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QUALITATIVE ANALYSIS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAY

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Abstract. This paper investigates the fractional differential equation (FDE) with proportional delay, a specific form of the fractional order time-dependent delay differential equation (FDDE). We use the Daftardar-Gejji and Jafari Method (DJM) to solve nonlinear FDEs with proportional delay involving the Caputo fractional derivative. We prove existence and uniqueness theorems for these equations and derive convergence results based on the Lipschitz condition. Moreover, we show that DJM solutions are continuously dependent on both the initial conditions and the fractional order. Finally, we derive and prove the convergence of power series solutions for the fractional order pantograph and Ambartsumian equations.

1. Introduction

Differential equations is one of the most fundamental branches of mathematics. Almost every applied research in mathematics reduces to the differential equations either ordinary or the partial. As we know differential equations are super use for modulating and simulating phenomena and describing every mysterious phenomena of nature. Fractional differential equations (FDE) has its origin in generalization of ordinary differential equations to non-integer order. The successive generalization has stimulated interest of researchers to study the role of FDE and apply to the problems of science and engineering to understand the complex and mysterious phenomena. In the present scenario research in FDE is multidisciplinary and is proved to be very useful in wide variety of diverse fields such as control systems, electric drives, continuum mechanics, heat transfer, quantum mechanics, signal analysis, biomathematics, bioengineering and many more. In the past many years FDEs has emerged as a strong and well organized tool in the study of many fields of science and engineering. Ordinary differential equations (ODEs) have played a very important role in the history of theoretical population dynamics and will no doubt continue to serve as indispensable tools in future investigation. The solution of any mathematical model of a real dynamical system obtained by using ODEs is considered to be the best first approximation. However, more realistic models must include some of the past history of the system. Indeed,

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the solutions of models with the past history are derived by using the delay differential equations (DDEs). The use of DDEs in modeling of the population dynamics is currently very active.

DDEs have been shown to be valuable in various fields, including control systems [13], traffic models [11], population biology [27], chemical kinetics [12] and erythropoiesis [4]. Lakshmikantham [17] developed the foundational theory for fractional delay differential equations (FDDEs). In studies [2, 5, 30], the existence and uniqueness of solutions for fractional neutral DDEs have been formulated. The existence of positive solutions for nonlinear FDDEs involving the Riemann-Liouville derivative has been discussed in [18,29]. Kexue et al. [15] established the existence and uniqueness of mild solutions for a class of abstract FDDEs using the solution operator approach. The existence and uniqueness of solutions for FDDEs under various conditions have been demonstrated using fixed-point theorems, as seen in references such as [1,14,19,26,28].

In 2006, Daftardar-Gejji and Jafari introduced a new iterative method (DJM) [9] for solving functional equations. This method is straightforward yet offers greater accuracy compared to other iterative techniques. DJM has been successfully applied to solve PDEs such as Newell-Whitehead-Segel equation [24], Fisher's equation [6]. The DJM has been employed to develop new numerical methods [10,21,22]. The analytical solutions of the Ambartsumian equation [25], pantograph equation [23], and nonlinear equations with proportional delays [7], along with their properties are discussed in [8] using the DJM.

Solving nonlinear FDEs with proportional delay is a significant challenge in mathematical analysis and its applications. This challenge motivates our work in finding solutions for FDEs with proportional delay. In this paper, we establish existence and uniqueness results for these equations and obtain their solutions using the DJM in the form of power series.

The structure of the paper is as follows: Section 2 covers the basic definitions and results, with a detailed discussion of DJM in Section 2.1. Section 3 focuses on the DJM solution for FDEs, while Section 3.1 presents the existence and uniqueness results. The series solutions for the pantograph equation and the Ambartsumian equation are described in Sections 4 and 5, respectively, and the conclusions are summarized in Section 6.

2. Preliminaries and notations

DEFINITION 2.1. [16] The Riemann-Liouville fractional integral of order $\mu > 0$ of $f \in C[0,\infty)$ is defined as

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > 0. \tag{2.1}$$

DEFINITION 2.2. [16] The (left sided) Caputo fractional derivative of $f, f \in$

$C^m_{-1}, m \in \mathbb{N} \cup \{0\}$, is defined as:

$$\begin{aligned} D^\mu f(t) &= \frac{d^m}{dt^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{d^m}{dt^m} f(t), \quad m-1 < \mu < m, \quad m \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Note that for $0 \leqslant m-1 < \alpha \leqslant m$ and $\beta > -1$

$$\begin{aligned} I^\alpha x^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \\ (I^\alpha D^\alpha f)(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}. \end{aligned} \tag{2.3}$$

DEFINITION 2.3. [16] The Mittag-Leffler function is defined as

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}, \quad \alpha > 0. \tag{2.4}$$

The multi-parameter Mittag-Leffler function $E_{(\alpha_1,\alpha_2,\cdots,\alpha_n),\beta}$ is defined as:

DEFINITION 2.4. [16]

$$E_{(\alpha_1,\cdots,\alpha_n),\beta}(z_1,z_2,\cdots,z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_j \geqslant 0}} (k; l_1,\cdots,l_n) \left[\frac{\prod_{j=1}^{n} z_j^{l_j}}{\Gamma(\beta + \sum_{j=1}^{n} \alpha_j l_j)} \right].$$

where, $(k; l_1, l_2, \cdots, l_n)$ is the multinomial coefficient defined as

$$(k; l_1, l_2, \cdots, l_n) = \frac{k!}{l_1! l_2! \cdots l_n!}. \tag{2.5}$$

2.1. Daftardar-Gejji and Jafari method

The Daftardar-Gejji and Jafari Method (DJM) [9] is employed to solve nonlinear equations of the form

$$u = f + L(u) + N(u), \tag{2.6}$$

where L and N represent linear and nonlinear operators, respectively, with f as a known function.

The DJM produces a solution expressed as a series

$$u = \sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i \tag{2.7}$$

where $G_0 = N(u_0)$ and $G_i = \left\{ N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}$, $i \geqslant 1$.

From Eq. (2.7), the DJM produces its series terms as follows:

$$u_0 = f, \quad u_{m+1} = L(u_m) + G_m, \quad m = 0, 1, 2, \cdots. \tag{2.8}$$

3. DJM solution for fractional differential equations

Let us consider the initial value problem (IVP)

$$\begin{aligned} D^{\alpha_i} y_i(t) &= f_i(t, \bar{y}(t), \bar{y}(qt)), \ \ 0 < \alpha_i \leqslant 1, \ \ 0 < q < 1 \\ y_i(0) &= {}^i y_0, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{3.1}$$

where D^{α_i} denotes Caputo fractional derivative, $\bar{y}(t) = (y_1(t), y_2(t) \cdots, y_n(t))$, $\bar{y}(qt) = (y_1(qt), y_2(qt) \cdots, y_n(qt))$ and $f = (f_1, f_2 \cdots, f_n)$ is a continuous function defined on the $(2n+1)$ dimensional rectangle

$$R = \{|t| \leqslant a, |y_i(t) - {}^i y_0| \leqslant b_i, |y_i(qt) - {}^i y_0| \leqslant b_i, a > 0, b_i > 0, 1 \leqslant i \leqslant n\}.$$

The IVP (3.1) can be transformed into the following system of integral equations

$$y_i(t) = y_i(0) + \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(x, \bar{y}(x), \bar{y}(qx))\, dx, \quad 1 \leqslant i \leqslant n. \tag{3.2}$$

The DJM solution for system (3.2) is

$$\bar{y}(t) = \sum_{m=0}^{\infty} \bar{\phi}_m(t), \tag{3.3}$$

where $\bar{\phi}_m = \left({}^1\phi_m, {}^2\phi_m, \cdots, {}^n\phi_m\right)$ and

$$\begin{aligned} {}^i\phi_0(t) &= {}^i y_0, \\ {}^i\phi_1(t) &= \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i\left(x, \bar{\phi}_0(x) \bar{\phi}_0(qx)\right) dx, \\ {}^i\phi_{m+1}(t) &= \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left[f_i\left(x, \sum_{i=0}^{m} \bar{\phi}_i(x), \sum_{i=0}^{m} \bar{\phi}_i(qx)\right) \right. \\ &\qquad \left. - f_i\left(x, \sum_{i=0}^{m-1} \bar{\phi}_i(x), \sum_{i=0}^{m-1} \bar{\phi}_i(qx)\right) \right] dx, \\ &\qquad\qquad i = 1, 2, \cdots, n \ \text{ and } \ m = 1, 2, 3, \cdots. \end{aligned}$$

The m-term approximate solution is represented as

$$\bar{u}_m(t) = \sum_{i=0}^{m-1} \bar{\phi}_i(t), \quad m = 1, 2, \cdots. \tag{3.4}$$

3.1. Existence and uniqueness theorems

THEOREM 1. *Let f is continuous and $||f|| \leqslant M$ on rectangle R. The approximate solution $\bar{u}_m(t) = \sum_{i=0}^{m-1} \bar{\phi}_i(x)$ obtained using DJM for the IVP (3.1) exists on the interval*

$I=[-\varepsilon,\varepsilon]$, where

$$\varepsilon = \min\left\{a, \left(\frac{\Gamma(\alpha_1+1)b_1}{M}\right)^{\frac{1}{\alpha_1}}, \cdots, \left(\frac{\Gamma(\alpha_n+1)b_n}{M}\right)^{\frac{1}{\alpha_n}},\right.$$
$$\left.\left(\frac{\Gamma(\alpha_1+1)b_1}{q^{\alpha_1}M}\right)^{\frac{1}{\alpha_1}}, \cdots, \left(\frac{\Gamma(\alpha_n+1)b_n}{q^{\alpha_n}M}\right)^{\frac{1}{\alpha_n}}\right\}.$$

If t is in interval I then $(t,\bar{u}_m(t),\bar{u}_m(qt))$ is in rectangle R and

$$||\bar{u}_m(t)-\bar{y}(0)|| \leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}, \quad ||\bar{u}_m(qt)-\bar{y}(0)|| \leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}, \forall m.$$

Proof. The approximate solution of IVP (3.1) is

$$\bar{u}_m(t) = \sum_{i=0}^{m-1}\bar{\phi}_i(x), \ \ m=1,2,\cdots., \tag{3.5}$$

where

$$
\begin{aligned}
{}^i\phi_0(t) &= {}^iy_0,\\
{}^i\phi_1(t) &= \int_0^t \frac{(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i\left(x,\bar{\phi}_0(x)\bar{\phi}_0(qx)\right)dx, \qquad (3.6)\\
{}^i\phi_{m+1}(t) &= \int_0^t \frac{(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left[f_i\left(x,\sum_{i=0}^{m}\bar{\phi}_i(x),\sum_{i=0}^{m}\bar{\phi}_i(qx)\right)\right.\\
&\quad \left. - f_i\left(x,\sum_{i=0}^{m-1}\bar{\phi}_i(x),\sum_{i=0}^{m-1}\bar{\phi}_i(qx)\right)\right]dx. \qquad (3.7)
\end{aligned}
$$

We prove the result using mathematical induction on m. We observe that the result holds true for $m=1$.

Now,

$$
\begin{aligned}
\bar{u}_2(t) &= \bar{\phi}_0(t)+\bar{\phi}_1(t)\\
&= \bar{y}(0)+\bar{\phi}_1(t)\\
\Rightarrow \bar{u}_2(t)-\bar{y}(0) &= \bar{\phi}_1(t)\\
\Rightarrow {}^iu_2(t)-{}^iy_0 &= {}^i\phi_1(t)
\end{aligned}
$$

From Eq. (3.6), we have

$$
\begin{aligned}
{}^iu_2(t)-{}^iy_0 &= \int_0^t \frac{(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i\left(x,\bar{\phi}_0(x),\bar{\phi}_0(qx)\right)dx\\
|{}^iu_2(t)-{}^iy_0| &\leqslant \int_0^t \frac{|(t-x)|^{\alpha_i-1}}{\Gamma(\alpha_i)} |f_i\left(x,\bar{\phi}_0(x),\bar{\phi}_0(qx)\right)|dx\\
&\leqslant M\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}\\
&\leqslant b_i, \quad 1\leqslant i\leqslant n
\end{aligned}
$$

and

$$\begin{aligned} {}^iu_2(qt) - {}^iy_0 &\leqslant M\frac{|qt|^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ &\leqslant M\frac{q^{\alpha_i}|t|^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ &\leqslant M\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)} \quad (\because 0<q<1) \\ &\leqslant b_i, \quad 1\leqslant i\leqslant n \end{aligned}$$

For any $t\in I$, the point $(t,\bar{\phi}_0(t),\bar{\phi}_0(qt))\in R$. Thus

$$||\bar{u}_2(t)-\bar{y}(0)||\leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}, \quad ||\bar{u}_2(qt)-\bar{y}(0)||\leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

Hence result is true for $m=2$.

Let's assume the result holds true for the positive integer $m-1$.

i. e. For any $t\in I$, the point $(t,\bar{u}_{m-2}(t),\bar{u}_{m-2}(qt))\in R$ and

$$||\bar{u}_{m-1}(t)-\bar{y}(0)||\leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}, ||\bar{u}_{m-1}(qt)-\bar{y}(0)||\leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

To prove result is true for m.

From (3.5), (3.6) and (3.7), we write

$$\begin{aligned} {}^iu_m(t) - {}^iy_0 &= \int_0^t \frac{(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i\left(x,\sum_{j=0}^{m-2}\bar{\phi}_j(x),\sum_{j=0}^{m-2}\bar{\phi}_j(qx)\right)dx \\ |{}^iu_m(t) - {}^iy_0| &\leqslant \int_0^t \frac{|(t-x)|^{\alpha_i-1}}{\Gamma(\alpha_i)} \left|f_i\left(x,\sum_{j=0}^{m-2}\bar{\phi}_j(x),\sum_{j=0}^{m-2}\bar{\phi}_j(qx)\right)\right|dx \\ &= \int_0^t \frac{|(t-x)|^{\alpha_i-1}}{\Gamma(\alpha_i)} |f_i(x,\bar{u}_{m-1}(x),\bar{u}_{m-1}(qx))|dx. \end{aligned}$$

By induction hypothesis the points $(t,\bar{u}_{m-1}(t),\bar{u}_{m-1}(qt))\in$R, $\forall t\in I$. Hence

$$\begin{aligned} |{}^iu_m(t) - {}^iy_0| &\leqslant M\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ &\leqslant b_i, \quad \text{for} \quad t\in I, 1\leqslant i\leqslant n \end{aligned}$$

and

$$\begin{aligned} |{}^iu_m(qt) - {}^iy_0| &\leqslant M\frac{|qt|^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ &= M\frac{q^{\alpha_i}|t|^{\alpha_i}}{\Gamma(\alpha_i+1)} \\ &= M\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)} \quad (\because 0<q<1) \\ &\leqslant b_i, \quad \text{for} \quad t\in I, 1\leqslant i\leqslant n \end{aligned}$$

i. e. The points $(t,\bar{u}_m(t),\bar{u}_m(qt)) \in R, \forall m \in I^+$ when $t \in I$ and

$$||\bar{u}_m(t)-\bar{y}(0)|| \leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)},\ ||\bar{u}_m(qt)-\bar{y}(0)|| \leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}, \forall m.$$

Hence by mathematical induction, the result is true for all positive integers m. □

THEOREM 2. *Let f be a continuous function defined on the rectangle R and $||f|| \leqslant M$ on R. Suppose that f satisfies Lipschitz type condition*

$$|f_i(t,\bar{y_1}(t),\bar{y_1}(qt)) - f_i(t,\bar{y_2}(t),\bar{y_2}(qt))| \leqslant L_1\,|\bar{y_1}(t)-\bar{y_2}(t)| + L_2\,|\bar{y_1}(qt)-\bar{y_2}(qt)|\,, \forall i,$$

then the series solution (3.3) converges on the interval $I=[-\tau,\tau]$ to a solution of the IVP (3.1).

Proof. The equivalent integral equation of IVP (3.1) is

$$y_i(t) = y_i(0) + \int_0^t \frac{(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i\,(x,\bar{y}(x),\bar{y}(qx))\,dx, \quad 1 \leqslant i \leqslant n.$$

Using DJM, we have

$$\begin{aligned}
||{}^i\phi_0(t)|| &= y_0,\\
|{}^i\phi_1(t)| &\leqslant \int_0^t \frac{|t-x|^{\alpha_i-1}}{\Gamma(\alpha_i)} |f_i\left(x,\bar{\phi}_0(x),\bar{\phi}_0(qx)\right)|dx\\
&\leqslant M\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}\\
\therefore ||\bar{\phi}_1(t)|| &\leqslant M\sum_{i=1}^{n}\frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}\\
&\leqslant M \sum_{r_1+r_2+\cdots+r_n=1} (1;r_1,r_2,\cdots,r_n)\frac{\Pi_{i=1}^{n}\left(|t|^{\alpha_i}\right)^{r_i}}{\Gamma\left(\sum_{i=1}^{n} r_i\alpha_i+1\right)}
\end{aligned}$$

Since f satisfies Lipschitz conditions in second and third variables, we have

$$\begin{aligned}
|{}^i\phi_2(t)| &\leqslant \int_0^t \frac{|t-x|^{\alpha_i-1}}{\Gamma(\alpha_i)} \Big| f_i\left(x,\bar{\phi}_0(x)+\bar{\phi}_1(x),\bar{\phi}_0(qx)+\bar{\phi}_1(qx)\right)\\
&\quad - f_i\left(x,\bar{\phi}_0(x),\bar{\phi}_0(qx)\right)\Big| dx\\
&\leqslant \int_0^t \frac{|t-x|^{\alpha_i-1}}{\Gamma(\alpha_i)} \left(L_1||\bar{\phi}_1(x)|| + L_2||\bar{\phi}_1(qx)||\right)dx\\
&\leqslant \sum_{j=1}^{n}\frac{M\,(L_1+q^{\alpha_j}L_2)\,|t|^{\alpha_i+\alpha_j}}{\Gamma(\alpha_i+\alpha_j+1)}\\
&\leqslant M\,(L_1+L_2)\sum_{j=1}^{n}\frac{|t|^{\alpha_i+\alpha_j}}{\Gamma(\alpha_i+\alpha_j+1)} \quad (\because 0<q<1)
\end{aligned}$$

Hence

$$\begin{aligned} ||\bar{\phi}_2(t)|| &\leqslant M(L_1+L_2)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|t|^{\alpha_i+\alpha_j}}{\Gamma(\alpha_i+\alpha_j+1)} \\ &\leqslant \frac{M}{(L_1+L_2)}\sum_{r_1+r_2+\cdots+r_n=2}(2;r_1,r_2,\cdots,r_n)\frac{\Pi_{i=1}^{n}\left(|t|^{\alpha_i}(L_1+L_2)\right)^{r_i}}{\Gamma\left(\sum_{i=1}^{n}r_i\alpha_i+1\right)}. \end{aligned} \tag{3.8}$$

By mathematical induction we prove that

$$||\bar{\phi}_m(t)|| \leqslant \frac{M}{(L_1+L_2)}\sum_{r_1+r_2+\cdots+r_n=m}(m;r_1,r_2,\cdots,r_n)\frac{\Pi_{i=1}^{n}\left(|t|^{\alpha_i}(L_1+L_2)\right)^{r_i}}{\Gamma\left(\sum_{i=1}^{n}r_i\alpha_i+1\right)}. \tag{3.9}$$

Assume that the inequality (3.9) is true for integer m.

Consider

$$\begin{aligned} |{}^{i}\phi_{m+1}(t)| &\leqslant \int_0^t \frac{|t-x|^{\alpha_i-1}}{\Gamma(\alpha_i)}\Big(L_1||\bar{\phi}_m(x)||+L_2||\bar{\phi}_m(qx)||\Big)\,dx \\ &\leqslant \frac{M(L_1+L_2)}{(L_1+L_2)}\sum_{r_1+r_2+\cdots+r_n=m}(m;r_1,r_2,\cdots,r_n) \\ &\quad\times\int_0^t \frac{|t-x|^{\alpha_i-1}}{\Gamma(\alpha_i)}\frac{\Pi_{j=1}^{n}\left(|x|^{\alpha_j}(L_1+L_2)\right)^{r_j}}{\Gamma\left(\sum_{j=1}^{n}r_j\alpha_j+1\right)}dx \\ &= \frac{M}{(L_1+L_2)}\sum_{r_1+r_2+\cdots+r_n=m}(m;r_1,r_2,\cdots,r_n)(L_1+L_2)^{m+1} \\ &\quad\times\frac{|t|^{\sum_{j=1,j\neq i}^{n}\alpha_j r_j}|t|^{\alpha_i(r_i+1)}}{\Gamma\left(\sum_{j=1,j\neq i}^{n}r_j\alpha_j+\alpha_i(r_i+1)+1\right)}. \end{aligned}$$

Hence we get

$$\begin{aligned} |{}^{i}\phi_{m+1}(t)| &\leqslant \frac{M}{(L_1+L_2)}\sum_{r_1+r_2+\cdots+r_n=m+1}(m;r_1,r_2,\cdots,r_i-1,\cdots,r_n) \\ &\quad\times\frac{\Pi_{j=1}^{n}\left(|t|^{\alpha_j}(L_1+L_2)\right)^{r_j}}{\Gamma\left(\sum_{j=1}^{n}r_j\alpha_j+1\right)}. \end{aligned}$$

Thus

$$||\bar{\phi}_{m+1}(t)|| \leqslant \frac{M}{(L_1+L_2)}\sum_{r_1+r_2+\cdots+r_n=m+1}(m+1;r_1,r_2,\cdots,r_n)\frac{\Pi_{i=1}^{n}\left(|t|^{\alpha_i}(L_1+L_2)\right)^{r_i}}{\Gamma\left(\sum_{i=1}^{n}r_i\alpha_i+1\right)}. \tag{3.10}$$

From inequality (3.10), it can be observed that the m^{th} term in Mittag-Leffler function

$$\frac{M}{(L_1+L_2)}E_{(\alpha_1,\alpha_2,\cdots,\alpha_n),1}\left((L_1+L_2)|t|^{\alpha_1},(L_1+L_2)|t|^{\alpha_2},\cdots,(L_1+L_2)|t|^{\alpha_n}\right)$$

is serves as an upper bound for the m^{th} term in DJM solution series (3.3). This establishes that the DJM solution series (3.3) is converges under the specified conditions. $\square$

THEOREM 3. *Let f be a continuous function defined on the strip $S: |t| \leqslant a,\ |y_i| < \infty,\ a > 0,\ 1 \leqslant i \leqslant n$ and suppose that f satisfies on S a Lipschitz type condition*

$$|f_i(t, \bar{y_1}(t), \bar{y_1}(qt)) - f_i(t, \bar{y_2}(t), \bar{y_2}(qt))| \leqslant L_1 |\bar{y_1}(t) - \bar{y_2}(t)| + L_2 |\bar{y_1}(qt) - \bar{y_2}(qt)|, \forall i,$$

Then the series solution (3.3) *converges on the interval $I = [-\tau, \tau]$ to a solution of the IVP* (3.1).

THEOREM 4. *Let ${}^j u_m = \sum_{i=0}^{m-1} {}^j\phi_i$, and ${}^j v_m = \sum_{i=0}^{m-1} {}^j\psi_i$ be m-term approximate DJM solutions defined on the interval I of the IVP:*

$$D^{\alpha_j} y_j(x) = f_j(t, \bar{y}(t), \bar{y}(qt)), \quad 0 < \alpha_i \leqslant 1, \quad 0 < q < 1, \quad 1 \leqslant j \leqslant n, \tag{3.11}$$

with initial conditions $\bar{y}(0) = \bar{c} = (c_1, \cdots, c_n)$ and $\bar{y}(0) = \bar{d} = (d_1, \cdots, d_n)$ respectively. Suppose that $f = (f_1, \cdots, f_n)$ satisfies Lipschitz condition in second and third variable with Lipschitz constants L_1 and L_2. Then

$$|{}^j u_m(t) - {}^j v_m(t)| \leqslant ||\bar{c} - \bar{d}|| \sum_{i=0}^{m-1} \frac{((L_1 + L_2)|t|^{\alpha_j})^i}{\Gamma(i\alpha_j + 1)}$$

and

$$|{}^j u_m(qt) - {}^j v_m(qt)| \leqslant ||\bar{c} - \bar{d}|| \sum_{i=0}^{m-1} \frac{((L_1 + L_2)|t|^{\alpha_j})^i}{\Gamma(i\alpha_j + 1)}, \quad 1 \leqslant j \leqslant n$$

Also,

$$|{}^j u_m(t) - {}^j v_m(t)| \leqslant ||\bar{c} - \bar{d}|| E_{\alpha_j}((L_1 + L_2)|t|^{\alpha_j})$$

and

$$|{}^j u_m(qt) - {}^j v_m(qt)| \leqslant ||\bar{c} - \bar{d}|| E_{\alpha_j}((L_1 + L_2)|t|^{\alpha_j}).$$

Proof. We prove the result using mathematical induction on m.

If $m = 1$ then $|{}^j u_1(t) - {}^j v_1(t)| = |c_j - d_j| \leqslant ||\bar{c} - \bar{d}||$ and $|{}^j u_1(qt) - {}^j v_1(qt)| = |c_j - d_j| \leqslant ||\bar{c} - \bar{d}||$.

If $m = 2$ then

$$\begin{aligned}
&{}^j u_2(t) - {}^j v_2(t) \\
&= ({}^j\phi_0 + {}^j\phi_1) - ({}^j\psi_0 + {}^j\psi_1) \\
&= \left(c_j + \int_0^t \frac{(t-x)^{\alpha_j - 1}}{\Gamma(\alpha_j)} f_j\left(x, \bar{\phi}_0(x), \bar{\phi}_0(qx)\right) dx \right) \\
&\quad - \left(d_j + \int_0^t \frac{(t-x)^{\alpha_j - 1}}{\Gamma(\alpha_j)} f_j\left(x, \bar{\psi}_0(x), \bar{\psi}_0(qx)\right) dx \right)
\end{aligned}$$

$$\begin{aligned}
&|{}^j u_2(t) - {}^j v_2(t)| \\
&\leqslant |c_j - d_j| + \int_0^t \frac{|t-x|^{\alpha_j - 1}}{\Gamma(\alpha_j)} \left| f_j\left(x, \bar{\phi}_0(x), \bar{\phi}_0(qx)\right) - f_j\left(x, \bar{\psi}_0(x), \bar{\psi}_0(qx)\right)\right| dx \\
&\leqslant ||\bar{c} - \bar{d}|| + \int_0^t \frac{|t-x|^{\alpha_j - 1}}{\Gamma(\alpha_j)} \left(L_1|\bar{\phi}_0(x) - \bar{\psi}_0(x)| + L_2|\bar{\phi}_0(qx) - \bar{\psi}_0(qx)|\right) dx \\
&\leqslant ||\bar{c} - \bar{d}|| + \int_0^t \frac{|t-x|^{\alpha_j - 1}}{\Gamma(\alpha_j)} \left(L_1||\bar{c} - \bar{d}|| + L_2 q||\bar{c} - \bar{d}||\right) dx \\
&\leqslant ||\bar{c} - \bar{d}|| + \int_0^t \frac{|t-x|^{\alpha_j - 1}}{\Gamma(\alpha_j)} (L_1 + L_2)\,||\bar{c} - \bar{d}|| dx
\end{aligned}$$

$$|{}^j u_2(t) - {}^j v_2(t)| \leqslant ||\bar{c} - \bar{d}|| + ||\bar{c} - \bar{d}|| \frac{(L_1 + L_2)\,|t|^{\alpha_j}}{\Gamma(\alpha_j + 1)}$$

and

$$\begin{aligned}
|{}^j u_2(qt) - {}^j v_2(qt)| &\leqslant ||\bar{c} - \bar{d}|| + ||\bar{c} - \bar{d}|| \frac{(L_1 + L_2)\,|qt|^{\alpha_j}}{\Gamma(\alpha_j + 1)} \\
&\leqslant ||\bar{c} - \bar{d}|| + ||\bar{c} - \bar{d}|| \frac{(L_1 + L_2)\,|t|^{\alpha_j}}{\Gamma(\alpha_j + 1)} \quad (\because 0 < q < 1).
\end{aligned}$$

Thus, the result is true for $m = 1$ and $m = 2$.

Assume that the result holds true for the positive integer $m - 1$

$$\text{i.e. } |{}^j u_{m-1}(t) - {}^j v_{m-1}(t)| \leqslant ||\bar{c} - \bar{d}|| \sum_{i=0}^{m-2} \frac{((L_1 + L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j + 1)}$$

and

$$|{}^j u_{m-1}(qt) - {}^j v_{m-1}(qt)| \leqslant ||\bar{c} - \bar{d}|| \sum_{i=0}^{m-2} \frac{((L_1 + L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j + 1)}.$$

Now, consider

$$\begin{aligned}
&|{}^j u_m(t) - {}^j v_m(t)| \\
&= |\sum_{i=0}^{m-1} \left({}^j\phi_i - {}^j\psi_i\right)| \\
&\leqslant |c_j - d_j| + I^{\alpha_j} \left| \left(f_j\left(t, \sum_{i=0}^{m-2} \bar{\phi}_i(t), \sum_{i=0}^{m-2} \bar{\phi}_i(qt)\right) - f_j\left(t, \sum_{i=0}^{m-2} \bar{\psi}_i(t), \sum_{i=0}^{m-2} \bar{\psi}_i(qt)\right)\right)\right| \\
&\leqslant |c_j - d_j| + I^{\alpha_j} \left(L_1 |\sum_{i=0}^{m-2} \bar{\phi}_i(t) - \sum_{i=0}^{m-2} \bar{\psi}_i(t)| + L_2 |\sum_{i=0}^{m-2} \bar{\phi}_i(qt) - \sum_{i=0}^{m-2} \bar{\psi}_i(qt)| \right) \\
&\leqslant |c_j - d_j| + I^{\alpha_j} \left(L_1 |{}^j u_{m-1}(t) - {}^j v_{m-1}(t)| + L_2 |{}^j u_{m-1}(qt) - {}^j v_{m-1}(qt)| \right) \\
&\leqslant ||\bar{c} - \bar{d}|| + I^{\alpha_j} \left(L_1 \|\bar{c} - \bar{d}\| \sum_{i=0}^{m-2} \frac{((L_1 + L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j + 1)} + L_2 \|\bar{c} - \bar{d}\| \sum_{i=0}^{m-2} \frac{((L_1 + L_2)\,|qt|^{\alpha_j})^i}{\Gamma(i\alpha_j + 1)} \right)
\end{aligned}$$

$$\leqslant ||\bar{c}-\bar{d}||+I^{\alpha_j}\left(L_1\|\bar{c}-\bar{d}\|\sum_{i=0}^{m-2}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}+L_2\|\bar{c}-\bar{d}\|\sum_{i=0}^{m-2}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}\right)$$

$$\leqslant \|\bar{c}-\bar{d}\|+((L_1+L_2)\,I^{\alpha_j}\left(\|\bar{c}-\bar{d}\|\sum_{i=0}^{m-2}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}\right).$$

$$\therefore |{}^ju_m(t)-{}^jv_m(t)|\leqslant ||\bar{c}-\bar{d}||\sum_{i=0}^{m-1}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}$$

and

$$\begin{aligned}|{}^ju_m(qt)-{}^jv_m(qt)| &\leqslant ||\bar{c}-\bar{d}||\sum_{i=0}^{m-1}\frac{((L_1+L_2)\,|qt|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}\\ &\leqslant ||\bar{c}-\bar{d}||\sum_{i=0}^{m-1}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}\quad (\because 0<q<1).\end{aligned}$$

As $m\longrightarrow\infty$, we get

$$\begin{aligned}|{}^ju_m(t)-{}^jv_m(t)| &\leqslant ||\bar{c}-\bar{d}||\sum_{i=0}^{\infty}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}\\ &= ||\bar{c}-\bar{d}||E_{\alpha_j}\left((L_1+L_2)\,|t|^{\alpha_j}\right)\end{aligned}$$

and

$$\begin{aligned}|{}^ju_m(qt)-{}^jv_m(qt)| &\leqslant ||\bar{c}-\bar{d}||\sum_{i=0}^{\infty}\frac{((L_1+L_2)\,|t|^{\alpha_j})^i}{\Gamma(i\alpha_j+1)}\\ &= ||\bar{c}-\bar{d}||E_{\alpha_j}\left((L_1+L_2)\,|t|^{\alpha_j}\right).\end{aligned}$$

Consequently, the DJM solution continuously dependent on the initial conditions. The uniqueness of the IVP solution is directly inferred from Theorem 4. □

THEOREM 5. *Let ${}^ju_m=\sum_{i=0}^{m-1}{}^j\phi_i$, and ${}^jv_m=\sum_{i=0}^{m-1}{}^j\psi_i$ be m-term approximate DJM solutions defined on the interval I of the IVPs:*

$$\begin{aligned}D^{\alpha_j}y_j(t) &= f_j(t,\bar{y}(t),\bar{y}(qt)),\quad 0<\alpha_i\leqslant 1,\quad 0<q<1,\quad 1\leqslant j\leqslant n,\\ \bar{y}(0) &= \bar{c}=(c_1,\cdots,c_n)\end{aligned}$$

and

$$\begin{aligned}D^{\alpha_j-\varepsilon}z_j(t) &= f_j(t,\bar{z}(t),\bar{z}(qt)),\quad 0<\alpha_i\leqslant 1,\quad 0<q<1,\quad 1\leqslant j\leqslant n,\\ \bar{z}(0) &= \bar{c},\quad 0<\varepsilon<min\{\alpha_j\}\leqslant 1,\end{aligned}$$

respectively. Suppose that $f=(f_1,\cdots,f_n)$ satisfies Lipschitz condition in second and third variable with Lipschitz constants L_1 and L_2. Then

$$|{}^ju(t)-{}^jv(t)|\leqslant\frac{M}{(L_1+L_2)}E_{(\alpha_j-\varepsilon,\alpha_j)}\left((L_1+L_2)t^{\alpha_j-\varepsilon},t^{\varepsilon}\right)\tag{3.12}$$

and

$$|{}^{j}u(qt)-{}^{j}v(qt)| \leqslant \frac{M}{(L_1+L_2)}E_{(\alpha_j-\varepsilon,\alpha_j)}\left((L_1+L_2)t^{\alpha_j-\varepsilon},t^{\varepsilon}\right) \tag{3.13}$$

where $1 \leqslant j \leqslant n$, ${}^{j}u(t)=\lim_{m\longrightarrow\infty}{}^{j}u_m(t)$, ${}^{j}v(t)=\lim_{m\longrightarrow\infty}{}^{j}v_m(t)$, ${}^{j}u(qt)=\lim_{m\longrightarrow\infty}{}^{j}u_m(qt)$ *and* ${}^{j}v(qt)=\lim_{m\longrightarrow\infty}{}^{j}v_m(qt)$.

Proof. We have

$$\begin{aligned}
|{}^{j}u_1(t)-{}^{j}v_1(t)| &= 0 \quad \text{and} \quad |{}^{j}u_1(qt)-{}^{j}v_1(qt)|=0,\\
|{}^{j}u_2(t)-{}^{j}v_2(t)| &= |I^{\alpha_j}f_j\left(t,\bar{\phi}_0(t),\bar{\phi}_0(qt)\right)-I^{\alpha_j-\varepsilon}f_j\left(t,\bar{\psi}_0(t),\bar{\psi}_0(qt)\right)|\\
&\leqslant M\left(\frac{|t|^{\alpha_j}}{\Gamma(\alpha_j+1)}+\frac{|t|^{\alpha_j-\varepsilon}}{\Gamma(\alpha_j-\varepsilon+1)}\right),
\end{aligned}$$

and

$$\begin{aligned}
&|{}^{j}u_2(qt)-{}^{j}v_2(qt)|\\
\leqslant\ & M\left(\frac{|qt|^{\alpha_j}}{\Gamma(\alpha_j+1)}+\frac{|qt|^{\alpha_j-\varepsilon}}{\Gamma(\alpha_j-\varepsilon+1)}\right),\\
\leqslant\ & M\left(\frac{|t|^{\alpha_j}}{\Gamma(\alpha_j+1)}+\frac{|t|^{\alpha_j-\varepsilon}}{\Gamma(\alpha_j-\varepsilon+1)}\right), \quad (\because 0<q<1)\\
&|{}^{j}u_3(t)-{}^{j}v_3(t)|\\
=\ & \Big|I^{\alpha_j}f_j\left(t,\bar{\phi}_0(t)+\bar{\phi}_1(t),\bar{\phi}_0(qt)+\bar{\phi}_1(qt)\right)\\
&-I^{\alpha_j-\varepsilon}f_j\left(t,\bar{\psi}_0(t)+\bar{\psi}_1(t),\bar{\psi}_0(t)+\bar{\psi}_1(t)\right)\Big|\\
\leqslant\ & \Big|I^{\alpha_j}f_j\left(t,\bar{\phi}_0(t)+\bar{\phi}_1(t),\bar{\phi}_0(qt)+\bar{\phi}_1(qt)\right)\\
&-I^{\alpha_j}f_j\left(t,\bar{\psi}_0(t)+\bar{\psi}_1(t),\bar{\psi}_0(t)+\bar{\psi}_1(t)\right)\Big|\\
&+\Big|I^{\alpha_j-\varepsilon}f_j\left(t,\bar{\phi}_0(t)+\bar{\phi}_1(t),\bar{\phi}_0(qt)+\bar{\phi}_1(qt)\right)\\
&-I^{\alpha_j-\varepsilon}f_j\left(t,\bar{\psi}_0(t)+\bar{\psi}_1(t),\bar{\psi}_0(qt)+\bar{\psi}_1(qt)\right)\Big|\\
&+\Big|I^{\alpha_j}f_j\left(t,\bar{\psi}_0(t)+\bar{\psi}_1(t),\bar{\psi}_0(qt)+\bar{\psi}_1(qt)\right)\\
&-I^{\alpha_j-\varepsilon}f_j\left(t,\bar{\phi}_0(t)+\bar{\phi}_1(t),\bar{\phi}_0(qt)+\bar{\phi}_1(qt)\right)\Big|\\
\leqslant\ & M(L_1+L_2)\left(\frac{|t|^{2\alpha_j}}{\Gamma(2\alpha_j+1)}+2\frac{|t|^{2\alpha_j-\varepsilon}}{\Gamma(2\alpha_j-\varepsilon+1)}+\frac{|t|^{2\alpha_j-2\varepsilon}}{\Gamma(2\alpha_j-2\varepsilon+1)}\right)\\
&+M\left(\frac{|t|^{\alpha_j}}{\Gamma(\alpha_j+1)}+\frac{|t|^{\alpha_j-\varepsilon}}{\Gamma(\alpha_j+1)}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&|{}^j u_3(qt) - {}^j v_3(qt)| \\
&\leqslant M(L_1+L_2)\left(\frac{|qt|^{2\alpha_j}}{\Gamma(2\alpha_j+1)} + 2\frac{|qt|^{2\alpha_j-\varepsilon}}{\Gamma(2\alpha_j-\varepsilon+1)} + \frac{|qt|^{2\alpha_j-2\varepsilon}}{\Gamma(2\alpha_j-2\varepsilon+1)}\right) \\
&\quad + M\left(\frac{|qt|^{\alpha_j}}{\Gamma(\alpha_j+1)} + \frac{|qt|^{\alpha_j-\varepsilon}}{\Gamma(\alpha_j+1)}\right) \\
&\leqslant M(L_1+L_2)\left(\frac{|t|^{2\alpha_j}}{\Gamma(2\alpha_j+1)} + 2\frac{|t|^{2\alpha_j-\varepsilon}}{\Gamma(2\alpha_j-\varepsilon+1)} + \frac{|t|^{2\alpha_j-2\varepsilon}}{\Gamma(2\alpha_j-2\varepsilon+1)}\right) \\
&\quad + M\left(\frac{|t|^{\alpha_j}}{\Gamma(\alpha_j+1)} + \frac{|t|^{\alpha_j-\varepsilon}}{\Gamma(\alpha_j+1)}\right) \quad (\because 0<q<1)
\end{aligned}
$$

and so on. Using induction,

$$
|{}^j u_n(t) - {}^j v_n(t)| \leqslant \frac{M}{(L_1+L_2)} \sum_{m=0}^{n-2} \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{\left(t^{\alpha_j-\varepsilon}(L_1+L_2)\right)^{m+1} (t^{\varepsilon})^{m+1-i}}{\Gamma(1+(\alpha_j-\varepsilon)i+\alpha_j(m+1-i))}.
$$

and

$$
\begin{aligned}
&|{}^j u_n(qt) - {}^j v_n(qt)| \\
&\leqslant \frac{M}{(L_1+L_2)} \sum_{m=0}^{n-2} \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{\left((qt)^{\alpha_j-\varepsilon}(L_1+L_2)\right)^{m+1} ((qt)^{\varepsilon})^{m+1-i}}{\Gamma(1+(\alpha_j-\varepsilon)i+\alpha_j(m+1-i))} \\
&\leqslant \frac{M}{(L_1+L_2)} \sum_{m=0}^{n-2} \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{\left(t^{\alpha_j-\varepsilon}(L_1+L_2)\right)^{m+1} (t^{\varepsilon})^{m+1-i}}{\Gamma(1+(\alpha_j-\varepsilon)i+\alpha_j(m+1-i))} \quad (\because 0<q<1)
\end{aligned}
$$

As $n \longrightarrow \infty$, we get required inequality (3.12) and (3.13). Consequently, the DJM solution continuously dependent on the fractional order. □

4. Series solution of pantograph equations: application in electric trains

A pantograph is a mechanism used in electric trains to draw current from overhead lines. The pantograph equation, developed by Ockendon and Taylor in 1971, has its roots in electrodynamics [20].

$$
D^{\alpha_i} y_i(t) = a_i y_i(t) + b_i y_i(qt)), \;\; y_i(0)=1, \;\; 0<\alpha_i \leqslant 1, \;\; 0<q<1, \;\; 1 \leqslant i \leqslant n. \tag{4.1}
$$

The integral equation corresponding to (4.1) is

$$
y_i(t) = 1 + I^{\alpha_i} a_i y_i(t) + I^{\alpha_i} b_i y_i(qt). \tag{4.2}
$$

Using DJM, we get

$$\begin{aligned}
{}^{i}\phi_0(t) &= 1, \\
{}^{i}\phi_1(t) &= a_i I^{\alpha_i}\, {}^{i}\phi_0(t) + b_i I^{\alpha_i}\, {}^{i}\phi_0(qt) \\
&= a_i \frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} + b_i \frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} \\
&= (a_i+b_i) \frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)} \\
{}^{i}\phi_2(t) &= a_i I^{\alpha_i}\, {}^{i}\phi_1(t) + b_i I^{\alpha_i}\, {}^{i}\phi_1(qt) \\
&= a_i (a_i+b_i) \frac{t^{2\alpha_i}}{\Gamma(2\alpha_i+1)} + b_i (a_i+b_i)\, q^{\alpha_i} \frac{t^{2\alpha_i}}{\Gamma(2\alpha_i+1)} \\
&= (a_i+b_i)(a_i+b_i q^{\alpha_i}) \frac{t^{2\alpha_i}}{\Gamma(2\alpha_i+1)}
\end{aligned}$$

$$\begin{aligned}
{}^{i}\phi_3(t) &= (a_i+b_i)(a_i+b_i q^{\alpha_i})\left(a_i+b_i q^{2\alpha_i}\right) \frac{t^{3\alpha_i}}{\Gamma(3\alpha_i+1)} \\
&\ \,\vdots \\
{}^{i}\phi_n(t) &= (a_i+b_i)(a_i+b_i q^{\alpha_i})\left(a_i+b_i q^{2\alpha_i}\right)\cdots\left(a_i+b_i q^{(n-1)\alpha_i}\right) \frac{t^{n\alpha_i}}{\Gamma(n\alpha_i+1)} \\
&= \prod_{j=0}^{n-1}(a_i+b_i q^{j\alpha_i}) \frac{t^{n\alpha_i}}{\Gamma(n\alpha_i+1)}
\end{aligned}$$

$\therefore$ The DJM solution of eq. (4.1) is

$$\begin{aligned}
y(t) &= {}^{i}\phi_0(t) + {}^{i}\phi_1(t) + {}^{i}\phi_2(t) + {}^{i}\phi_3(t) + \cdots. \\
y(t) &= \sum_{n=0}^{\infty}\prod_{j=0}^{n-1}(a_i+b_i q^{j\alpha_i}) \frac{t^{n\alpha_i}}{\Gamma(n\alpha_i+1)},
\end{aligned} \tag{4.3}$$

where

$$\prod_{j=0}^{n-1}(a_i+b_i q^{j\alpha_i}) = 1 \quad \text{for} \quad n=0.$$

THEOREM 6. *If $0<q<1$, then the power series (4.3) converges for all finite values of t.*

5. Series solution of Ambartsumian equations: application in astronomy

Ambartsumian [3] developed a delay differential equation to model the variations in the surface brightness of the Milky Way. The equation is expressed as:

$$D^{\alpha_i} y_i(t) = -y_i(t) + \frac{1}{q} y_i\left(\frac{t}{q}\right), \quad y_i(0) = \lambda_i, \quad q>1. \tag{5.1}$$

The integral equation corresponding to (5.1) is

$$y_i(t) = \lambda_i - I^{\alpha_i} y_i(t) + \frac{1}{q} I^{\alpha_i} y_i\left(\frac{t}{q}\right) \tag{5.2}$$

Applying DJM, We get

$$\begin{aligned}
{}^i\phi_0(t) &= \lambda_i, \\
{}^i\phi_1(t) &= -I^{\alpha_i}{}^i\phi_0(t) + \frac{1}{q} I^{\alpha_i}{}^i\phi_0\left(\frac{t}{q}\right) \\
&= -\frac{\lambda_i t^{\alpha_i}}{\Gamma(\alpha_i+1)} + \frac{1}{q}\frac{\lambda_i t^{\alpha_i}}{\Gamma(\alpha_i+1)} \\
&= \left(\frac{1}{q} - 1\right)\frac{\lambda_i t^{\alpha_i}}{\Gamma(\alpha_i+1)} \\
{}^i\phi_2(t) &= -I^{\alpha_i}{}^i\phi_1(t) + \frac{1}{q} I^{\alpha_i}{}^i\phi_1\left(\frac{t}{q}\right) \\
&= \left(\frac{1}{q} - 1\right)\left(\frac{1}{q^{\alpha_i+1}} - 1\right)\frac{\lambda_i t^{2\alpha_i}}{\Gamma(2\alpha_i+1)} \\
{}^i\phi_3(t) &= -I^{\alpha_i}{}^i\phi_2(t) + \frac{1}{q} I^{\alpha_i}{}^i\phi_2\left(\frac{t}{q}\right) \\
&= \left(\frac{1}{q} - 1\right)\left(\frac{1}{q^{\alpha_i+1}} - 1\right)\left(\frac{1}{q^{2\alpha_i+1}} - 1\right)\frac{\lambda_i t^{3\alpha_i}}{\Gamma(3\alpha_i+1)} \\
&\;\;\vdots \\
{}^i\phi_n(t) &= \prod_{j=1}^{n}\left(\frac{1}{q^{(j-1)\alpha_i+1}} - 1\right)\frac{\lambda_i t^{n\alpha_i}}{\Gamma(n\alpha_i+1)}
\end{aligned}$$

$\therefore$ The DJM solution of eq. (4.1) is

$$\begin{aligned}
y(t) &= {}^i\phi_0(t) + {}^i\phi_1(t) + {}^i\phi_2(t) + {}^i\phi_3(t) + \cdots. \\
y(t) &= \sum_{n=0}^{\infty}\prod_{j=1}^{n}\left(\frac{1}{q^{(j-1)\alpha_i+1}} - 1\right)\frac{\lambda_i t^{n\alpha_i}}{\Gamma(n\alpha_i+1)},
\end{aligned} \tag{5.3}$$

where

$$\prod_{j=1}^{n}\left(\frac{1}{q^{(j-1)\alpha_i+1}} - 1\right) = 1 \quad \text{for} \quad n = 0.$$

THEOREM 7. *If $q > 1$, then the power series (5.3) converges for all finite values of t.*

6. Conclusions

This paper focuses on solving non-linear fractional differential equations with proportional delay through the Daftardar-Gejji and Jafari method (DJM). We established the existence and uniqueness theorems for FDEs with proportional delay and used the Lipschitz condition to derive convergence results. Furthermore, we proved that the DJM solution exhibits continuous dependence on initial conditions and fractional order, and applied the method to obtain series solutions for the pantograph and Ambartsumian equations.

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