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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO TERMINAL VALUE PROBLEMS FOR FRACTIONAL–ORDER DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS

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Abstract. Our aim is to define the Dzhrbashyan Nersesyan Liouville fractional derivative, elucidate its properties, and apply Banach's fixed point theorem to prove the existence and uniqueness of solutions to terminal value problems for fractional differential equations with advanced arguments involving the Dzhrbashyan Nersesyan Liouville fractional derivative. We provide some examples to showcase the practical application of our results.

1. Introduction

Fractional differential equations are encountered in various scientific fields, such as electromagnetic theory, biology, electrical circuits, electroanalytical chemistry, viscoelasticity, calculus of probabilities, and among others (see [11], [13], [14], [17], and the references cited in [22]).

On the other hand, number theory, theoretical physics, models of cellular proliferation in biology, control theory, and the absorption of light in an interstellar medium all include the use of differential equations with advanced arguments (see [1], [2], [3], [7], [9], [20], [23], and the references mentioned in them).

The existence of solutions for terminal value problems related to fractional order differential equations on semi-infinite intervals is contingent upon their equivalence to a Volterra integral equation; however, the composition formula for the Liouville-Weyl fractional integral and derivative is inapplicable over semi-infinite intervals. For this reason, the authors in [11] introduced the modified Liouville fractional integral and derivative, which are alterations in the kernel of the Liouville-Weyl fractional integrals and derivatives on the half-axis, to tackle this problem. The authors investigated the properties of the modified Liouville fractional integral and derivative within the realm of integrable functions and, utilizing Banach's fixed point theorem, established the existence and uniqueness of solutions for the following problem:

$$\begin{cases} \left(D_{-}^{\alpha}\right)(x) = \widetilde{f}\left(x,u\left(x\right)\right), x \in \left[1,+\infty\right), \\ \lim\limits_{x\to+\infty} \left(\mathscr{D}_{-}^{\alpha-k}u\right)(x) = b_k \in \mathbb{R},\ k = 1,\ldots,n = -\left[-\alpha\right], \end{cases}$$

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where $\left(D_{-}^{\alpha}\right)(u)(x)=\left(-x^{2}\frac{d}{dx}\right)^{n}\left(J_{-}^{n-\alpha}u\right)(x)=\left(-x^{2}\frac{d}{dx}\right)^{n}\int\limits_{x}^{+\infty}(xt)^{n-\alpha}\frac{u(t)}{t^{2}(t-x)^{n-\alpha}}dt$ is the modified Liouville fractional derivative of order $\alpha>0$, $\widetilde{f}:[1,+\infty)\times O\to\mathbb{R}$, with O is an open subset of $\mathbb{R}$ and $\widetilde{f}$ is such that $\widetilde{f}(x,y)\in \pounds(1;+\infty)$ for any $y\in O$, and globally Lipschitz with respect to the second variable.

After this study, the authors in [10] employed the modified Liouville fractional integral and derivative in the study of integral equations, while the authors in [4] and [5] utilized both the modified Liouville fractional integral and derivative, as well as Caputo's modified fractional derivative, in the study of terminal value problems over infinite intervals.

On the other hand, in [6], the authors present new ways to think about a number of results in the theory of Dirichlet series and quasi-analytic classes of functions. They do this by introducing the Dzhrbashyan Nersesyan fractional integral and derivative on infinite intervals.

This paper defines the Dzhrbashyan Nersesyan Liouville fractional derivative utilizing the Dzhrbashyan Nersesyan fractional integral presented in [6]. It demonstrates some properties of this derivative and from these results, we establish the equivalence of the terminal value problems for fractional differential equations and of the Volterra integral equation. Subsequently, we apply Banach's fixed point theorem to investigate the existence and uniqueness of solutions for the following terminal value problems

$$\begin{cases} D_{-}^{\alpha}u(x)=f\left(x,u(x),u(\theta(x))\right),\ x\in[\sigma,+\infty), \\ \lim\limits_{x\to+\infty}e^{(\alpha-1)x}u(x)=a, \end{cases} \tag{1}$$

where D_{-}^{α} is the Dzhrbashyan Nersesyan Liouville fractional derivative of order α with $0<\alpha<1$, $f:[\sigma,+\infty)\times\mathbb{R}^{2}\to\mathbb{R}$ and $\theta:[\sigma,+\infty)\to[\sigma,+\infty)$ are continuous such that $\theta(x)\geqslant x$, $\sigma\geqslant 0$ and $a\in\mathbb{R}$, and

$$\begin{cases} \left(D_{-}^{\alpha}u\right)(x)=f\left(x,u(x),u(\theta(x))\right),\ x\in[\sigma,+\infty), \\ \lim\limits_{x\to+\infty}\left(D_{-}^{\alpha-k}u\right)(x)=a_{k},\ k=1,\ldots,n-1, \\ \lim\limits_{x\to+\infty}\left(J_{-}^{n-\alpha}u\right)(x)=a_{n}, \end{cases} \tag{2}$$

where $\alpha>0$, a_k a real numbers for $k=1,\ldots,n=-[-\alpha]$, $J_{-}^{n-\alpha}$ is the Dzhrbashyan Nersesyan Liouville fractional integral of order $n-\alpha$, $f:[\sigma,+\infty)\times\mathbb{R}^{2}\to\mathbb{R}$ and $\theta:[\sigma,+\infty)\to[\sigma,+\infty)$ are continuous such that $\theta(x)\geqslant x$.

Here's the outline for this paper: The remainder of the paper makes use of the definitions and preliminary results provided in Section 2. In Section 3, we present and prove the main results. In Section 4, we provide some examples that show how our results can be applied and in Section 5, we give a conclusion.

2. Preliminaries

This section contains some definitions and preliminary results that will be utilized throughout the rest of this paper.

DEFINITION 1. ([18]) For $\beta > 0$ and $x \in \mathbb{R}$, the Mittag-Leffler function E_β is defined by

$$E_\beta(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(\beta n + 1)},$$

where Γ is the Euler gamma function defined by

$$\Gamma(p) = \int_0^{+\infty} e^{-t} t^{p-1} dt,$$

with $p > 0$.

DEFINITION 2. ([18]) For $\beta_1 > 0$, $\beta_2 > 0$ and $x \in \mathbb{R}$, the Mittag-Leffler function E_{β_1,β_2} is defined by

$$E_{\beta_1,\beta_2}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(\beta_1 n + \beta_2)}.$$

DEFINITION 3. ([12]) The generalized Mittag-Leffler function $E_{\beta,m,l}$ is defined by

$$E_{\beta,m,l}(x) = \sum_{n=0}^{+\infty} c_k x^k,$$

where $c_0 = 1$, $c_k = \prod_{j=0}^{k-1} \frac{\Gamma(\beta(jm+l)+1)}{\Gamma(\beta(jm+l+1)+1)}$ with $k \in \mathbb{N}^*$, $\beta > 0$, $m > 0$, $\beta(jm+l) \notin \mathbb{Z}^-$ and $x \in \mathbb{R}$.

DEFINITION 4. ([19]) The deformed Mittag-Leffler function F_β is defined by

$$F_\beta(x,y) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(\beta n + 1)} y^{\frac{n(n-1)}{2}},$$

where $\beta > 0$, $x \in \mathbb{R}$ and $y \in [-1,1]$.

REMARK 1. The function F_β is called the deformed Mittag-Leffler function since it reduces to the Mittag-Leffler function E_β when $y = 1$.

DEFINITION 5. ([4]) The deformed Mittag-Leffler function F_{β_1,β_2} is defined by

$$F_{\beta_1,\beta_2}(x,y;\iota) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(\beta_1 n + \beta_2)} y^{\frac{n(n+1)}{2}\iota - n},$$

where $\beta_1 > 0$, $\beta_2 > 0$, $x \in \mathbb{R}$, $y \in]0,1]$ and $\iota > 0$.

REMARK 2. It is observed that when y is equal to 1, the deformed Mittag function F_{β_1,β_2} simplifies to the Mittag-Leffler function E_{β_1,β_2}.

DEFINITION 6. For all $\sigma \geqslant 0$, we note $\pounds(\sigma;+\infty)$ the following space

$$\pounds(\sigma;+\infty) = \left\{ g : \ \|g\|_{\pounds(\sigma;+\infty)} = \int_{\sigma}^{+\infty} e^{-x} |g(x)| \, dx < \infty \right\}.$$

DEFINITION 7. ([6]) Let $g : [\sigma,+\infty) \to \mathbb{R}$ be a function such that $g \in \pounds(\sigma;+\infty)$. For $\alpha > 0$, the Dzhrbashyan Nersesyan fractional integral of order α of g is defined by

$$\left(J_{-}^{\alpha} g\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} g(t) \, dt, \text{ for a.e. } x \in [\sigma,+\infty).$$

REMARK 3. If $\alpha = n \in \mathbb{N}^*$, the above definition coincide the n-th integral of the form

$$\begin{aligned} \left(J_{-}^{n} g\right)(x) &= \int_{x}^{+\infty} \left(e^{-x} - e^{-t_1}\right) e^{-t_1} dt_1 \int_{t_1}^{+\infty} \left(e^{-x} - e^{-t_2}\right) e^{-t_2} dt_2 \\ &\quad \ldots \int_{t_{n-1}}^{+\infty} \left(e^{-x} - e^{-t_{n-1}}\right) e^{-t_{n-1}} g(t_{n-1}) \, dt_{n-1} \\ &= \frac{1}{\Gamma(n)} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{n-1} e^{-t} g(t) \, dt. \end{aligned}$$

EXAMPLE 1. If $g(x) = b \in \mathbb{R}$, then for all $\alpha > 0$ and $x \geqslant \sigma$, we have

$$\begin{aligned} \left(J_{-}^{\alpha} g\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} b dt \\ &= \frac{b}{\Gamma(\alpha+1)} e^{-\alpha x}. \end{aligned}$$

EXAMPLE 2. If $g(x) = e^{\beta x}$ with $\beta < 1$, then for all $\alpha > 0$ and $x \geqslant \sigma$, we have

$$\begin{aligned} \left(J_{-}^{\alpha} g\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} e^{\beta t} dt \\ &= \frac{\Gamma(-\beta+1)}{\Gamma(\alpha-\beta+1)} e^{(\beta-\alpha)x}. \end{aligned}$$

EXAMPLE 3. For all $\alpha > 0$, $\mu > 0$ and $x \geqslant \sigma$, we have

$$\begin{aligned}\left(J_{-}^{\alpha}E_{\alpha}\left(\mu e^{-\alpha t}\right)\right)(x) &= \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}e^{-t}E_{\alpha}\left(\mu e^{-\alpha t}\right)dt \\ &= \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{+\infty}\frac{\mu^{n}}{\Gamma(\alpha n+1)}\int\limits_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}e^{-t}e^{-n\alpha t}dt \\ &= \sum_{n=0}^{+\infty}\frac{\mu^{n}e^{-(n+1)\alpha x}}{\Gamma((n+1)\alpha+1)} \\ &= \frac{1}{\mu}\left(E_{\alpha}\left(\mu e^{-\alpha x}\right)-1\right).\end{aligned}$$

EXAMPLE 4. For all $\alpha > 0$ with $n = -[-\alpha]$, $\mu > 0$ and $x \geqslant \sigma$, we have

$$\begin{aligned}&\left(J_{-}^{\alpha}e^{(n-\alpha)t}E_{\alpha,\alpha-n+1}\left(\mu e^{-\alpha t}\right)\right)(x) \\ &= \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{+\infty}\frac{e^{-t}e^{(n-\alpha)t}E_{\alpha,\alpha-n+1}\left(\mu e^{-\alpha t}\right)}{\left(e^{-x}-e^{-t}\right)^{1-\alpha}}dt \\ &= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{+\infty}\frac{\mu^{k}}{\Gamma(\alpha(k+1)-n+1)}\int\limits_{x}^{+\infty}\frac{e^{-(\alpha(k+1)-n+1)t}}{\left(e^{-x}-e^{-t}\right)^{1-\alpha}}dt \\ &= \sum_{k=0}^{+\infty}\frac{\mu^{k}e^{-(\alpha(k+2)-n)x}}{\Gamma(\alpha(k+2)-n+1)} \\ &= \frac{e^{(n-\alpha)x}}{\mu}\left(E_{\alpha,\alpha-n+1}\left(\mu e^{-\alpha x}\right)-\frac{1}{\Gamma(\alpha-n+1)}\right).\end{aligned}$$

We have the following results.

LEMMA 1. *For all* $\alpha > 0$, $\beta > 0$ *and* $g \in \pounds(\sigma;+\infty)$, *we have*

$$J_{-}^{\alpha}J_{-}^{\beta}g = J_{-}^{\alpha+\beta}g.$$

Proof. We have

$$\begin{aligned}\left(J_{-}^{\alpha}J_{-}^{\beta}g\right)(x) &= \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}e^{-t}\left(J_{-}^{\beta}g\right)(t)\,dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}e^{-t} \\ &\quad\times\int\limits_{t}^{+\infty}\left(e^{-t}-e^{-\tau}\right)^{\beta-1}e^{-\tau}g(\tau)\,d\tau dt\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{x}^{+\infty} e^{-\tau} g(\tau) \int_{x}^{\tau} \left(e^{-x} - e^{-t}\right)^{\alpha-1}$$
$$\times \left(e^{-t} - e^{-\tau}\right)^{\beta-1} e^{-t} dt d\tau$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{x}^{+\infty} e^{-\tau} g(\tau) \left(e^{-x} - e^{-\tau}\right)^{\alpha+\beta-1}$$
$$\times \int_{0}^{1} v^{\alpha-1} (1-v)^{\beta-1} dv d\tau$$
$$= \frac{1}{\Gamma(\alpha+\beta)} \int_{x}^{+\infty} \left(e^{-x} - e^{-\tau}\right)^{\alpha+\beta-1} e^{-\tau} g(\tau) d\tau$$
$$= \left(J_{-}^{\alpha+\beta} g\right)(x). \quad \square$$

REMARK 4. The idea of the proof of Lemma 1 is similar to that of the first formula (2.21) in [18].

LEMMA 2. *If* $\alpha > 0$*, then the Dzhrbashyan Nersesyan fractional integral of order* α *is bounded from* $\pounds(\sigma;+\infty)$ *to* $\pounds(\sigma;+\infty)$*, and we have*

$$\left\|J_{-}^{\alpha} g\right\|_{\pounds(\sigma;+\infty)} \leqslant \frac{e^{-\alpha\sigma}}{\Gamma(\alpha+1)} \|g\|_{\pounds(\sigma;+\infty)}.$$

Proof. We have

$$\left\|J_{-}^{\alpha} g\right\|_{\pounds(\sigma;+\infty)}$$
$$= \int_{\sigma}^{+\infty} e^{-x} \left|\left(J_{-}^{\alpha} g\right)(x)\right| dx$$
$$= \int_{\sigma}^{+\infty} e^{-x} \left|\frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} g(t) dt\right| dx$$
$$\leqslant \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{+\infty} e^{-x} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} |g(t)| dt dx$$
$$= \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{+\infty} e^{-t} |g(t)| \int_{\sigma}^{t} e^{-x} \left(e^{-x} - e^{-t}\right)^{\alpha-1} dx dt$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_{\sigma}^{+\infty} e^{-t} |g(t)| \left(e^{-\sigma} - e^{-t}\right)^{\alpha} dt$$
$$\leqslant \frac{e^{-\alpha\sigma}}{\Gamma(\alpha+1)} \|g\|_{\pounds(\sigma;+\infty)}. \quad \square$$

NOTATION. For $\sigma \geqslant 0$, $\alpha > 0$ and $n = -[-\alpha]$, we note $\mathfrak{C}_{n-\alpha}[\sigma,+\infty)$ the following set

$$\mathfrak{C}_{n-\alpha}[\sigma,+\infty) = \left\{u \in C([\sigma,+\infty),\mathbb{R}),\ \lim_{x\to+\infty} e^{(\alpha-n)x} u(x) \text{ exists and finite}\right\}.$$

Note that $\left(\mathfrak{C}_{n-\alpha}[\sigma,+\infty), \|.\|_{\mathfrak{C}_{n-\alpha}}\right)$ is a Banach space, where

$$\|u\|_{\mathfrak{C}_{n-\alpha}} = \sup_{x\in[\sigma,+\infty)} \left|e^{(\alpha-n)x} u(x)\right|.$$

We have the following result.

LEMMA 3. *Let $\alpha > 0$ and $n = -[-\alpha]$, then the Dzhrbashyan Nersesyan fractional integral of order α is bounded from $\mathfrak{C}_{n-\alpha}[\sigma,+\infty)$ to $\mathfrak{C}_{n-\alpha}[\sigma,+\infty)$, and we have*

$$\left\|J_{-}^{\alpha} g\right\|_{\mathfrak{C}_{n-\alpha}} \leqslant \frac{\Gamma(\alpha-n+1) e^{-\alpha\sigma}}{\Gamma(2\alpha-n+1)} \|g\|_{\mathfrak{C}_{n-\alpha}}.$$

Proof. Let $\alpha > 0$ and $g \in \mathfrak{C}_{n-\alpha}[\sigma,+\infty)$, we have

$$\left|e^{(\alpha-n)x} \left(J_{-}^{\alpha} g\right)(x)\right| = \frac{e^{(\alpha-n)x}}{\Gamma(\alpha)} \left| \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} g(t)\, dt \right|$$
$$\leqslant \frac{e^{(\alpha-n)x} \|g\|_{\mathfrak{C}_{n-\alpha}}}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{(n-\alpha)t} e^{-t} dt$$
$$= \frac{\Gamma(\alpha-n+1) \|g\|_{\mathfrak{C}_{n-\alpha}}}{\Gamma(2\alpha-n+1)} e^{-\alpha x}$$
$$\leqslant \frac{\Gamma(\alpha-n+1) \|g\|_{\mathfrak{C}_{n-\alpha}}}{\Gamma(2\alpha-n+1)} e^{-\alpha\sigma}.$$

Which implies that

$$\left\|J_{-}^{\alpha} g\right\|_{\mathfrak{C}_{n-\alpha}} \leqslant \frac{\Gamma(\alpha-n+1) e^{-\alpha\sigma}}{\Gamma(2\alpha-n+1)} \|g\|_{\mathfrak{C}_{n-\alpha}}. \quad \square$$

DEFINITION 8. We define the set of functions $\mathscr{AC}[\sigma;+\infty)$ as follows

$$\mathscr{AC}[\sigma;+\infty)=\left\{g:[\sigma,+\infty)\to\mathbb{R}:\ g(x)=c+\int_x^{+\infty}e^{-t}\varphi(t)\,dt\right\},$$

where $\sigma\geqslant 0$, c is an arbitrary constant and $\varphi\in\pounds(\sigma;+\infty)$, and we denote by $\mathscr{AC}^n[\sigma;+\infty)$ with $n\in\mathbb{N}^*$ the class of functions g continuously differentiable on $[\sigma,+\infty)$ up to order $n-1$, and $\left(-e^x\dfrac{d}{dx}\right)^{n-1}g\in\mathscr{AC}[\sigma;+\infty)$.

DEFINITION 9. Let $\alpha>0$ and $g:[\sigma,+\infty)\to\mathbb{R}$ be a function such that $J_-^{n-\alpha}g\in\mathscr{AC}^n[\sigma;+\infty)$ with $n=-[-\alpha]$. The Dzhrbashyan Nersesyan Liouville fractional derivative of order α of g is defined by

$$\begin{aligned}\left(D_-^{\alpha}g\right)(x)&=\left(-e^x\frac{d}{dx}\right)^n\left(J_-^{n-\alpha}g\right)(x)\\&=\frac{1}{\Gamma(n-\alpha)}\left(-e^x\frac{d}{dx}\right)^n\int_x^{+\infty}\frac{e^{-t}g(t)}{\left(e^{-x}-e^{-t}\right)^{\alpha-n+1}}dt,\ \text{for a.e. } x\in[\sigma,+\infty).\end{aligned}$$

EXAMPLE 5. If $g(x)=b\in\mathbb{R}$, then for all $0<\alpha<1$ and $x\geqslant\sigma$, we have

$$\begin{aligned}\left(D_-^{\alpha}g\right)(x)&=-e^x\frac{d}{dx}\left(J_-^{1-\alpha}g\right)(x)\\&=-be^x\frac{d}{dx}\frac{e^{(\alpha-1)x}}{\Gamma(2-\alpha)}\\&=\frac{b}{\Gamma(1-\alpha)}e^{\alpha x}.\end{aligned}$$

EXAMPLE 6. For all $\alpha>0$ and $\beta<1$ with $\alpha+\beta<1$ and $x\geqslant\sigma$, we have

$$\begin{aligned}\left(D_-^{\alpha}e^{\beta t}\right)(x)&=\left(-e^x\frac{d}{dx}\right)^n\left(J_-^{n-\alpha}e^{\beta t}\right)(x)\\&=\left(-e^x\frac{d}{dx}\right)^n\frac{\Gamma(-\beta+1)}{\Gamma(n-\alpha-\beta+1)}e^{(\beta+\alpha-n)x}\\&=\left(-e^x\frac{d}{dx}\right)^{n-1}\frac{\Gamma(-\beta+1)}{\Gamma(n-\alpha-\beta)}e^{(\beta+\alpha+1-n)x}\\&=\left(-e^x\frac{d}{dx}\right)^{n-2}\frac{\Gamma(-\beta+1)}{\Gamma(n-\alpha-\beta-1)}e^{(\beta+\alpha+2-n)x}\\&=\frac{\Gamma(-\beta+1)}{\Gamma(1-\alpha-\beta)}e^{(\beta+\alpha)x}.\end{aligned}$$

REMARK 5. For all $k = 1, 2, \ldots, -[-\alpha]$, we have

$$\left(D_-^{\alpha} e^{(k-\alpha)t}\right)(x) = 0.$$

EXAMPLE 7. For all $\alpha > 0$ with $n = -[-\alpha]$, $\mu > 0$ and $x \geqslant 0$, we have

$$\begin{aligned}
&\left(D_-^{\alpha} e^{(n-\alpha)t} E_{\alpha,\alpha-n+1}\left(\mu e^{-\alpha t}\right)\right)(x) \\
&= \left(-e^x \frac{d}{dx}\right)^n \left(J_-^{n-\alpha} e^{(n-\alpha)t} E_{\alpha,\alpha-n+1}\left(\mu e^{-\alpha t}\right)\right)(x) \\
&= \left(-e^x \frac{d}{dx}\right)^n \left(J_-^{n-\alpha} \sum_{k=0}^{\infty} \frac{\mu^k e^{(-(k+1)\alpha+n)t}}{\Gamma(k\alpha+\alpha-n+1)}\right)(x) \\
&= \left(-e^x \frac{d}{dx}\right)^n \sum_{k=0}^{\infty} \frac{\mu^k e^{-k\alpha x}}{\Gamma(k\alpha+1)} \\
&= \sum_{k=1}^{+\infty} \frac{\mu^k e^{-(k\alpha-n)x}}{\Gamma(k\alpha-n+1)} \\
&= \mu e^{(n-\alpha)x} \sum_{k=0}^{+\infty} \frac{\mu^k e^{-k\alpha x}}{\Gamma(k\alpha+\alpha-n+1)} \\
&= \mu e^{(n-\alpha)x} E_{\alpha,\alpha-n+1}\left(\mu e^{-\alpha x}\right).
\end{aligned}$$

DEFINITION 10. ([6]) Let $g : [\sigma; +\infty) \to \mathbb{R}$ be a function with $\sigma \geqslant 0$ and $0 < \alpha \leqslant 1$. The Dzhrbashyan Nersesyan fractional derivative of order α of g is defined by

$$\begin{aligned}
\left({}^{DN}D_-^{\alpha} g\right)(x) &= -J_-^{1-\alpha}\left(e^t g'(t)\right)(x) \\
&= \frac{-1}{\Gamma(1-\alpha)} \int_x^{+\infty} \left(e^{-x} - e^{-t}\right)^{-\alpha} g'(t)\, dt.
\end{aligned}$$

We have the following results.

LEMMA 4. *Let* $\alpha > 0$. *If* $g \in \pounds(\sigma; +\infty)$, *then we have*

$$\left(D_-^{\alpha} \circ J_-^{\alpha} g\right)(x) = g(x).$$

Proof. Using the definition of Dzhrbashyan Nersesyan Liouville fractional derivative, we have

$$\left(D_-^{\alpha} \circ J_-^{\alpha} g\right)(x) = \left(-e^x \frac{d}{dx}\right)^n \left(J_-^{n-\alpha} J_-^{\alpha} g\right)(x),$$

where $n = -[-\alpha]$.

Then from Lemma 1, we get

$$\left(D_-^{\alpha} \circ J_-^{\alpha} g\right)(x) = \left(-e^x \frac{d}{dx}\right)^n \left(J_-^{n} g\right)(x),$$

and consequently from Remark 3, it follows that

$$\left(D_{-}^{\alpha}\circ J_{-}^{\alpha}g\right)(x)=g(x). \quad \square$$

THEOREM 1. *Let* $0<\alpha<1$. *If* $g\in £(\sigma;+\infty)$ *and* $J_{-}^{1-\alpha}g\in\mathscr{AC}[\sigma;+\infty)$, *then we have*

$$\left(J_{-}^{\alpha}\circ D_{-}^{\alpha}g\right)(x)=g(x)-\frac{\left(J_{-}^{1-\alpha}g\right)(+\infty)}{\Gamma(\alpha)}e^{(1-\alpha)x}, \textit{for a.e. } x\in[\sigma,+\infty).$$

Proof. We have

$$\begin{aligned}
\left(J_{-}^{\alpha+1}\circ D_{-}^{\alpha}g\right)(x) &= \frac{1}{\Gamma(\alpha+1)}\int_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha}e^{-t}\left(D_{-}^{\alpha}g\right)(t)\,dt\\
&= -\frac{1}{\Gamma(\alpha+1)}\int_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha}\left(\frac{d}{dt}J_{-}^{1-\alpha}g\right)(t)\,dt\\
&= -\frac{J_{-}^{1-\alpha}g(+\infty)}{\Gamma(\alpha+1)}e^{-\alpha x}\\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}\left(J_{-}^{1-\alpha}g\right)(t)\,dt\\
&= -\frac{J_{-}^{1-\alpha}g(+\infty)}{\Gamma(\alpha+1)}e^{-\alpha x}+\int_{x}^{+\infty}e^{-t}g(t)\,dt.
\end{aligned}$$

That is

$$\left(J_{-}^{\alpha+1}\circ D_{-}^{\alpha}g\right)(x)=-\frac{J_{-}^{1-\alpha}g(+\infty)}{\Gamma(\alpha+1)}e^{-\alpha x}+\int_{x}^{+\infty}e^{-t}g(t)\,dt. \tag{3}$$

Applying the operator D_{-}^{1} to the both sides of (3) and using the previous Theorem, we obtain

$$\left(J_{-}^{\alpha}\circ D_{-}^{\alpha}g\right)(x)=-\frac{\left(J_{-}^{1-\alpha}g\right)(+\infty)}{\Gamma(\alpha)}e^{(1-\alpha)x}+g(x). \quad \square$$

The preceding Theorem admits the following generalization.

THEOREM 2. *Let* $\alpha\geqslant 1$ *and* $n=-[-\alpha]$. *If* $g\in £(\sigma;+\infty)$ *and* $J_{-}^{n-\alpha}g\in\mathscr{AC}^{n}[\sigma;+\infty)$, *then we have*

$$\left(J_{-}^{\alpha}\circ D_{-}^{\alpha}g\right)(x)=g(x)-\sum_{k=1}^{n-1}\frac{\left(D_{-}^{\alpha-k}g\right)(+\infty)}{\Gamma(\alpha-k+1)}e^{-(\alpha-k)x}-\frac{\left(J_{-}^{n-\alpha}g\right)(+\infty)}{\Gamma(\alpha-n+1)}e^{-(\alpha-n)x}.$$

Proof. We have

$$\left(J_{-}^{\alpha+n}\circ D_{-}^{\alpha}g\right)(x)=\frac{1}{\Gamma(\alpha+n)}\int_{x}^{+\infty}\frac{e^{-t}\left(-e^{t}\frac{d}{dt}\right)^{n}\left(J_{-}^{n-\alpha}g\right)(t)}{\left(e^{-x}-e^{-t}\right)^{1-\alpha-n}}dt$$
$$=-\frac{1}{\Gamma(\alpha+n)}\int_{x}^{+\infty}\frac{\frac{d}{dt}\left(-e^{t}\frac{d}{dt}\right)^{n-1}\left(J_{-}^{n-\alpha}g\right)(t)}{\left(e^{-x}-e^{-t}\right)^{1-\alpha-n}}dt.$$

Using an integration by parts, we obtain

$$\left(J_{-}^{\alpha+n}\circ D_{-}^{\alpha}g\right)(x)=-\frac{\left(D_{-}^{\alpha-1}g\right)(+\infty)}{\Gamma(\alpha+n)}e^{-(\alpha+n-1)x}$$
$$+\frac{1}{\Gamma(\alpha+n-1)}\int_{x}^{+\infty}\frac{e^{-t}\left(-e^{t}\frac{d}{dt}\right)^{n-1}\left(J_{-}^{n-\alpha}g\right)(t)}{\left(e^{-x}-e^{-t}\right)^{2-\alpha-n}}dt.$$

Using the above argument $(n-1)$ times, we obtain

$$\left(J_{-}^{\alpha+n}\circ D_{-}^{\alpha}g\right)(x)=-\sum_{k=1}^{n-1}\frac{\left(D_{-}^{\alpha-k}g\right)(+\infty)}{\Gamma(\alpha+n-k+1)}e^{-(\alpha+n-k)x}$$
$$-\frac{\left(J_{-}^{n-\alpha}g\right)(+\infty)}{\Gamma(\alpha+1)}e^{-\alpha x}+\left(J_{-}^{n}g\right)(x).$$

Now applying the operator D_{-}^{n} to the both sides of the previous equality and using Lemma 4, we obtain

$$\left(J_{-}^{\alpha}\circ D_{-}^{\alpha}g\right)(x)=-\sum_{k=1}^{n-1}\frac{\left(D_{-}^{\alpha-k}g\right)(+\infty)}{\Gamma(\alpha-k+1)}e^{-(\alpha-k)x}-\frac{\left(J_{-}^{n-\alpha}g\right)(+\infty)}{\Gamma(\alpha-n+1)}e^{-(\alpha-n)x}+g(x). \quad \square$$

LEMMA 5.

(i) If $\lim_{x\to+\infty}e^{(\alpha-1)x}g(x)=b_1$ *with* $b_1\in\mathbb{R}$, *then* $\lim_{x\to+\infty}\left(J_{-}^{1-\alpha}g\right)(x)=b_1\Gamma(\alpha)$.

(ii) If $\lim_{x\to+\infty}\left(J_{-}^{1-\alpha}g\right)(x)=b_1$ *with* $b_1\in\mathbb{R}$ *and if* $\lim_{x\to+\infty}e^{(\alpha-1)x}g(x)$ *exists, then* $\lim_{x\to+\infty}e^{(\alpha-1)x}g(x)=\frac{b_1}{\Gamma(\alpha)}$.

Proof.

(i) Let $\varepsilon>0$ arbitrarily chosen, then there exists $\delta(\varepsilon)>0$ such that

$$\left|e^{(\alpha-1)x}g(x)-b_1\right|<\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\varepsilon, \tag{4}$$

for all $x>\delta(\varepsilon)$.

Since

$$\left(J_{-}^{1-\alpha}e^{(1-\alpha)t}\right)(x)=\Gamma(\alpha),$$

we obtain

$$\begin{aligned}\left|\left(J_{-}^{1-\alpha}g\right)(x)-b_1\Gamma(\alpha)\right| &= \left|\left(J_{-}^{1-\alpha}g\right)(x)-b_1\left(J_{-}^{1-\alpha}e^{(1-\alpha)t}\right)(x)\right| \\ &= \frac{1}{\Gamma(1-\alpha)}\left|\int_x^{+\infty}\frac{\left(g(t)-b_1e^{(1-\alpha)t}\right)}{\left(e^{-x}-e^{-t}\right)^{\alpha}e^t}dt\right| \\ &= \frac{1}{\Gamma(1-\alpha)}\left|\int_x^{+\infty}\frac{e^{(1-\alpha)t}\left(e^{(\alpha-1)t}g(t)-b_1\right)}{\left(e^{-x}-e^{-t}\right)^{\alpha}e^t}dt\right|.\end{aligned}$$

Now if we choose $x>\delta(\varepsilon)$ and by using (4) and (2), we obtain

$$\left|\left(J_{-}^{1-\alpha}g\right)(x)-b_1\Gamma(\alpha)\right|<\varepsilon.$$

Which means that

$$\lim_{x\to+\infty}\left(J_{-}^{1-\alpha}g\right)(x)=b_1\Gamma(\alpha).$$

(ii) Suppose that $\lim_{x\to+\infty}\left(J_{-}^{1-\alpha}g\right)(x)=b_1$ and $\lim_{x\to+\infty}e^{(\alpha-1)x}g(x)=b_2$ with $b_2\in\mathbb{R}$, then by the preceding result (i), we have

$$\lim_{x\to+\infty}\left(J_{-}^{1-\alpha}g\right)(x)=b_2\Gamma(\alpha),$$

and consequently it follows that

$$b_2=\frac{b_1}{\Gamma(\alpha)}.$$

Which means that

$$\lim_{x\to+\infty}e^{(\alpha-1)x}g(x)=\frac{b_1}{\Gamma(\alpha)}. \quad \square$$

REMARK 6. The proof of Lemma 5 is similar to that of Lemma 3.2 in [12].

Now, we consider the following problems

$$\begin{cases}\left(D_{-}^{\alpha}u\right)(x)=F\left(x,u(x),u(\theta(x))\right),\ x\in[\sigma,+\infty),\\ \lim_{x\to+\infty}e^{(\alpha-1)x}u(x)=b_3,\end{cases} \tag{5}$$

where $0<\alpha<1$, $\sigma\geqslant 0$, $F:[\sigma,+\infty)\times G\to\mathbb{R}$ is a function such that $F(x,y,z)\in\mathfrak{C}_{1-\alpha}[\sigma,+\infty)$ for any $(y,z)\in G$ with G is an open set in $\mathbb{R}^2$, $\theta:[\sigma,+\infty)\to[\sigma,+\infty)$

continuous such that $\theta(x) \geqslant x$, for all $x \in [\sigma, +\infty)$ and $b_3 \in \mathbb{R}$, and

$$\begin{cases} \left(D_-^{\alpha} u\right)(x) = F(x, u(x), u(\theta(x))), \ x \in [\sigma, +\infty), \\ \lim\limits_{x \to +\infty} \left(D_-^{\alpha-k} u\right)(x) = a_k, \ k = 1, \dots, n-1, \\ \lim\limits_{x \to +\infty} \left(J_-^{n-\alpha} u\right)(x) = a_n, \end{cases} \tag{6}$$

where $\alpha > 0$, a_k a real numbers for $k = 1, \dots, n = -[-\alpha]$, $F : [\sigma, +\infty) \times G \to \mathbb{R}$ is a function such that $F(x, y, z) \in \mathfrak{C}_{n-\alpha}[\sigma, +\infty)$ for any $(y, z) \in G$ with G is an open set in $\mathbb{R}^2$, $\theta : [\sigma, +\infty) \to [\sigma, +\infty)$ continuous such that $\theta(x) \geqslant x$, for all $x \in [\sigma, +\infty)$.

We have the following results.

THEOREM 3. *$u \in \mathfrak{C}_{1-\alpha}[\sigma, +\infty)$ is a solution for the problem (5) if, and only if, is a solution of the following Volterra equation*

$$u(x) = b_3 e^{(1-\alpha)x} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{e^{-t} F(t, u(t), u(\theta(t)))}{(e^{-x} - e^{-t})^{1-\alpha}} dt. \tag{7}$$

Proof. The proof is an immediate consequence of Theorem 1, Lemma 4 and Lemma 5. □

REMARK 7. Theorem 3 remains valid if we suppose $F(x, y, z) \in \pounds[\sigma, +\infty)$ for any $(y, z) \in G$ with G is an open set in $\mathbb{R}^2$ and in this case $u \in \pounds(\sigma; +\infty)$ is a solution for the problem (5) if, and only if, is a solution of the integral equation (7).

THEOREM 4. *$u \in \mathfrak{C}_{n-\alpha}[\sigma, +\infty)$ is a solution for the problem (6) if, and only if, is a solution of the following Volterra equation*

$$u(x) = \sum_{k=1}^{n} \frac{a_k}{\Gamma(\alpha - k + 1)} e^{-(\alpha-k)x} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{e^{-t} F(t, u(t), u(\theta(t)))}{(e^{-x} - e^{-t})^{1-\alpha}} dt. \tag{8}$$

Proof. The proof directly follows from Theorem 2 and Lemma 4. □

REMARK 8. Theorem 4 remains valid if we suppose $F(x, y, z) \in \pounds[\sigma, +\infty)$ for any $(y, z) \in G$ with G is an open set in $\mathbb{R}^2$ and in this case $u \in \pounds(\sigma; +\infty)$ is a solution for the problem (5) if, and only if, is a solution of the integral equation (8).

We now introduce the classical Banach fixed point theorem within a complete metric space.

THEOREM 5. ([12]) *Let (U, d) be a nonempty complete metric space, let $0 \leqslant \omega < 1$, and let $T : U \to U$ be the map such that, for every $u,\ v \in U$, the relation*

$$d(Tu, Tv) \leqslant \omega d(u, v),$$

holds. Then the operator T has a unique fixed point $u^* \in U$.

Furthermore, if T^k ($k \in \mathbb{N}^*$) is the sequence of operators defined by

$$T^1 = T,\ T^k = TT^{k-1} \text{ if } k > 1,$$

then, for any $u_0 \in U$, the sequence $\left\{T^k u_0\right\}_{k=1}^{+\infty}$ converges to the above fixed point u^*.

3. Main results

In this section, the main results of this work are stated and proven.

Consider problem (1) and suppose that $f : [\sigma, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and fulfills the following assumptions.

(H1) The function $f(x,y,z) \in \mathfrak{C}_{1-\alpha}[\sigma,+\infty)$ for any $(y,z) \in \mathbb{R}^2$.

(H2) There exist $K_1 > 0$ and $K_2 > 0$ such that for all $x \in [\sigma,+\infty)$ and y_j, $z_j \in \mathbb{R}$ for $j = 1,2$, we have

$$|f(x,y_1,z_1) - f(x,y_2,z_2)| \leqslant K_1 |y_1 - y_2| + K_2 |z_1 - z_2|.$$

THEOREM 6. *Assume that the hypotheses (H1) and (H2) are satisfied, then the problem (1) admits a solution $u \in \mathfrak{C}_{1-\alpha}[\sigma;+\infty]$ such that $D_{-}^{\alpha} u \in \mathfrak{C}_{1-\alpha}[\sigma;+\infty]$ and this solution is unique.*

Proof. Consider the following operator

$$Q : \mathfrak{C}_{1-\alpha}[\sigma,+\infty) \to \mathfrak{C}_{1-\alpha}[\sigma,+\infty)$$

$$u \mapsto (Qu)(x) = ae^{(1-\alpha)x} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(e^{-x} - e^{-t}\right)^{\alpha-1} e^{-t} f(t,u(t),u(\theta(t)))\, dt,$$

and take into account the following norm

$$\|v\|_* = \sup_{x \in [\sigma,+\infty)} \frac{\left|e^{(\alpha-1)x} v(x)\right|}{E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)},$$

with $\kappa > 0$ and $v \in \mathfrak{C}_{1-\alpha}[\sigma,+\infty)$.

Since the function $y \mapsto E_{\alpha,\alpha}(y)$ is increasing, then for all $x \geqslant \sigma$, we have

$$1 \leqslant E_{\alpha,\alpha}(\kappa e^{-\alpha x}) \leqslant E_{\alpha,\alpha}(\kappa e^{-\alpha\sigma}),$$

which implies

$$\frac{\|v\|_{\mathfrak{C}_{1-\alpha}}}{E_{\alpha,\alpha}(\kappa e^{-\alpha\sigma})} \leqslant \|v\|_* \leqslant \|v\|_{\mathfrak{C}_{1-\alpha}},$$

and then the norms $\|.\|_{\mathfrak{C}_{1-\alpha}}$ and $\|.\|_*$ are equivalent.

Now let's prove operator Q is a contraction on the Banach space $(\mathfrak{C}_{1-\alpha}[\sigma,+\infty), \|.\|_*)$.

For all $v_1, v_2 \in \mathfrak{C}_{1-\alpha}[\sigma,+\infty)$ and all $x \in [\sigma,+\infty)$, one has

$$\begin{aligned}
&\frac{\left|e^{(\alpha-1)x}\left((Qv_1)(x)-(Qv_2)(x)\right)\right|}{E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)} \\
&= \frac{\left|e^{(\alpha-1)x}\int\limits_x^{+\infty}\left(e^{-x}-e^{-\tau}\right)^{\alpha-1}e^{-\tau}\left(f\left(\tau,v_1(\tau),v_1(\theta(\tau))\right)-f\left(\tau,v_2(\tau),v_2(\theta(\tau))\right)\right)d\tau\right|}{\Gamma(\alpha)E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)} \\
&\leqslant \frac{K_1 e^{(\alpha-1)x}}{\Gamma(\alpha)E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)}\left|\int\limits_x^{+\infty}\left(e^{-x}-e^{-\tau}\right)^{\alpha-1}e^{-\tau}\left|v_1(\tau)-v_2(\tau)\right|d\tau\right| \\
&\quad + \frac{K_2 e^{(\alpha-1)x}}{\Gamma(\alpha)E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)}\left|\int\limits_x^{+\infty}\left(e^{-x}-e^{-\tau}\right)^{\alpha-1}e^{-\tau}\left|v_1(\theta(\tau))-v_2(\theta(\tau))\right|d\tau\right| \\
&\leqslant \frac{(K_1+K_2)e^{(\alpha-1)x}\left\|v_1-v_2\right\|_*}{\Gamma(\alpha)E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)}\int\limits_x^{+\infty}\left(e^{-x}-e^{-\tau}\right)^{\alpha-1}e^{-\tau}e^{(1-\alpha)\tau}E_{\alpha,\alpha}\left(\kappa e^{-\alpha\tau}\right)d\tau \\
&= \frac{(K_1+K_2)\left\|v_1-v_2\right\|_*}{\kappa E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)}\left(E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)-\frac{1}{\Gamma(\alpha)}\right) \\
&= \frac{(K_1+K_2)\left\|v_1-v_2\right\|_*}{\kappa}\left(1-\frac{1}{\Gamma(\alpha)E_{\alpha,\alpha}\left(\kappa e^{-\alpha x}\right)}\right) \\
&< \frac{(K_1+K_2)\left\|v_1-v_2\right\|_*}{\kappa}.
\end{aligned}$$

Which implies that

$$\left\|Qv_1-Qv_2\right\|_* \leqslant \frac{(K_1+K_2)}{\kappa}\left\|v_1-v_2\right\|_*.$$

Now if we choose $\kappa > K_1+K_2$, we obtain

$$\left\|Qv_1-Qv_2\right\|_* < \left\|v_1-v_2\right\|_*.$$

Then according to Theorem 5, it follows the existence of a unique fixed point for Q and consequently from Theorem 3, we conclude that the problem (1) admits a unique solution. □

REMARK 9. The hypothesis (H2) does not guarantee the uniqueness of solutions to terminal value problems for fractional-order differential equations with deviating arguments. For example the following problem

$$\begin{cases}\left(D_-^{\frac{1}{4}}u\right)(x) = 2\dfrac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}u\left(\dfrac{x}{2}\right), \ x \in [0,+\infty), \\ \lim\limits_{x\to+\infty} e^{-\frac{3x}{4}}u(x) = 0,\end{cases}$$

where $0<\alpha<1$, admits two solutions $u\equiv 0$ and $u(x)=e^{\frac{-x}{2}}$, for all $x\in[0,+\infty)$.

REMARK 10. The hypothesis (H2) is sufficient but not necessary for the existence of a unique solution for the problem (1). For example the following problem

$$\begin{cases} \left(D_{-}^{\alpha}u\right)(x)=\dfrac{\Gamma\left(\alpha+\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)}\left(e^{2\left(\alpha-\frac{3}{4}\right)x}u(2x)-e^{-\frac{3}{2}x}\right)+\Gamma(\alpha+1),\ x\in[\sigma,+\infty), \\ \lim\limits_{x\to+\infty}e^{(\alpha-1)x}u(x)=0, \end{cases} \tag{9}$$

where $\frac{3}{4}<\alpha<1$, admits $u(x)=e^{-\alpha x}$, for all $x\in[\sigma,+\infty)$ as a unique solution.

First, it is not difficult to prove that the function defined by $u(x)=e^{-\alpha x}$, for all $x\in[\sigma,+\infty)$ is a solution.

Now suppose that the problem (9) admits two solutions u_1 and u_2 and we put by definition

$$v(x)=u_1(x)-u_2(x),\ \text{for all } x\in[\sigma,+\infty).$$

Then, we have

$$\begin{cases} \left(D_{-}^{\alpha}v\right)(x)=\dfrac{\Gamma\left(\alpha+\frac{1}{2}\right)e^{2\left(\alpha-\frac{3}{4}\right)x}}{2\Gamma\left(\frac{1}{2}\right)}v(2x),\ x\in[\sigma,+\infty), \\ \lim\limits_{x\to+\infty}e^{(\alpha-1)x}v(x)=0, \end{cases} \tag{10}$$

v is a solution for the problem (10) if, and only if, is a solution of the following Volterra equation

$$v(x)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{2\Gamma(\alpha)\Gamma\left(\frac{1}{2}\right)}\int_x^{+\infty}\frac{e^{-t}e^{2\left(\alpha-\frac{3}{4}\right)t}v(2t)}{\left(e^{-x}-e^{-t}\right)^{1-\alpha}}dt.$$

We define the following operator Consider the following operator

$$\begin{aligned} &T:\mathfrak{C}_{1-\alpha}[\sigma,+\infty)\to\mathfrak{C}_{1-\alpha}[\sigma,+\infty) \\ &u\mapsto(Tv)(x)=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{2\Gamma(\alpha)\Gamma\left(\frac{1}{2}\right)}\int_x^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}e^{-t}e^{2\left(\alpha-\frac{3}{4}\right)t}v(2t)\,dt, \end{aligned}$$

For all $v_1,\ v_2\in\mathfrak{C}_{1-\alpha}[\sigma,+\infty)$ and all $x\in[\sigma,+\infty)$, one has

$$\begin{aligned} \|Tv_1-Tv_2\|_{\mathfrak{C}_{1-\alpha}} &\leqslant \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\|v_1-v_2\|_{\mathfrak{C}_{1-\alpha}}}{2\Gamma(\alpha)\Gamma\left(\frac{1}{2}\right)}\int_x^{+\infty}\left(e^{-x}-e^{-t}\right)^{\alpha-1}e^{-t}e^{\frac{t}{2}}dt \\ &=\frac{e^{\left(\frac{1}{2}-\alpha\right)x}}{2}\|v_1-v_2\|_{\mathfrak{C}_{1-\alpha}} \\ &<\|v_1-v_2\|_{\mathfrak{C}_{1-\alpha}}. \end{aligned}$$

Then according to Theorem 5, it follows the existence of a unique fixed point for T and consequently from Theorem 3, we conclude that the problem (9) admits a unique solution.

COROLLARY 1. *Suppose that* $f(x,y,z) \in \mathfrak{C}_{n-\alpha}[\sigma,+\infty)$ *for any* $(y,z) \in G$ *with* G *is an open set in* $\mathbb{R}^2$ *and the assumption (H2) is satisfied, then the problem (2) admits a solution* $u \in \mathfrak{C}_{n-\alpha}[\sigma;+\infty]$ *such that* $D_{-}^{\alpha}u \in \mathfrak{C}_{n-\alpha}[\sigma;+\infty]$ *and this solution is unique.*

Proof. Consider the following operator

$$Q : \mathfrak{C}_{n-\alpha}[\sigma,+\infty) \to \mathfrak{C}_{n-\alpha}[\sigma,+\infty)$$

$$u \mapsto (Qu)(x) = \sum_{k=1}^{n} \frac{a_k}{\Gamma(\alpha-k+1)} e^{-(\alpha-k)x} + \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \frac{e^{-t} f(t,u(t),u(\theta(t)))\,dt}{(e^{-x}-e^{-t})^{1-\alpha}},$$

and take into account the following norm

$$\|v\|_{**} = \sup_{x\in[\sigma,+\infty)} \frac{\left|e^{(\alpha-n)x} v(x)\right|}{E_{\alpha,\alpha-n+1}(\kappa e^{-\alpha x})},$$

with $\kappa > 0$ and $v \in \mathfrak{C}_{n-\alpha}[\sigma,+\infty)$.

Since for all $x \geqslant \sigma$, we have

$$1 \leqslant E_{\alpha,\alpha-n+1}(\kappa e^{-\alpha x}) \leqslant E_{\alpha,\alpha-n+1}(\kappa e^{-\alpha\sigma}),$$

which implies

$$\frac{\|v\|_{\mathfrak{C}_{n-\alpha}}}{E_{\alpha,\alpha-n+1}(\kappa e^{-\alpha\sigma})} \leqslant \|v\|_{**} \leqslant \|v\|_{\mathfrak{C}_{n-\alpha}},$$

and then the norms $\|.\|_{\mathfrak{C}_{n-\alpha}}$ and $\|.\|_{**}$ are equivalent.

The rest of the proof is similar to that of the previous Theorem. □

REMARK 11. Theorem 6 can be generalized to the following problem

$$\begin{cases} (D_{-}^{\alpha}u)(x) = f(x,u(x),u(\theta_1(x),\dots,u(\theta_m(x))), x \in [\sigma,+\infty), \\ \lim_{x\to+\infty} e^{(\alpha-1)x} u(x) = a, \end{cases} \tag{11}$$

where $0 < \alpha < 1$, a a real number, $\theta_i : [\sigma,+\infty) \to [\sigma,+\infty)$ are continuous and $\theta_i(x) \geqslant x$ for all $i = 1,\dots,m$ and $f : [\sigma,+\infty) \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuous such that $f(x,y) \in \mathfrak{C}_{1-\alpha}[\sigma,+\infty)$ for any $y = (y_1,\dots,y_{m+1}) \in \mathbb{R}^{m+1}$ and satisfies the following condition.

There exist $K_i > 0$ for $i = 1,\dots,m+1$ such that

$$|f(x,u_1,\dots,u_{m+1}) - f(x,v_1,\dots,v_{m+1})| \leqslant \sum_{i=1}^{m+1} K_i |u_i - v_i|, \tag{12}$$

for all $x \in [\sigma,+\infty)$ and $u_i,\ v_i \in \mathbb{R}$ for $i = 1,\dots,m+1$.

NOTATION. For $\sigma \geqslant 0$, we note $\mathfrak{C}[\sigma,+\infty)$ the following set

$$\mathfrak{C}[\sigma,+\infty)=\left\{u\in C([\sigma,+\infty),\mathbb{R}),\ \lim_{x\to+\infty}u(x) \text{ exists and finite}\right\}.$$

REMARK 12. Theorem 6 can be generalized to the following problem

$$\begin{cases} \left({}^{\mathrm{DN}}D_{-}^{\alpha}u\right)(x)=f(x,u(x),u(\theta_1(x),\ldots,u(\theta_m(x)),\ x\in[\sigma,+\infty),\\ \lim\limits_{x\to+\infty}u(x)=a,\end{cases}$$

where $0<\alpha<1$, a a real number, $\theta_i:[\sigma,+\infty)\to[\sigma,+\infty)$ continuous and $\theta_i(x)\geqslant x$ for all $i=1,\ldots,m$ and $f:[\sigma,+\infty)\times\mathbb{R}^{m+1}\to\mathbb{R}$ continuous such that $f(x,y)\in\mathfrak{C}[\sigma,+\infty)$ for any $y=(y_1,\ldots,y_{m+1})\in\mathbb{R}^{m+1}$ and satisfies the condition (12).

REMARK 13. Corollary 1 can be generalized to the following problem

$$\begin{cases} \left(D_{-}^{\alpha}u\right)(x)=f(x,u(x),u(\theta_1(x),\ldots,u(\theta_m(x)),\ x\in[\sigma,+\infty),\\ \lim\limits_{x\to+\infty}\left(D_{-}^{\alpha-k}u\right)(x)=a_k,\ k=1,\ldots,n-1,\\ \lim\limits_{x\to+\infty}\left(J_{-}^{n-\alpha}u\right)(x)=a_n,\end{cases}\tag{13}$$

where $\alpha>0$, a_k a real numbers for $k=1,\ldots,n=-[-\alpha]$, $\theta_i:[\sigma,+\infty)\to[\sigma,+\infty)$ continuous and $\theta_i(x)\geqslant x$ for all $i=1,\ldots,m$ and $f:[\sigma,+\infty)\times\mathbb{R}^{m+1}\to\mathbb{R}$ continuous such that $f(x,y)\in\mathfrak{C}_{n-\alpha}[\sigma,+\infty)$ for any $y=(y_1,\ldots,y_{m+1})\in\mathbb{R}^{m+1}$ and satisfies the condition (12).

4. Examples

EXAMPLE 8. Consider the problem

$$\begin{cases} \left(D_{-}^{\alpha}u\right)(x)=\lambda u(x),\ x\in[\sigma,+\infty),\\ \lim\limits_{x\to+\infty}e^{(\alpha-1)x}u(x)=\dfrac{a}{\Gamma(\alpha)},\end{cases}\tag{14}$$

where $0<\alpha<1$, λ and a are real numbers.

First, we note that the function $(x,y,z)\mapsto\lambda y$ satisfy the assumptions (H1) and (H2) and then from Theorem 6, it follows that the problem (14) admits a unique solution.

Now, from Theorem 3, $u\in\mathfrak{C}_{1-\alpha}[\sigma,+\infty)$ is a solution for the problem (14) if, and only if, is a solution of the following Volterra equation

$$u(x)=\frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_x^{+\infty}\frac{e^{-t}\lambda u(t)}{(e^{-x}-e^{-t})^{1-\alpha}}dt.$$

This integral equation is solved by using the method of successive approximations. Using this approach, we put

$$u_0(x) = \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)},$$
$$u_{n+1}(x) = \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_x^{+\infty} \frac{e^{-t}\lambda u_n(t)}{(e^{-x}-e^{-t})^{1-\alpha}}dt, \text{ for all } n \in \mathbb{N}.$$

Then, we have

$$\begin{aligned} u_1(x) &= \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_x^{+\infty} \left(e^{-x}-e^{-\tau}\right)^{\alpha-1} e^{-\tau}\lambda u_0(\tau)\,d\tau \\ &= \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{\lambda ae^{(1-2\alpha)x}}{\Gamma(2\alpha)}, \\ u_2(x) &= \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_x^{+\infty} \left(e^{-x}-e^{-\tau}\right)^{\alpha-1} e^{-\tau}\lambda u_1(\tau)\,d\tau \\ &= \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{\lambda ae^{(1-2\alpha)x}}{\Gamma(2\alpha)} + \frac{ae^{(1-3\alpha)x}}{\Gamma(3\alpha)}, \end{aligned}$$

$$\begin{aligned} u_3(x) &= \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_x^{+\infty} \left(e^{-x}-e^{-\tau}\right)^{\alpha-1} e^{-\tau}\lambda u_2(\tau)\,d\tau \\ &= \frac{ae^{(1-\alpha)x}}{\Gamma(\alpha)} + \frac{\lambda ae^{(1-2\alpha)x}}{\Gamma(2\alpha)} + \frac{\lambda^2 ae^{(1-3\alpha)x}}{\Gamma(3\alpha)} + \frac{\lambda^3 ae^{(1-4\alpha)x}}{\Gamma(4\alpha)}. \end{aligned}$$

By recurrence, we obtain

$$u_n(x) = ae^{(1-\alpha)x}\sum_{k=0}^{n} \frac{\lambda^k e^{-\alpha kx}}{\Gamma(\alpha+k\alpha)},$$

and then the unique solution to problem (14) is given by

$$\begin{aligned} u(x) &= ae^{(1-\alpha)x}\sum_{k=0}^{+\infty} \frac{\lambda^k e^{-\alpha kx}}{\Gamma(\alpha+k\alpha)} \\ &= ae^{(1-\alpha)x}E_{\alpha,\alpha}\left(\lambda e^{-\alpha x}\right). \end{aligned}$$

EXAMPLE 9. Consider the problem

$$\begin{cases} (D_-^{\alpha}u)(x) = \lambda u(x) + \widetilde{\mathfrak{F}}(x),\ x \in [\sigma, +\infty), \\ \lim\limits_{x\to+\infty} \left(D_-^{\alpha-k}g\right)(x) = a_k,\ k = 1, \ldots, n-1, \\ \lim\limits_{x\to+\infty} \left(J_-^{n-\alpha}g\right)(+\infty) = a_n, \end{cases} \tag{15}$$

where $\alpha > 0$, λ and a_k a real numbers for $k = 1, \ldots, n = -[-\alpha]$ and $\widetilde{\mathfrak{F}} \in \mathfrak{C}_{n-\alpha}[\sigma; +\infty]$.

According to Corollary 1, problem (15) admits a unique solution and we use the technique of successive approximations to show that this solution is given by

$$u(x) = \sum_{k=1}^{n} a_k e^{(k-\alpha)x} E_{\alpha,\alpha-k+1}\left(\lambda e^{-\alpha x}\right) + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(e^{-x} - e^{-\tau}\right)^{\alpha-1} e^{-\tau} E_{\alpha,\alpha}\left(\lambda \left(e^{-x} - e^{-\tau}\right)^{\alpha}\right) \widetilde{\mathfrak{F}}(\tau)\, d\tau.$$

EXAMPLE 10. Consider the problem

$$\begin{cases} \left({}^{\mathrm{DN}}D_{-}^{\alpha} u\right)(x) = \lambda u(x) + \widetilde{\mathfrak{F}}(x), \ x \in [\sigma, +\infty), \\ \lim\limits_{x \to +\infty} u(x) = a, \end{cases} \tag{16}$$

where $0 < \alpha < 1$, λ and a are real numbers and $\widetilde{\mathfrak{F}} \in \mathfrak{C}[\sigma; +\infty]$.

First, we see that according to Remark 12, problem (16) admits a unique solution. We then demonstrate that, by applying the successive approximation technique, this solution is given by

$$u(x) = aE_{\alpha}\left(\lambda e^{-\alpha x}\right) + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(e^{-x} - e^{-\tau}\right)^{\alpha-1} e^{-\tau} E_{\alpha,\alpha}\left(\lambda \left(e^{-x} - e^{-\tau}\right)^{\alpha}\right) \widetilde{\mathfrak{F}}(\tau)\, d\tau.$$

EXAMPLE 11. Consider the problem

$$\begin{cases} \left(D_{-}^{\alpha} u\right)(x) = \lambda u(x+q), \ x \in [\sigma, +\infty), \\ \lim\limits_{x \to +\infty} \left(D_{-}^{\alpha-k} g\right)(x) = a_k, \ k = 1, \ldots, n-1, \\ \lim\limits_{x \to +\infty} \left(J_{-}^{n-\alpha} g\right)(+\infty) = a_n, \end{cases} \tag{17}$$

where $\alpha > 0$, $q > 0$, λ and a_k are a real numbers for $k = 1, \ldots, n = -[-\alpha]$.

First, we note from Remark 11, the problem (17) admits a unique solution.

Now, from Theorem 4, $u \in \mathfrak{C}_{n-\alpha}[\sigma, +\infty)$ is a solution for the problem (17) if, and only if, is a solution of the following Volterra equation

$$u(x) = \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha - k + 1)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{e^{-t} \lambda u(t+q)}{\left(e^{-x} - e^{-t}\right)^{1-\alpha}} dt.$$

This integral equation is solved by using the method of successive approximations.

Using this approach, we put

$$u_0(x) = \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)},$$

$$u_{m+1}(x) = \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{e^{-t}\lambda u_m(t+q)}{(e^{-x}-e^{-t})^{1-\alpha}} dt, \text{ for all } m \in \mathbb{N}.$$

Then, we have

$$\begin{aligned} u_1(x) &= \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(e^{-x}-e^{-\tau}\right)^{\alpha-1} e^{-\tau}\lambda u_0(\tau+q)\,d\tau \\ &= \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \lambda \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)q} e^{(k-2\alpha)x}}{\Gamma(2\alpha-k+1)} \end{aligned}$$

$$\begin{aligned} u_2(x) &= \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(e^{-x}-e^{-\tau}\right)^{\alpha-1} e^{-\tau}\lambda u_1(\tau+q)\,d\tau \\ &= \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \lambda \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)q} e^{(k-2\alpha)x}}{\Gamma(2\alpha-k+1)} + \lambda^2 \sum_{k=1}^{n} \frac{a_k e^{(2k-3\alpha)q} e^{(k-3\alpha)x}}{\Gamma(3\alpha-k+1)}, \\ u_3(x) &= \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(e^{-x}-e^{-\tau}\right)^{\alpha-1} e^{-\tau}\lambda u_2(\tau+q)\,d\tau \\ &= \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)x}}{\Gamma(\alpha-k+1)} + \lambda \sum_{k=1}^{n} \frac{a_k e^{(k-\alpha)q} e^{(k-2\alpha)x}}{\Gamma(2\alpha-k+1)} + \lambda^2 \sum_{k=1}^{n} \frac{a_k e^{(2k-3\alpha)q} e^{(k-3\alpha)x}}{\Gamma(3\alpha-k+1)} \\ &\quad + \lambda^3 \sum_{k=1}^{n} \frac{a_k e^{(3k-6\alpha)q} e^{(k-4\alpha)}x}{\Gamma(4\alpha-k+1)} \end{aligned}$$

By recurrence, we obtain

$$u_m(x) = \sum_{k=1}^{n} a_k \sum_{j=0}^{m} \frac{e^{\left(jk-\frac{j(j+1)}{2}\alpha\right)q} \lambda^j e^{(k-(j+1)\alpha)x}}{\Gamma((j+1)\alpha-k+1)},$$

and then the unique solution to problem (17) is given by

$$\begin{aligned} u(x) &= \sum_{k=1}^{n} a_k e^{(k-\alpha)x} \sum_{j=0}^{+\infty} \frac{e^{\left(j-\frac{j(j+1)}{2k}\alpha\right)kq} \lambda^j e^{-\alpha jx}}{\Gamma(j\alpha+\alpha-k+1)} \\ &= \sum_{k=1}^{n} a_k e^{(k-\alpha)x} F_{\alpha,\alpha-k+1}\left(\lambda e^{-\alpha x}, e^{-kq}; \frac{\alpha}{k}\right). \end{aligned}$$

EXAMPLE 12. Consider the problem

$$\begin{cases} \left({}^{\mathrm{DN}}D_{-}^{\alpha}u\right)(x) = \lambda u(x+q),\ x \in [\sigma,+\infty), \\ \lim\limits_{x\to+\infty} u(x) = a, \end{cases} \tag{18}$$

where $0 < \alpha < 1$, $q > 0$, λ and a are a real numbers.

First, we see that according to Remark 12, problem (18) admits a unique solution. We then demonstrate that, by applying the successive approximation technique, this solution is given by

$$u(x) = aF_{\alpha}(\lambda e^{-\alpha x}, e^{-q}).$$

EXAMPLE 13. Consider the problem

$$\begin{cases} \left(D_{-}^{\alpha}u\right)(x) = \lambda_1 u(x) + \lambda_2 \sum\limits_{i=1}^{m} u(x+q_i),\ x \in [\sigma,+\infty), \\ \lim\limits_{x\to+\infty} e^{(\alpha-1)x} u(x) = \dfrac{a}{\Gamma(\alpha)}, \end{cases} \tag{19}$$

where $0 < \alpha < 1$, $q_i > 0$ for all $i = 1,\ldots,m$ and λ_1, λ_2 and a are a real numbers.

First, we see that according to Remark 11, problem (19) admits a unique solution. We then demonstrate that, by applying the successive approximation technique, this solution is given by

$$u(x) = ae^{(1-\alpha)x}\left(\frac{1}{\Gamma(\alpha)} + \sum_{n=1}^{+\infty} \frac{e^{-n\alpha x}}{\Gamma(n\alpha+\alpha)} \prod_{j=1}^{n}\left(\lambda_1 + \lambda_2 \sum_{i=1}^{m} e^{(1-j\alpha)q_i}\right)\right).$$

EXAMPLE 14. Consider the problem

$$\begin{cases} \left({}^{\mathrm{DN}}D_{-}^{\alpha}u\right)(x) = \lambda_1 u(x) + \lambda_2 \sum\limits_{i=1}^{m} u(x+q_i),\ x \in [\sigma,+\infty), \\ \lim\limits_{x\to+\infty} u(x) = a, \end{cases} \tag{20}$$

where $0 < \alpha < 1$, $q_i > 0$ for all $i = 1,\ldots,m$ and λ_1, λ_2 and a are a real numbers.

First, we see that according to Remark 12, problem (20) admits a unique solution. We then demonstrate that, by applying the successive approximation technique, this solution is given by

$$u(x) = a\left(1 + \sum_{n=1}^{+\infty} \frac{e^{-n\alpha x}}{\Gamma(n\alpha+1)} \prod_{j=1}^{n}\left(\lambda_1 + \lambda_2 \sum_{i=1}^{m} e^{-j\alpha q_i}\right)\right).$$

5. Conclusion

This study defines the Dzhrbashyan Nersesyan Liouville fractional derivative, illustrates its properties, and utilizes Banach's fixed point theorem to establish the existence and uniqueness of solutions to terminal value problems for fractional-order differential equations with advanced arguments. Several examples are provided to demonstrate the application of our results. We note that our results can be applied to the following problems

$$\begin{cases} \left({}_{\psi}D_{-}^{\alpha}u\right)(x) = f\left(x, u(x), u(\theta(x))\right), x \in I, \\ \lim\limits_{x\to+\infty} \left(D_{-}^{\alpha-k}g\right)(x) = a_k,\ k = 1, \ldots, n-1, \\ \lim\limits_{x\to+\infty} \left(J_{-}^{n-\alpha}g\right)(+\infty) = a_n, \end{cases}$$

where $\psi : I \to \mathbb{R}$ is of class C^1 and strictly decreasing, I is an interval of $\mathbb{R}^+$, ${}_{\psi}D_{-}^{\alpha}$ is the Dzhrbashyan Nersesyan Liouville fractional derivative of order α with respect to ψ defined by

$$\begin{aligned} \left({}_{\psi}D_{-}^{\alpha}u\right)(x) &= \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} \left({}_{\psi}J_{-}^{n-\alpha}u\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} \int_{x}^{+\infty} \frac{\psi'(t)\,u(t)}{\left(\psi(x)-\psi(t)\right)^{\alpha-n+1}}dt, \end{aligned}$$

with $0 < \alpha < n-1$, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ and $\theta : [\sigma, +\infty) \to [\sigma, +\infty)$ are continuous such that $\theta(x) \geqslant x$ and a_k a real numbers for $k = 1, \ldots, n = -[-\alpha]$, and

$$\begin{cases} \left(D_{-}^{\alpha}u\right)(x) = f\left(x, u(x), u(\theta(x))\right), x \in I, \\ \lim\limits_{x\to+\infty} \psi^{1-\alpha}(x)\,u(x) = a, \end{cases}$$

where $0 < \alpha < 1$ and $a \in \mathbb{R}$.

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