamma(\gamma)}\int_0^t\phi'(s)(\phi(t)-\phi(s))^{\gamma-1}\left[Av(s)+f\big(s,v(s),\int_0^s\rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\big)\right.\\&\quad\left.+\mathscr{B}(s)c(s)\right]ds.\end{aligned}$$

Hence, the integral equation (4) holds.

Conversely, through direct computation, it is evident that if v satisfies the integral equation (4), then the equation (1) is fulfilled. which completes the proof. □

LEMMA 7. *If the fractional equation (4) is satisfied, then we have*

$$\begin{aligned}v(t)&=S_{\gamma,\phi}(t)(v_0+\Phi(v))\\&\quad+\int_0^{\phi(t)-\phi(0)}R_{\gamma,\phi}(\phi(t)-\phi(s))\left[f\big(s,v(s),\int_0^s\rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\big)\right.\\&\quad\left.+B(s)c(s)\right]ds.\end{aligned}$$

Where,

$$S_{\gamma,\phi}(t)=I^{1-\gamma,\phi}R_{\gamma,\phi}(t)$$

and

$$\mathscr{R}_{\gamma,\phi}(t)=\int_0^{+\infty}\varphi_\gamma(\rho)T\left(t^\gamma(\phi(\rho)-\phi(0))\right)t^{\gamma-1}\phi'(\rho)d\rho.$$

Proof. Let λ be a strictly positive real number. Applying generalized Laplace transforms to the fractional equation (4) and employing Lemma 1, we derive

$$\widehat{v}(\lambda) = \frac{1}{\lambda}(v_0 + \Phi(v)) + \frac{1}{\lambda^\gamma}\big(A\widehat{v}(\lambda) + \widehat{f}(\lambda) + \widehat{Bc}(\lambda)\big),$$

where $\widehat{v}(\lambda) = \int_0^\infty \phi'(t)e^{-\lambda(\phi(t)-\phi(0))}v(t)dt$,

$$\widehat{f}(\lambda) = \int_0^\infty \phi'(t)e^{-\lambda(\phi(t)-\phi(0))}f\big(t, v(t), \int_0^t \rho(t,s)\sigma(t,s,v(s))ds\big)dt,$$

and

$$\widehat{Bc}(\lambda) = \int_0^\infty \phi'(t)e^{-\lambda(\phi(t)-\phi(0))}B(t)c(t)dt.$$

It implies that

$$\widehat{v}(\lambda) = \lambda^{\gamma-1}(\lambda^\gamma I - A)^{-1}(v_0 + \Phi(v)) + (\lambda^\gamma I - A)^{-1}\big(\widehat{f}(\lambda) + \widehat{Bc}(\lambda)\big).$$

Thus,

$$\widehat{v}(\lambda) = \lambda^{\gamma-1}\int_0^{+\infty} e^{-\lambda^\gamma s}T(s)(v_0 + \Phi(v))ds + \int_0^{+\infty} e^{-\lambda^\gamma s}T(s)\big(\widehat{f}(\lambda) + \widehat{Bc}(\lambda)\big)ds.$$

Choosing $s = \tau^\gamma$ with $\tau = \phi(t) - \phi(0)$, we get

$$\begin{aligned}
\widehat{v}(\lambda) &= \gamma\lambda^{\gamma-1}\int_0^{+\infty} \tau^{\gamma-1}e^{-(\lambda\tau)^\gamma}T(\tau^\gamma)(v_0 + \Phi(v))d\tau \\
&\quad + \gamma\int_0^{+\infty} \tau^{\gamma-1}e^{-(\lambda\tau)^\gamma}T(\tau^\gamma)\big(\widehat{f}(\lambda) + \widehat{Bc}(\lambda)\big)d\tau \\
&:= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \gamma\lambda^{\gamma-1}\int_0^{+\infty}\Big[(\phi(t)-\phi(0))^{\gamma-1}e^{-\lambda^\gamma(\phi(t)-\phi(0))^\gamma} \\
&\qquad \times T((\phi(t)-\phi(0))^\gamma)(v_0+\Phi(v))\phi'(t)\Big]dt \\
&= \gamma\lambda^{\gamma-1}\int_0^{+\infty}\int_0^{+\infty}\Big[e^{-\lambda(\phi(t)-\phi(0))(\phi(\rho)-\phi(0))}\omega_\gamma(\rho)\phi'(\rho) \\
&\qquad \times T((\phi(t)-\phi(0))^\gamma)(v_0+\Phi(v))(\phi(t)-\phi(0))^{\gamma-1}\phi'(t)\Big]dtd\rho \\
&= \gamma\lambda^{\gamma-1}\int_0^{+\infty}\int_0^{+\infty}\Big[e^{-\lambda(\phi(t)-\phi(0))} \\
&\qquad \times T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^\gamma \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^\gamma}\omega_\gamma(\rho)(v_0+\Phi(v))\phi'(\rho)\phi'(t))\Big]dtd\rho
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^{\gamma-1} \int_0^{+\infty} \Big[e^{-\lambda(\phi(t)-\phi(0))} \int_0^{+\infty} \Big(\gamma \omega_\gamma(\rho) \\
&\quad \times T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)} \right)^{\gamma} \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}} (v_0+\Phi(v)) \phi'(\rho) d\rho \Big) \phi'(t) \Big] dt \\
&= \lambda^{\gamma-1} \mathscr{L}_\phi (\mathscr{R}_{\gamma,\phi}(\phi(t)-\phi(0)))(v_0+\Phi(v)),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{R}_{\gamma,\phi}(\phi(t)-\phi(0)) = \int_0^{+\infty} \Big[& \gamma \omega_\gamma(\rho) \\
& \times T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)} \right)^{\gamma} \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}} (v_0+\Phi(v)) \phi'(\rho) \Big] d\rho.
\end{aligned}
$$

We can write $\mathscr{R}_{\gamma,\phi}(t) = \int_0^{+\infty} \varphi_\gamma(\rho) T\left(t^\gamma(\phi(\rho)-\phi(0))\right) t^{\gamma-1} \phi'(\rho) d\rho$, such that

$$
\varphi_\gamma(\rho) = (\phi(\rho)-\phi(0))^{-\frac{1}{\gamma}} \omega_\gamma \Big[\phi^{-1} \Big(\big(\frac{1}{\phi(\rho)-\phi(0)} \big)^{\frac{1}{\gamma}} + \phi(0) \Big) \Big].
$$

On other hand, we have $\mathscr{L}_\phi\big(\frac{\left(\phi(t)-\phi(0)\right)^{-\gamma}}{\Gamma(1-\gamma)}\big)(\lambda) = \lambda^{\gamma-1}$. Then,

$$
\begin{aligned}
I_1 &= \mathscr{L}_\phi \Big(\frac{(\phi(.)-\phi(0))^{-\gamma}}{\Gamma(1-\gamma)} * R_{\gamma,\phi}(.) \Big)(t)(v_0+\Phi(v)) \\
&= \mathscr{L}_\phi \big(I^{1-\gamma,\phi} R_{\gamma,\phi}(t) \big)(v_0+\Phi(v)).
\end{aligned}
$$

Let $\phi_t^\tau = \phi(\tau)+\phi(t)-\phi(0)$, and let us calculate I_2, we have

$$
I_2 = \int_0^\infty e^{-\lambda^\gamma(\phi(t)-\phi(0))} T((\phi(t)-\phi(0))) \big(\widehat{f}(\lambda) + \widehat{Bc}(\lambda) \big) \phi'(t) dt
$$

Then,

$$
\begin{aligned}
I_2 &= \iint_0^\infty \Big[e^{-\lambda^\gamma(\phi(t)-\phi(0))} e^{-\lambda(\phi(s)-\phi(0))} T((\phi(t)-\phi(0))) \\
&\quad \times f\Big(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big) \phi'(s)\phi'(t) \Big] dsdt \\
&\quad + \iint_0^\infty \Big[e^{-\lambda^\gamma(\phi(t)-\phi(0))} e^{-\lambda(\phi(s)-\phi(0))} \\
&\quad \times T((\phi(t)-\phi(0))) B(s)c(s)\phi'(s)\phi'(t) \Big] dsdt \\
&= \iint_0^\infty \Big[\gamma(\phi(t)-\phi(0))^{\gamma-1} e^{-(\lambda(\phi(t)-\phi(0)))^\gamma} e^{-\lambda(\phi(s)-\phi(0))} \\
&\quad \times T((\phi(t)-\phi(0))^\gamma)\, f\Big(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big) \phi'(s)\phi'(t) \Big] dtds \\
&\quad + \iint_0^\infty \Big[\gamma(\phi(t)-\phi(0))^{\gamma-1} e^{-(\lambda(\phi(t)-\phi(0)))^\gamma} e^{-\lambda(\phi(s)-\phi(0))} \\
&\quad \times T((\phi(t)-\phi(0))^\gamma) B(s)c(s)\phi'(s)\phi'(t) \Big] dsdt
\end{aligned}
$$

$$
\begin{aligned}
&= \iiint_0^\infty \Big[\gamma(\phi(t)-\phi(0))^{\gamma-1} e^{-\lambda(\phi(t)-\phi(0))(\phi(\rho)-\phi(0))} e^{-\lambda(\phi(s)-\phi(0))}\omega_\gamma(\rho) \\
&\quad \times T((\phi(t)-\phi(0))^\gamma) f\Big(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big)\phi'(\rho)\phi'(s)\phi'(t)\Big] d\rho ds dt \\
&\quad + \iiint_0^\infty \Big[\gamma(\phi(t)-\phi(0))^{\gamma-1} e^{-\lambda(\phi(t)-\phi(0))(\phi(\rho)-\phi(0))} e^{-\lambda(\phi(s)-\phi(0))}\omega_\gamma(\rho) \\
&\quad \times T(\phi(t)-\phi(0))^\gamma B(s)c(s)\phi'(\rho)\phi'(s)\phi'(t)\Big] d\rho ds dt.
\end{aligned}
$$

which implies that,

$$
\begin{aligned}
I_2 &= \iiint_0^\infty \Big[\gamma e^{-\lambda(\phi(t)+\phi(s)-2\phi(0))} T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^\gamma \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^\gamma}\omega_\gamma(\rho) \\
&\quad \times f\Big(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big)\phi'(\rho)\phi'(s)\phi'(t)\Big] d\rho ds dt \\
&\quad + \iiint_0^\infty \Big[\gamma e^{-\lambda(\phi(t)+\phi(s)-2\phi(0))} T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^\gamma \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^\gamma}\omega_\gamma(\rho) \\
&\quad \times B(s)c(s)\phi'(\rho)\phi'(s)\phi'(t)\Big] d\rho ds dt.
\end{aligned}
$$

So,

$$
\begin{aligned}
I_2 &= \iiint_0^\infty \Bigg[\gamma e^{-\lambda(\phi(\tau)-\phi(0))} T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^\gamma \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^\gamma}\omega_\gamma(\rho) \\
&\quad \times f\Big(\phi^{-1}(\phi_t^\tau), v(\phi^{-1}(\phi_t^\tau)), \int_0^{\phi^{-1}(\phi_t^\tau)} \rho(\phi^{-1}(\phi_t^\tau),\tau)\sigma(\phi^{-1}(\phi_t^\tau),\tau,v(\tau))d\tau\Big) \\
&\quad \times \phi'(\rho)\phi'(\tau)\phi'(t)\Bigg] d\rho d\tau dt \\
&\quad + \iiint_0^\infty \Bigg[\gamma e^{-\lambda(\phi(\tau)-\phi(0))} T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^\gamma \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^\gamma}\omega_\gamma(\rho) \\
&\quad \times B\left(\phi^{-1}(\phi_t^\tau)\right) c\left(\phi^{-1}(\phi_t^\tau)\right)\phi'(\rho)\phi'(\tau)\phi'(t)\Bigg] d\rho\, d\tau\, dt.
\end{aligned}
$$

And thus,

$$
\begin{aligned}
I_2 &= \iiint_0^\infty \Bigg[\gamma e^{-\lambda(\phi(\tau)-\phi(0))} T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^\gamma \frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^\gamma}\omega_\gamma(\rho) \\
&\quad \times f\Big(\phi^{-1}(\phi_t^\tau), v(\phi^{-1}(\phi_t^\tau)), \int_0^{\phi^{-1}(\phi_t^\tau)} \rho(\phi^{-1}(\phi_t^\tau),\xi)\sigma(\phi^{-1}(\phi_t^\tau),\xi,v(\xi))d\xi\Big) \\
&\quad \times \phi'(\rho)\phi'(\tau)\phi'(t)\Bigg] d\rho dt d\tau
\end{aligned}
$$

$$+\iiint_0^{\infty}\left[\gamma e^{-\lambda(\phi(\tau)-\phi(0))}T\left(\frac{\phi(t)-\phi(0)}{\phi(\rho)-\phi(0)}\right)^{\gamma}\frac{(\phi(t)-\phi(0))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}}\omega_{\gamma}(\rho)\right.$$
$$\times B\left(\phi^{-1}\left(\phi_t^{\tau}\right)\right)c\left(\phi^{-1}\left(\phi_t^{\tau}\right)\right)\phi'(\rho)\phi'(\tau)\phi'(t)d\rho dt d\tau.$$

Hence,

$$I_2=\int_0^{\infty}e^{-\lambda(\phi(\tau)-\phi(0))}\left(\int_0^{\tau}\int_0^{+\infty}\gamma T\left(\frac{\phi(\tau)-\phi(s)}{\phi(\rho)-\phi(0)}\right)^{\gamma}\frac{(\phi(\tau)-\phi(s))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}}\omega_{\gamma}(\rho)\right.$$
$$\left.\times f\left(s,v(s),\int_0^{s}\rho(s,\xi)\sigma(s,\xi,v(\xi))d\xi\right)\phi'(\rho)\phi'(s)d\rho\,ds\right)\phi'(\tau)d\tau$$
$$+\int_0^{\infty}e^{-\lambda(\phi(\tau)-\phi(0))}\left(\int_0^{\tau}\int_0^{+\infty}\gamma T\left(\frac{\phi(\tau)-\phi(s)}{\phi(\rho)-\phi(0)}\right)^{\gamma}\frac{(\phi(\tau)-\phi(s))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}}\omega_{\gamma}(\rho)\right.$$
$$\left.\times B(s)c(s)\phi'(\rho)\phi'(s)d\rho\,ds\right)\phi'(\tau)d\tau.$$

It follows that,

$$\widehat{v}(\lambda)=\mathscr{L}_{\phi}\left(I^{1-\gamma,\phi}R_{\gamma,\phi}(t)\right)$$
$$+\int_0^{\infty}e^{-\lambda(\phi(t)-\phi(0))}\left[\int_0^{t}\int_0^{+\infty}\gamma T\left(\frac{\phi(t)-\phi(s)}{\phi(\rho)-\phi(0)}\right)^{\gamma}\frac{(\phi(t)-\phi(s))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}}\omega_{\gamma}(\rho)\right.$$
$$\left.\times f\left(s,v(s),\int_0^{s}\rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right)\phi'(\rho)\phi'(s)d\rho\,ds\right]\phi'(t)dt$$
$$+\int_0^{\infty}e^{-\lambda(\phi(t)-\phi(0))}\left[\int_0^{t}\int_0^{+\infty}\gamma T\left(\frac{\phi(t)-\phi(s)}{\phi(\rho)-\phi(0)}\right)^{\gamma}\frac{(\phi(t)-\phi(s))^{\gamma-1}}{(\phi(\rho)-\phi(0))^{\gamma}}\omega_{\gamma}(\rho)\right.$$
$$\left.\times B(s)c(s)\phi'(\rho)\phi'(s)d\rho\,ds\right]\phi'(t)dt.$$

By applying the inverse Laplace transform, we derive

$$\begin{aligned}v(t)&=I^{1-\gamma,\phi}R_{\gamma,\phi}(t)(v_0+\Phi(v))+\int_0^{t}R_{\gamma,\phi}(\phi(t)-\phi(s))\\&\quad\times\left[f\left(s,v(s),\int_0^{s}\rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right)+B(s)c(s)\right]\phi'(s)ds\\&=S_{\gamma,\phi}(t)(v_0+\Phi(v))+\int_0^{t}R_{\gamma,\phi}(\phi(t)-\phi(s))\\&\quad\times\left[f\left(s,v(s),\int_0^{s}\rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right)+B(s)c(s)\right]\phi'(s)ds.\end{aligned}$$

Which completes the proof. □

Throughout the paper, we assume the following about A.

H(A): A is an infinitesimal generator of a strongly continuous semigroup $\{T(s):s\geqslant 0\}$ in X, and there is a real constant $M>1$ with

$$\sup_{s\in[0,\infty)}\{|T(s)|_{L(X)}\}\leqslant M.$$

PROPOSITION. *We assume that $(H(A))$ holds, then*

1. *For a fixed strictly positive real number t, the operators $\{R_{\gamma,\phi}(t)\}_{t>0}$ and $\{S_{\gamma,\phi}(t)\}_{t>0}$ are linear.*
2. *the operators $\{R_{\gamma,\phi}(t)\}_{t>0}$ and $\{S_{\gamma,\phi}(t)\}_{t>0}$ are strongly continuous.*
3. *For $y \in X$, then $\|R_{\gamma,\phi}(t)y\| \leqslant \frac{t^{\gamma-1}M}{\Gamma(1+\gamma)}\|y\|$ and $\|S_{\gamma,\phi}(t)y\| \leqslant \frac{M}{\gamma}\|y\|$.*

Proof. Since $\displaystyle\int_0^{+\infty} \gamma\rho\omega_\gamma(\rho)\phi'(\rho)d\rho = \frac{1}{\Gamma(1+\gamma)}$. Then,

$$\| R_{\gamma,\phi}(t)x \| \leqslant \frac{t^{\gamma-1}M}{\Gamma(1+\gamma)}\| x \|.$$

From the above inequality it follows that

$$\begin{aligned}\left\|S_{\gamma,\phi}(\phi(t)-\phi(0))y\right\| &= \left\|I^{1-\gamma,\phi}R_{\gamma,\phi}(\phi(t)-\phi(0))y\right\| \\ &\leqslant \frac{MI^{1-\gamma,\phi}(\phi(t)-\phi(0))^{\gamma-1}}{\Gamma(1+\gamma)}\|y\| \\ &\leqslant \frac{M\Gamma(\gamma)}{\Gamma(1+\gamma)}\|y\| \\ &\leqslant \frac{M}{\gamma}\|y\|.\end{aligned}$$

Which implies that $\|S_{\gamma,\phi}(t)y\| \leqslant \frac{M}{\gamma}\|y\|$. Moreover, let $x \in X_0$ and $0 < t_1 < t_2 \leqslant T$, from a simple calculation, it follows that $\lim_{t_1\to t_2}\left\|R_{\gamma,\phi}(t_1)y - R_{\gamma,\phi}(t_2)y\right\| = 0$ and $\lim_{t_1\to t_2}\left\|S_{\gamma,\phi}(t_1)x - S_{\gamma,\phi}(t_2)y\right\| = 0$. □

Now, we introduce the following hypotheses:

(H1) $\Phi : X \to X$ is continuous and there exist two positive real constants K and d such that $\|\Phi(v)-\Phi(u)\| \leqslant K\|v-u\|$ and $\|\Phi(v)\| \leqslant K\|v\|+d$, for every $v \in \mathscr{C}$.

(H2) The function $f : [0,b]\times X\times X \to X$ is of Carathéodory type, meaning that $f(t,.,.)$ is continuous for almost every $t \in [0,b]$ and $f(.,v,Gv)$ is measurable for every $v \in X$.

(H3)
 (i) There is a non-decreasing continuous function $\varphi_f : X \to X$ and $m_f \in \mathscr{C}([0,b],X)$ such that $\|f(t,u,v)\| \leqslant m_f\varphi_f(\|v\|+\|u\|)$, for every $u,v \in X$ almost every $t \in [0,b]$.
 (ii) There exists an integrable function $\eta : [0,b] \to [0,+\infty)$ with $\mu(f(t,C_1,C_2) \leqslant \eta(t)[\sup_{-\infty<\theta\leqslant 0}\mu(C_1(\theta)+\mu(C_2)]$ for almost every $t \in [0,b]$ and each bounded subsets $C_1,C_2 \subset X$, μ is the Hausdorff noncompactness measure and

$$\int_0^t \eta(s)ds \leqslant \kappa^*.$$

(H4) (i) $\sigma(t,s,.) : X \to X$ is continuous function for any $(t,s) \in \Delta$, and for every $v \in X$, $\sigma(.,.,v) : \Delta \to X$ is measurable function. Moreover, there is a function $u : \Delta \to \mathbb{R}^+$ such that $\sup_{t\in[0,b]} \int_0^t u(t,s)ds := u^* < \infty$ and $\|\sigma(t,s,v)\| \leqslant u(t,s)\|v\|$, $v \in X$.

(ii) For every C_1, bounded subset of X and $0 \leqslant s \leqslant t \leqslant b$, there is a $\varphi : \Delta \to R^+$ such that $\mu(\sigma(t,s,C_1)) \leqslant \varphi(s,t)\mu(C_1)$ where $\sup_{t\in[0,b]} \int_0^t \varphi(s,t)ds := \varphi^*$.

(H5) For any $t \in [0,b]$, the function $\rho(t,.)$ is measurable on $[0,t]$ and $\sup\{|\rho(t,s)|, 0 \leqslant s \leqslant t\} = \ell(t)$ is bounded on $[0,b]$.

THEOREM 2. *Under the conditions* $(H1)$–$(H5)$ *and* $(H(A))$, *the problem* (1) *admits at least one mild solution, provided that*

$$M\Big(\frac{K}{\gamma} + \frac{2\kappa^*}{\Gamma(1+\gamma)}(\phi(b)-\phi(0))^\gamma\big[1+2l\varphi^*\big]\Big) < 1.$$

Proof. We define a mapping $\mathit{\Gamma} : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$\begin{aligned} \mathit{\Gamma} v(t) = S_{\gamma,\phi}(t)(v_0 + \Phi(v)) + \int_0^t \Big[R_{\gamma,\phi}(\phi(t)-\phi(s)) \\ \times \Big(f\Big(s,v(s),\int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big) + B(s)c(s)\Big)\Big]\phi'(s)ds. \end{aligned}$$

The proof consists of several steps.

Step 1: We will show that the operator $\mathit{\Gamma}$ is continuous. Let $\{v_n\}$ be a sequence such that v_n converge to v in $\mathscr{C}$. Then, by $(H2)$, when $n \to \infty$ we have

$$f\Big(s,v_n(s),\int_0^s \rho(s,\tau)\sigma(s,\tau,v_n(\tau))d\tau\Big) \to f\Big(s,v(s),\int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big),$$

for every $s \in [0,b]$. To simplify the expressions, we pose that

$$\begin{aligned} K_{f_n} &= f\Big(s,v_n(s),\int_0^s \rho(s,\tau)\sigma(s,\tau,v_n(\tau))d\tau\Big), \\ K_f &= f\Big(s,v(s),\int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\Big). \end{aligned}$$

For any $t \in [0,b]$, we know that

$$\begin{aligned} &\|\mathit{\Gamma} v_n(t) - \mathit{\Gamma} v(t)\| \\ &\quad = \|S_{\gamma,\phi}(t)(v_0+\Phi(v_n)) - S_{\gamma,\phi}(t)(v_0+\Phi(v)) + \int_0^t R_{\gamma,\phi}(\phi(t)-\phi(s)) \\ &\quad\quad \times \Big[K_{f_n} + B(s)c(s) - K_f - B(s)c(s)\Big]\phi'(s)ds\| \end{aligned}$$

$$\begin{aligned}
&\leqslant \|S_{\gamma,\phi}(t)(\Phi(v_n)) - \Phi(v)))\| \\
&\quad + \int_0^t \|R_{\gamma,\phi}(\phi(t)-\phi(s))\|\phi'(s)\|K_{f_n} - K_f\| ds \\
&\leqslant \frac{MK}{\gamma}\|v_n - v\| + \frac{M}{\Gamma(1+\gamma)}\int_0^t (\phi(t)-\phi(s))^{\gamma-1}\phi'(s)\|K_{f_n} - K_f\| ds.
\end{aligned}$$

Thus, we infer that $\|\Gamma v_n - \Gamma v\|_\infty \to 0$ as $n \to \infty$, which implies that the mapping Γ is continuous on $\mathscr{C}([0,b],X)$.

Step 2: Γ is a bounded operator. We claim that $\Gamma(B_r) \subset B_r$ for every $v \in B_r$, for each $t \in [0,b]$, we know that

$$\begin{aligned}
\|\Gamma v(t)\| &\leqslant \frac{M}{\gamma}(\|v_0\| + K\|v\| + d) + \frac{M}{\Gamma(1+\gamma)}\int_0^t (\phi(t)-\phi(s))^{\gamma-1} \\
&\quad \times \left[\left\|f\left(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right)\right\| + \|B(s)c(s)\|\right]\phi'(s)ds \\
&\leqslant \frac{M}{\gamma}(\|v_0\| + K\|v\| + d) \\
&\quad + \frac{M(\phi(b)-\phi(0))^\gamma}{\Gamma(1+\gamma)}\left[m_f\varphi_f\left(\|v\| + l\int_0^s u(s,\tau)d\tau\|v\|\right) + \|Bc\|_{L^1}\right] \\
&\leqslant \frac{M}{\gamma}(\|v_0\| + Kr + d) + \frac{M(\phi(b)-\phi(0))^\gamma}{\Gamma(1+\gamma)}\left[m_f\varphi_f r(1+lu^*) + \|Bc\|_{L^1}\right] \\
&\leqslant r.
\end{aligned}$$

Step 3: $\Gamma(B_r)$ is equicontinuous. For any $\tau_2, \tau_1, \in [0,b]$ such that $\tau_1 \leqslant \tau_2$, we have

$$\begin{aligned}
\|\Gamma v(\tau_2) - \Gamma v(\tau_1)\| \leqslant\ & \|S_{\gamma,\phi}(\tau_2) - S_{\gamma,\phi}(\tau_1)\|(\|v_0\| + K\|v\| + d) \\
& + \left\| \int_0^{\tau_1} (R_{\gamma,\phi}(\phi(\tau_2)-\phi(s)) - R_{\gamma,\phi}(\phi(\tau_1)-\phi(s))) \right. \\
& \times \left[f\left(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right) + B(s)c(s)\right]\phi'(s)ds \\
& + \int_{\tau_1}^{\tau_2} R_{\gamma,\phi}(\phi(\tau_2)-\phi(s)) \\
& \left. \times \left[f\left(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right) + B(s)c(s)\right])\phi'(s)ds \right\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|\Gamma v(\tau_2) - \Gamma v(\tau_1)\| \leqslant\ & \|S_{\gamma,\phi}(\tau_2) - S_{\gamma,\phi}(\tau_1)\|(\|v_0\| + K\|v\| + d) \\
& + \int_0^{\tau_1} \|R_{\gamma,\phi}(\phi(\tau_2)-\phi(s)) - R_{\gamma,\phi}(\phi(\tau_1)-\phi(s))\| \\
& \times \left[m_f\varphi_f r(1+lu^*) + \|Bc\|_{L^1}\right]\phi'(s)ds \\
& + \int_{\tau_1}^{\tau_2} \|R_{\gamma,\phi}(\phi(\tau_2)-\phi(s))\|\left[m_f\varphi_f r(1+lu^*) + \|Bc\|_{L^1}\right]\phi'(s)ds.
\end{aligned}$$

Step 4: The Mönch condition is satisfied. Suppose that $V \subset B_r$ is a countable set and V is subset of $\overline{conv}(\{0\} \cup \Gamma(V))$. We aim to demonstrate that $\mu(V) = 0$ where μ denotes the Hausdorff denotes the Hausdorff noncompactness measure. Without loss of generality, we may suppose $V = \{u\}_{k=1}^{\infty}$, and it can be easily verified that V is both bounded and equicontinuous. For every $t \in [0,b]$, we have the following

$$\begin{aligned}\Gamma v(t) &= S_{\gamma,\phi}(t)(v_0 + \Phi(v)) + \int_0^t R_{\gamma,\phi}(\phi(t) - \phi(s)) \\ &\quad \times \left[f\left(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau\right) + B(s)c(s)\right]\phi'(s)ds \\ &= \Gamma_1 v(t) + \Gamma_2 v(t),\end{aligned}$$

where $\Gamma_1 v(t) = S_{\gamma,\phi}(t)(v_0 + \Phi(v))$ and

$$\begin{aligned}\Gamma_2 v(t) = &\int_0^t R_{\gamma,\phi}(\phi(t) - \phi(s)) \\ &\times \left[f\left(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau)\right)d\tau) + B(s)c(s)\right]\phi'(s)ds.\end{aligned}$$

Moreover, $\Gamma_1 : V \to \mathscr{C}([0,b],X)$ is Lipschitz continuous with constant KM_γ due to the hypothesis $(H1)$. In fact v and u two elements of V, we have the following

$$\begin{aligned}\|\Gamma_1 v(t) - \Gamma_1 u(t)\| &\leqslant \sup_{t\in[0,b]} \|S_{\gamma,\phi}(t)\| \|\Phi(v) - \Phi(u)\| \\ &\leqslant \frac{MK}{\gamma}\|v - u\|_\infty.\end{aligned}$$

So, from Lemmas 3, 4 and 5 and hypotheses $(H3)(ii)$, $(H4)(ii)$, we have

$$\begin{aligned}\mu(\{\Gamma v_n\}_{k=1}^{\infty}) &\leqslant \mu(\{\Gamma_1 v_n\}_{k=1}^{\infty}) + \mu(\{\Gamma_2 v_n\}_{k=1}^{\infty}) \\ &\leqslant \frac{MK}{\gamma}\mu(\{v_n\}_{k=1}^{\infty}) + \mu\left(\int_0^t R_{\gamma,\phi}(\phi(t) - \phi(s))\right. \\ &\quad \left. \times \left[f\left(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau)\right)d\tau) + B(s)c(s)\right]\phi'(s)ds\right) \\ &\leqslant \frac{MK}{\gamma}\mu(\{v_n\}_{k=1}^{\infty}) + \frac{2M}{\Gamma(1+\gamma)}\int_0^t \mu\left((\phi(t) - \phi(s))^{\gamma-1}\right. \\ &\quad \times \left[f(s, v(s), \int_0^s \rho(s,\tau)\sigma(s,\tau,v(\tau))d\tau) + B(s)c(s)\right)\phi'(s)\Big]ds.\end{aligned}$$

So,

$$\begin{aligned}\mu(\{\Gamma v_n\}_{k=1}^{\infty}) \leqslant &\frac{MK}{\gamma}\mu(\{v_n\}_{k=1}^{\infty}) + \frac{2M}{\Gamma(1+\gamma)}\int_0^t (\phi(t) - \phi(s))^{\gamma-1}\eta(s) \\ &\times \left[\sup_{0<s\leqslant b} \mu(\{v_n\}_{k=1}^{\infty}(s)) + \mu\left(\int_0^s \rho(s,\tau)\sigma(s,\tau,\{v_n\}_{k=1}^{\infty}(\tau))d\tau\right)\right]ds\end{aligned}$$

$$\leqslant \frac{MK}{\gamma}\mu(\{v_n\}_{k=1}^{\infty}) + \frac{2M}{\Gamma(1+\gamma)}\int_0^t (\phi(t)-\phi(s))^{\gamma-1}\eta(s)$$
$$\times \Big[\sup_{0<s\leqslant b}\mu(\{v_n\}_{k=1}^{\infty}(s)) + 2l\int_0^s \varphi(s,\tau)d\tau\mu\{v_n\}_{k=1}^{\infty}(\tau)\Big]ds$$
$$\leqslant \frac{MK}{\gamma}\mu(\{v_n\}_{k=1}^{\infty}) + \frac{2M}{\Gamma(1+\gamma)}\int_0^t (\phi(t)-\phi(s))^{\gamma-1}\eta(s)$$
$$\times \Big[\sup_{0<s\leqslant b}\mu(\{v_n\}_{k=1}^{\infty}(s)) + 2l\varphi^*\mu\{v_n\}_{k=1}^{\infty}(\tau)\Big]ds$$
$$\leqslant \frac{MK}{\gamma}\mu(\{v_n\}_{k=1}^{\infty}) + \frac{2M\kappa^*}{\Gamma(1+\gamma)}(\phi(b)-\phi(0))^{\gamma}\big[1+2l\varphi^*\big]$$
$$\times \sup_{0<s\leqslant b}\mu_C(V(s))$$
$$\leqslant M\Big(\frac{K}{\gamma} + \frac{2\kappa^*}{\Gamma(1+\gamma)}(\phi(b)-\phi(0))^{\gamma}\big[1+2l\varphi^*\big]\Big)\sup_{0<s\leqslant b}\mu_C(V(s)).$$

From above, we have

$$\mu_C(\mathit{\Gamma}(V)) \leqslant M\Big[\frac{K}{\gamma} + \frac{2\kappa^*}{\Gamma(1+\gamma)}(\phi(b)-\phi(0))^{\gamma}[1+2l\varphi^*]\Big]\mu_C(V)$$
$$\leqslant \varpi^*\mu_C(V),$$

where $\varpi^* = M\Big[\frac{K}{\gamma} + \frac{2\kappa^*}{\Gamma(1+\gamma)}(\phi(b)-\phi(0))^{\gamma}\big[1+2l\varphi^*\big]\Big]$. Thus

$$\mu_C(V) \leqslant \mu_C((\overline{conv\{0\}}) \cup \mathit{\Gamma}(V)) = \mu_C(\mathit{\Gamma}(V)) \leqslant \varpi^*\mu_C(V),$$

which leads to the conclusion that $\mu_C(V) = 0$. Therefore, by applying Theorem 1, there exists a fixed point v of the mapping $\mathit{\Gamma}$ in B_r, which serves as a solution to the fractional problem (1). □

4. Application

In this section, we consider $Y = L^2([0,\pi],\mathbb{R})$ and A is an operator defined by $Av = v$" with the domain

$$D(A) = \{y \in Y : v\text{"} \in Y,\ v(\pi) = v(0) = 0\}.$$

It is widely recognized that A is the bounded linear operator of $\{T(t), t \geqslant 0\}$, a compact semigroup on Y with $\|T(t)\| \leqslant e^{-t}$ for every $t \geqslant 0$. Let's consider a nonlocal differential problem given below

$$\begin{cases} {}^{C}D_{0^+}^{\gamma,\phi}v(t,w) = \frac{\partial^2}{\partial w^2}v(t,w) + \int_0^t \sigma(t,s,v(s,w))ds + \mathscr{P}(t,v(t,w)) + c(t,w), \\ v(0,w) + \sum_{j=1}^{n} c_j u(t_j,w) = v_0(w), \quad w \in [0,\pi]. \end{cases} \tag{5}$$

with $0 = t_0 < t_1 \leqslant t_2 \ldots\ldots \leqslant t_n \leqslant b$ are fixed real numbers, $v_0 \in Y$, $\mathscr{P} \in \mathscr{C}([a,b]\times\mathbb{R},\mathbb{R})$.

To represent the problem (5) in the abstract form problem (1), we suppose that

(i) $f:[0,b]\times Y\to Y$ defined by $f(t,v)(w)=\int_0^t \sigma(t,s,v(s,w))ds+\mathscr{P}(t,v(t,w),)$ for $t\in[0,b]$ and $w\in[0,\pi]$.

(ii) The function $\Phi:\mathscr{C}([0,b],Y)\to Y$ is continuous and defined by $\Phi(v)(w)=v_0(w)-\sum_{j=1}^{n}c_ju(t_j,w)$, $t\in[0,b]$, $w\in[0,\pi]$, where $v(t)(w)=v(t,w)$, for all $t\geqslant 0$ and $w\in[0,\pi]$.

Now, we say that $v\in\mathscr{C}(Y)$ is a mild solution of the problem (5) if $v(.)$ is a mild solution of the associated abstract problem (1).

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

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Data availability. The data used to support the findings of this study are included in the references within the article.

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REFERENCES

[1] R. ALMEIDA, A. B. MALINOWSKA, M. T. T. MONTEIRO, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Math. Methods Appl. Sci., **41**, no. 1, 336–352 (2018).

[2] J. BANAS, *Measure of noncompactness in Banach space*, Comment. Math. Univ. Carol., **21**, no. 1, 131–143 (1980).

[3] D. BOTHE, *Multivalued perturbation of m-accretive differential inclusions*, J. Math., **108**, 109–138 (1998).

[4] K. HILAL, A. KAJOUNI, N. CHEFNAJ, *Existence of Solution for a Conformable Fractional Cauchy Problem with Nonlocal Condition*, Int. J. Differ. Equ., (2022).

[5] F. JARAD, T. ABDELJAWAD, *Generalized fractional derivatives and Laplace transform*, Discrete Contin. Dyn. Syst. Ser., **13**, no. 3, 709–722 (2019).

[6] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Elsevier, **204**, (2006).

[7] F. MAINARDI, *On the initial value problem for the fractional diffusion-wave equation*, Waves Stab. Contin. Media, World Sci., **1994**, 246–251 (1994).

[8] K. MALAR, A. ANGURAJ, *Existence Results of Abstract Impulsive Integrodifferential Systems with Measure of Non-compactness*, J. Stat. Sci. Appl., **4**, no. 03–04, 108–117 (2016).

[9] H. MALGHI, A. TAQBIBT, M. EL OMARI, A. QUAFFOU, K. HILAL, *Initial value problems for nonlinear neutral delay ψ-Caputo fractional hybrid differential equations with bounded delayq*, J. Math. Sci. (2025), https://doi.org/10.1007/s10958-024-07485-3.

[10] H. MALGHI, A. TAQBIBT, K. HILAL, A. QUAFFOU, *Ulam-Hyers stability analysis for hybrid ϕ-Caputo time-fractional systems of order δ in (1; 2]*, Gulf Journal of Mathematics, **18** (1), 95–117 (2024), https://doi.org/10.56947/gjom.v18i1.2313.

[11] S. MELLIANI, A. EL ALLAOUI, L. S. CHADLI, *A general class of periodic boundary value problems for controlled nonlinear impulsive evolution equations on Banach spaces*, Adv. Differ. Equ., (2016).

[12] H. MÖNCH, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal., **4**, 985–999 (1980).

[13] I. PODLUBNY, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier (1998).

[14] A. TAQBIBT, C. CHEFNAJ, K. HILAL, S. MELLIANI, *ϕ-Caputo fractional differential equations with maxima on time scales*, J. Math. Sci. (2024), https://doi.org/10.1007/s10958-024-07034-y.

[15] Y. ZHOU, *Basic theory of fractional differential equations*, World Scientific (2023).

[16] J. ZUO, A. TAQBIBT, M. CHAIB, J. VANTERLER, M. ELOMARI, *An existence and uniqueness of mild solutions of fractional evolution problems*, Computational and Applied Mathematics, **43** (8) (2024), https://doi.org/10.1007/s40314-024-02943-9.

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Najat Chefnaj
LMACS
Sultan Moulay Slimane University
FST, Beni Mellal, 23000, Morocco
e-mail: chefnaj.fst@gmail.com

Abdellah Taqbibt
LMACS
Sultan Moulay Slimane University
FST, Beni Mellal, 23000, Morocco
e-mail: abdellah.taqbibt@usms.ma

Khalid Hilal
LMACS
Sultan Moulay Slimane University
FST, Beni Mellal, 23000, Morocco
e-mail: hilalkhalid2005@yahoo.fr

M'hamed Elomari
LMACS
Sultan Moulay Slimane University
FST, Beni Mellal, 23000, Morocco
e-mail: m.elomari@usms.ma