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EXISTENCE RESULTS FOR PANTOGRAPH DIFFERENTIAL EQUATIONS WITH HADAMARD FUNCTIONAL FRACTIONAL DERIVATIVE

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Abstract. This study examines the existence of solutions for various types of fractional pantograph differential equations by employing the Hadamard functional fractional derivative. The results are derived by means of the Banach and Krasnoselskii fixed point theorems. Theoretical results are validated through illustrative examples.

1. Introduction

In recent years, there has been a considerable rise in enthusiasm concerning the application of fractional calculus to describe, and manage various engineering frameworks [13, 17, 26, 27]. The effectiveness of fractional calculus has been demonstrated in the examination of multiple diffusion phenomena, such as heat transmission [25], gas exchange and water movement through porous structures [25, 31]. Bagley and Torvik [6, 7] initially introduced fractional calculus as an essential tool for modeling tissue viscoelasticity. Several researchers have established the existence and uniqueness of solutions for different fractional differential equations incorporating diverse fractional derivatives [2, 8, 9, 12, 22, 35].

Pantograph differential equations find applications in multiple disciplines, including physics, engineering, biology and economics. They are particularly valuable for modeling systems with distributed delays or long-term memory effects. Through the study of these equations, researchers have gained important insights into the dynamic behaviors of complex systems with delayed interactions [18, 19, 21, 30]. Balachandran et al. [10] investigated the existence of solutions for nonlinear fractional pantograph differential equations. Rafeeq et al. [34] discussed the Caputo-Hadamard fractional pantograph equation of two distinct orders with Dirichlet boundary conditions. Thabet et al. [43] examined the analytical investigation of the ABC-fractional pantograph implicit differential equation concerning another function. In [42], the authors made an analytical study of the multi-order ρ-Hilfer fractional pantograph implicit differential equation on unbounded domains. Abdelnebi et al. [3] explored the existence,

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uniqueness and stability of solutions for a pantograph problem involving three successive derivatives of Caputo-Hadamard-type. Several researchers have explored the existence and uniqueness of solutions for such equations involving various fractional derivatives [15, 18, 20, 21].

Moreover, the implementation of the ψ-derivative (functional derivative) provides a framework for formulating and analyzing fractional differential equations, enabling extensive possibilities for constructing mathematical models and numerical methodologies. This flexibility renders it particularly useful in places where standard derivatives fail to accurately represent the complex dynamics of a system. Balachandran et al. [11] introduced Hadamard fractional integrals and derivatives. Nyamoradi et al. [28] analyzed a Hadamard-type fractional differential equation by incorporating a logarithmic-type integro-initial conditions. Further, using standard fixed point theorems, they investigated the existence and uniqueness of solutions for a new class of Hadamard fractional differential equations on a half-line with logarithmic-type initial conditions [29]. Krushna et al. [23] examined the Hadamard fractional boundary value problems with the help of fixed point method. Employing the ψ-Caputo fractional derivative many researchers have utilized the fixed point principles to investigate the existence and uniqueness problems for nonlinear differential equations of fractional order. Further details on these derivatives can be found in [4, 5, 33].

To the best of our knowledge, no studies have been made on fractional pantograph differential equations with Hadamard functional fractional derivative. Motivated by this, we aim to examine the existence and uniqueness of solutions for fractional pantograph differential equations of the form

$$ {}^{H}D_{0^+}^{\alpha,\psi}\mathfrak{u}(t) = \mathfrak{f}\big(t,\mathfrak{u}(t),\mathfrak{u}(\lambda t)\big),\ t \in \mathrm{J} = [0,\mathfrak{b}],\ 0 < \lambda < 1 \tag{1} $$

$$ \mathfrak{u}(0) = \mathfrak{u}_0, \tag{2} $$

where ${}^{H}D_{0^+}^{\alpha,\psi}$ is the Hadamard functional fractional derivative of order $0 < \alpha < 1$ in a Banach space X and $\mathfrak{f} : \mathrm{J} \times X \times X \to X$ is a continuous function.

This paper is organized as follows: Section 2 introduces the notations, fundamental concepts, and preliminary results required for the study. Section 3 employs classical fixed point theorems to establish the existence and uniqueness of solutions for the fractional pantograph differential equations with nonlocal conditions, boundary value problem of fractional pantograph differential equations and neutral fractional pantograph differential equations. Finally, Section 4 presents illustrative examples to validate the theoretical findings.

2. Preliminaries

In this section, we provide notations, definitions, and introductory information that will be used in this work. Let $(X, |\cdot|)$ be a Banach space and $\mathscr{C}(\mathrm{J},X) = \mathscr{E}$ be the Banach space of all continuous functions from J into X with norm

$$ \|\mathfrak{u}\| = \sup\{|\mathfrak{u}(t)| : t \in \mathrm{J}\}. $$

Furthermore, $B_{\mathfrak{r}}(\mathfrak{u}_o, X)$ denotes the closed ball with center at $\mathfrak{u}_o$ and radius $\mathfrak{r}$ in X.

DEFINITION 1. [11] (Hadamard functional fractional integrals) Let $[a,b]$ be an interval, f be an integrable function defined on $[a,b]$ and $\psi \in C^1[a,b]$ be an increasing function with $\psi'(x) \neq 0$, for all $x \in [a,b]$. Then the left-sided Hadamard functional fractional integral of order $\alpha > 0$ is defined by

$$ {}^H I_{a^+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln \psi(x) - \ln \psi(t))^{\alpha-1} f(t) dt, \tag{3} $$

and the right-sided Hadamard functional fractional integral of order α is defined by

$$ {}^H I_{b^-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\psi'(t)}{\psi(t)} (\ln \psi(t) - \ln \psi(x))^{\alpha-1} f(t) dt. $$

DEFINITION 2. [11] (Hadamard functional fractional derivatives) Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $[a,b]$ be an interval, f be an integrable function defined on $[a,b]$ and $\psi \in C^1[a,b]$ be an increasing function with $\psi'(x) \neq 0$, for all $x \in [a,b]$. The left-sided Hadamard functional fractional derivative of f of order α is defined by

$$ \begin{aligned} {}^H D_{a^+}^{\alpha,\psi} f(x) &= \left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n {}^H I_{a^+}^{n-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n \int_a^x \frac{\psi'(t)}{\psi(t)} (\ln \psi(x) - \ln \psi(t))^{n-\alpha-1} f(t) dt \end{aligned} \tag{4} $$

and the right-sided Hadamard functional fractional derivative of f of order α is defined by

$$ {}^H D_{b^-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\psi(x)}{\psi'(x)} \frac{d}{dx} \right)^n \int_x^b \frac{\psi'(t)}{\psi(t)} (\ln \psi(t) - \ln \psi(x))^{n-\alpha-1} f(t) dt. $$

THEOREM 1. [41] (Banach fixed point theorem) *If X is a Banach space and $F : X \to X$ is a contraction mapping, then F has a unique fixed point.*

THEOREM 2. [41] (Krasnoselskii fixed point theorem) *Let K be a nonempty closed convex subset of a Banach space X. If A and B are two operators such that*

i) $Ax + By \in K$, for any $x, y \in K$,

ii) A is compact and continuous,

iii) B is contraction mapping,

then there exists $z \in K$ such that $z = Az + Bz$.

Equation (1)–(2) is observed to be equal to the following integral equation

$$ \mathfrak{u}(t) = \mathfrak{u}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln \psi(t) - \ln \psi(\tau))^{\alpha-1} \mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda \tau)) d\tau. $$

3. Existence results

We introduce the following conditions to demonstrate our findings.

(H1) The function $\mathfrak{f}$ is continuous and there exists a constant $\mathfrak{L}_1 > 0$ such that

$$|\mathfrak{f}(t,\mathfrak{u}_1,\mathfrak{v}_1) - \mathfrak{f}(t,\mathfrak{u}_2,\mathfrak{v}_2)| \leqslant \mathfrak{L}_1 \left(|\mathfrak{u}_1 - \mathfrak{u}_2| + |\mathfrak{v}_1 - \mathfrak{v}_2|\right),$$

for each $t \in \mathsf{J} = [0,b]$ and $\mathfrak{u}_1,\mathfrak{v}_1,\mathfrak{u}_2,\mathfrak{v}_2 \in X$.

For brevity, let us assume $M = \max_{t\in \mathsf{J}} |\mathfrak{f}(t,0,0)|$.

Our first result is based on the Banach fixed point theorem.

THEOREM 3. *Suppose* (H1) *holds. If*

$$2\left(\frac{2\mathfrak{L}_1\ (\ln\psi(\mathfrak{b}) - \ln\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right) < 1,$$

then there exists a unique solution to the problem (1)–(2).

Proof. Define an operator $\mathsf{F} : \mathscr{E} \to \mathscr{E}$ by

$$\mathsf{F}\mathfrak{u}(t) = \mathfrak{u}_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.$$

Choose $\mathfrak{r} \geqslant 2\left(|\mathfrak{u}_0| + \dfrac{M\ (\ln\psi(\mathfrak{b}) - \ln\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)$. Then we may demonstrate that $\mathsf{F}B_{\mathfrak{r}} \subset B_{\mathfrak{r}}$, where $B_{\mathfrak{r}} = \{\mathfrak{u} \in \mathscr{E} : \|\mathfrak{u}\| \leqslant \mathfrak{r}\}$. Using the assumptions, we obtain

$$\begin{aligned}
|\mathsf{F}\mathfrak{u}(t)| &\leqslant |\mathfrak{u}_0| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}|\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau)) - \mathfrak{f}(\tau,0,0)|d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}|\mathfrak{f}(\tau,0,0)|d\tau \\
&\leqslant |\mathfrak{u}_0| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(|\mathfrak{u}(\tau)| + |\mathfrak{u}(\lambda\tau)|)d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}Md\tau \\
\|\mathsf{F}\mathfrak{u}\| &\leqslant |\mathfrak{u}_0| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}2\mathfrak{L}_1\|\mathfrak{u}\|d\tau \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}d\tau \\
&\leqslant |\mathfrak{u}_0| + \frac{2\mathfrak{L}_1\|\mathfrak{u}\|}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}d\tau \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}d\tau
\end{aligned}$$

$$\leqslant |\mathfrak{u}_0| + \frac{2\mathfrak{L}_1\mathfrak{r}}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b}) - \ln\psi(0))^\alpha + \frac{M}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b}) - \ln\psi(0))^\alpha$$
$$\leqslant \frac{\mathfrak{r}}{2} + \frac{\mathfrak{r}}{2} = \mathfrak{r}.$$

Therefore, F maps $B_\mathfrak{r}$ into itself. Let us consider $\mathfrak{u},\mathfrak{v} \in \mathscr{E}$, then for any $t \in \mathtt{J}$, we obtain

$$\begin{aligned}|\mathsf{F}\mathfrak{u}(t) - \mathsf{F}\mathfrak{v}(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \\ &\quad |\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau)) - \mathfrak{f}(\tau,\mathfrak{v}(\tau),\mathfrak{v}(\lambda\tau))|d\tau \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \\ &\quad \mathfrak{L}_1\left(|\mathfrak{u}(\tau) - \mathfrak{v}(\tau)| + |\mathfrak{u}(\lambda\tau) - \mathfrak{v}(\lambda\tau)|\right)d\tau \\ \|\mathsf{F}\mathfrak{u} - \mathsf{F}\mathfrak{v}\| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(2\|\mathfrak{u} - \mathfrak{v}\|)d\tau \\ &\leqslant \left(\frac{2\mathfrak{L}_1\left(\ln\psi(\mathfrak{b}) - \ln\psi(0)\right)^\alpha}{\Gamma(\alpha+1)}\right)\|\mathfrak{u} - \mathfrak{v}\|.\end{aligned}$$

Hence, F is contraction and has a unique fixed point as a result of the Banach fixed point theorem, which is the unique solution to the problem (1)–(2). □

3.1. Fractional pantograph differential equations with nonlocal conditions

The study of the nonlocal Cauchy problem was first introduced by Byszewski [14]. Since then, numerous researchers have explored various types of equations, including nonlinear differential equations, integrodifferential equations, functional differential equations, and fractional differential equations [1, 12].

Consider the equation (1) with nonlocal conditions of the form

$$\mathfrak{u}(0) + \mathfrak{g}(\mathfrak{u}) = \mathfrak{u}_0, \tag{5}$$

where $\mathfrak{g} : \mathscr{E} \to X$ is a continuous function that satisfies the following assumption:

(H2) $\mathfrak{g}$ is continuous and there exists a constant $\mathfrak{L}_2 > 0$ such that

$$|\mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})| \leqslant \mathfrak{L}_2\|\mathfrak{u} - \mathfrak{v}\|, \text{ for } \mathfrak{u},\mathfrak{v} \in \mathscr{E}.$$

The following integral equation can be seen to be equivalent to the equations (1) and (5)

$$\mathfrak{u}(t) = \mathfrak{u}_0 - \mathfrak{g}(\mathfrak{u}) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t) - \ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.$$

THEOREM 4. *Suppose* (H1)–(H2) *hold. If*

$$2\left(\mathfrak{L}_2 + \frac{2\mathfrak{L}_1\left(\ln\psi(\mathfrak{b}) - \ln\psi(0)\right)^\alpha}{\Gamma(\alpha+1)}\right) < 1,$$

then there exists a unique solution to the problem (1) *and* (5).

Proof. Define an operator $\mathsf{P} : \mathscr{E} \to \mathscr{E}$ by

$$\mathsf{P}\mathfrak{u}(t) = \mathfrak{u}_0 - \mathfrak{g}(\mathfrak{u}) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau)) d\tau.$$

Choose $\mathfrak{r} \geqslant 2\left(|\mathfrak{u}_0| + |\mathfrak{g}(0)| + \frac{M\ (\ln\psi(\mathfrak{b}) - \ln\psi(0))^\alpha}{\Gamma(\alpha+1)}\right)$. Thus, we may demonstrate that $\mathsf{P}B_\mathfrak{r} \subset B_\mathfrak{r}$. Based on our assumptions, we have

$$\begin{aligned}
|\mathsf{P}\mathfrak{u}(t)| &\leqslant |\mathfrak{u}_0| + |\mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(0)| + |\mathfrak{g}(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \\
&\quad |\mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau)) - \mathfrak{f}(\tau, 0, 0)| d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} |\mathfrak{f}(\tau, 0, 0)| d\tau \\
&\leqslant |\mathfrak{u}_0| + \mathfrak{L}_2\|\mathfrak{u}\| + |\mathfrak{g}(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \\
&\quad \mathfrak{L}_1(|\mathfrak{u}(\tau)| + |\mathfrak{u}(\lambda\tau)|) d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} M d\tau \\
\|\mathsf{P}\mathfrak{u}\| &\leqslant |\mathfrak{u}_0| + \mathfrak{L}_2\|\mathfrak{u}\| + |\mathfrak{g}(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} 2\mathfrak{L}_1\|\mathfrak{u}\| d\tau \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} d\tau \\
&\leqslant |\mathfrak{u}_0| + \mathfrak{L}_2\|\mathfrak{u}\| + |\mathfrak{g}(0)| + \frac{2\mathfrak{L}_1\|\mathfrak{u}\|}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} d\tau \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} d\tau \\
&\leqslant |\mathfrak{u}_0| + |\mathfrak{g}(0)| + \mathfrak{L}_2\mathfrak{r} + \frac{2\mathfrak{L}_1\mathfrak{r}}{\Gamma(\alpha+1)} (\ln\psi(\mathfrak{b}) - \ln\psi(0))^\alpha \\
&\quad + \frac{M}{\Gamma(\alpha+1)} (\ln\psi(\mathfrak{b}) - \ln\psi(0))^\alpha \\
&\leqslant \frac{\mathfrak{r}}{2} + \frac{\mathfrak{r}}{2} = \mathfrak{r}.
\end{aligned}$$

Therefore, F maps $B_\mathfrak{r}$ into itself. Let us consider $\mathfrak{u}, \mathfrak{v} \in \mathscr{E}$, then for any $t \in \mathsf{J}$, we obtain

$$\begin{aligned}
|\mathsf{P}\mathfrak{u}(t) - \mathsf{P}\mathfrak{v}(t)| \leqslant\ & |\mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \\
& |\mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau)) - \mathfrak{f}(\tau, \mathfrak{v}(\tau), \mathfrak{v}(\lambda\tau))| d\tau \\
\leqslant\ & \mathfrak{L}_2\|\mathfrak{u} - \mathfrak{v}\| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t) - \ln\psi(\tau))^{\alpha-1} \\
& \mathfrak{L}_1 (|\mathfrak{u}(\tau) - \mathfrak{v}(\tau)| + |\mathfrak{u}(\lambda\tau) - \mathfrak{v}(\lambda\tau)|) d\tau
\end{aligned}$$

$$\begin{aligned}\|\mathsf{P}\mathfrak{u}-\mathsf{P}\mathfrak{v}\| &\leqslant \mathfrak{L}_2\|\mathfrak{u}-\mathfrak{v}\| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(2\|\mathfrak{u}-\mathfrak{v}\|)d\tau \\ &\leqslant \left(\mathfrak{L}_2 + \frac{2\mathfrak{L}_1\,(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\|\mathfrak{u}-\mathfrak{v}\|.\end{aligned}$$

Thus, P is a contraction mapping. Due to the Banach fixed point theorem, P has a unique fixed point, which is the unique solution to the problem (1)–(5). □

3.2. Boundary value problem for fractional pantograph differential equations

In recent years, many researchers have investigated boundary value problems for fractional differential equations under various boundary conditions [23, 35–40]. Here we explore the fractional pantograph differential equations (1) along with boundary condition

$$c\mathfrak{u}(0)+d\mathfrak{u}(b)=\mathsf{e}, \tag{6}$$

where c, d and e are real constants with $c+d \neq 0$. The integral equation that corresponds to (1) and (6) is represented by

$$\begin{aligned}\mathfrak{u}(t) = \frac{\mathsf{e}}{c+d} - \frac{d}{c+d}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau \\ + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.\end{aligned}$$

THEOREM 5. *Assume that* (H1) *holds. If*

$$2\left(\left(1+\frac{|d|}{|c+d|}\right)\frac{2\mathfrak{L}_1(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right) < 1,$$

then (1) *and* (6) *has a unique solution.*

Proof. Define the operator $\mathsf{Q} : \mathscr{E} \to \mathscr{E}$ by

$$\begin{aligned}\mathsf{Q}\mathfrak{u}(t) = \frac{\mathsf{e}}{c+d} - \frac{d}{c+d}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau \\ + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.\end{aligned}$$

Choose $\mathfrak{r} \geqslant 2\left(\frac{|\mathsf{e}|}{|c+d|} + \left(1+\frac{|d|}{|c+d|}\right)(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha}\right).$ □

Thus, we may demonstrate that $\mathsf{Q}B_{\mathfrak{r}} \subset B_{\mathfrak{r}}$. Based on our assumptions, we obtain

$$
\begin{aligned}
|\mathsf{Q}\mathfrak{u}(t)| \leqslant & \frac{|\mathsf{e}|}{|c+d|} + \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1} \\
& |\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))-\mathfrak{f}(\tau,0,0)|d\tau \\
& + \frac{|d|}{|c+d|\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1} \\
& |\mathfrak{f}(\tau,0,0)|d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}| \\
& \mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))-\mathfrak{f}(\tau,0,0)|d\tau \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}|\mathfrak{f}(\tau,0,0)|d\tau \\
\leqslant & \frac{|\mathsf{e}|}{|c+d|} + \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(|\mathfrak{u}(\tau)|+|\mathfrak{u}(\lambda\tau)|)d\tau \\
& + \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}Md\tau \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(|\mathfrak{u}(\tau)|+|\mathfrak{u}(\lambda\tau)|)d\tau \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}Md\tau \\
\|\mathsf{Q}\mathfrak{u}\| \leqslant & \frac{|\mathsf{e}|}{|c+d|} + \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}2\mathfrak{L}_1(\|\mathfrak{u}\|)d\tau \\
& + \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}Md\tau \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}2\mathfrak{L}_1(\|\mathfrak{u}\|)d\tau \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}Md\tau \\
\leqslant & \frac{|\mathsf{e}|}{|c+d|} + \frac{|d|}{|c+d|}\frac{2\mathfrak{L}_1\|\mathfrak{u}\|}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha} \\
& + \frac{|d|}{|c+d|}\frac{M}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha} \\
& + \frac{2\mathfrak{L}_1\|\mathfrak{u}\|}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha} + \frac{M}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha} \\
\leqslant & \frac{|\mathsf{e}|}{|c+d|} + \left(\frac{|d|}{|c+d|}+1\right)\frac{2\mathfrak{L}_1\mathfrak{r}}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha} \\
& + \left(\frac{|d|}{|c+d|}+1\right)\frac{M}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha} \leqslant \frac{\mathfrak{r}}{2}+\frac{\mathfrak{r}}{2} \\
\leqslant & \mathfrak{r}.
\end{aligned}
$$

Thus, $\mathrm{Q}B_{\mathfrak{r}} \subset B_{\mathfrak{r}}$. Let $\mathfrak{u},\mathfrak{v} \in \mathscr{E}$. Then

$$\begin{aligned}
|\mathrm{Q}\mathfrak{u}(t)-\mathrm{Q}\mathfrak{v}(t)| \leqslant\ & \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1} \\
& |\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))-\mathfrak{f}(\tau,\mathfrak{v}(\tau),\mathfrak{v}(\lambda\tau))|d\tau \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1} \\
& |\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))-\mathfrak{f}(\tau,\mathfrak{v}(\tau),\mathfrak{v}(\lambda\tau))|d\tau \\
\leqslant\ & \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1} \\
& \mathfrak{L}_1\left(|\mathfrak{u}(\tau)-\mathfrak{v}(\tau)|+|\mathfrak{u}(\lambda\tau)-\mathfrak{v}(\lambda\tau)|\right)d\tau \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1} \\
& \mathfrak{L}_1\left(|\mathfrak{u}(\tau)-\mathfrak{v}(\tau)|+|\mathfrak{u}(\lambda\tau)-\mathfrak{v}(\lambda\tau)|\right)d\tau \\
\|\mathrm{Q}\mathfrak{u}-\mathrm{Q}\mathfrak{v}\| \leqslant\ & \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1\left(\|\mathfrak{u}-\mathfrak{v}\|+\|\mathfrak{u}-\mathfrak{v}\|\right)d\tau \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1\left(\|\mathfrak{u}-\mathfrak{v}\|+\|\mathfrak{u}-\mathfrak{v}\|\right)d\tau \\
\leqslant\ & \frac{|d|}{|c+d|}\frac{1}{\Gamma(\alpha)}\int_0^b \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(b)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(2\|\mathfrak{u}-\mathfrak{v}\|)d\tau \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_1(2\|\mathfrak{u}-\mathfrak{v}\|)d\tau \\
\leqslant\ & \left(\left(1+\frac{|d|}{|c+d|}\right)\frac{2\mathfrak{L}_1(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\|\mathfrak{u}-\mathfrak{v}\|.
\end{aligned}$$

Thus the operator Q is contraction and due to the Banach fixed point theorem, Q has a unique fixed point, which is the unique solution to the problem (1)–(6).

3.3. Neutral fractional pantograph differential equations

Neutral differential equations appear in various fields of applied mathematics and have recently garnered significant interest, as noted in [16, 24]. In recent years, numerous studies have examined neutral fractional differential equations under diverse conditions [18, 32].

Consider the neutral fractional pantograph differential equation of the form:

$$ {}^{H}D_{0^+}^{\alpha,\psi}\left(\mathfrak{u}(t)-\mathfrak{h}(t,\mathfrak{u}(\lambda t))\right)=\mathfrak{f}\big(t,\mathfrak{u}(t),\mathfrak{u}(\lambda t)\big),\ \ t\in \mathrm{J}=[0,\mathfrak{b}], \tag{7}$$

$$\mathfrak{u}(0)=\mathfrak{u}_0, \tag{8}$$

where $\mathfrak{h}:\mathrm{J}\times X\to X$ is a continuous function and the equations (7)–(8) is equivalent to the integral equation

$$\begin{aligned}\mathfrak{u}(t) &= \mathfrak{u}_0 + \mathfrak{h}(t,\mathfrak{u}_0) - \mathfrak{h}(t,\mathfrak{u}(\lambda t)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.\end{aligned}$$

We need the following hypotheses:

(H3) The function $\mathfrak{f}$ is completely continuous and there exists a constant $\mathfrak{L}_3 > 0$ such that

$$|\mathfrak{f}(t,\mathfrak{u},\mathfrak{v})| \leqslant \mathfrak{L}_3, \quad \text{for each } t \in \mathtt{J} \text{ and } \mathfrak{u},\mathfrak{v} \in X.$$

(H4) The function $\mathfrak{h}$ is continuous and there exists a constant $\mathfrak{L}_4$ such that $0 < \mathfrak{L}_4 < 1$ and

$$|\mathfrak{h}(t,\mathfrak{u}_1) - \mathfrak{h}(t,\mathfrak{u}_2)| \leqslant \mathfrak{L}_4\left(|\mathfrak{u}_1 - \mathfrak{u}_2|\right),$$

for each $t \in \mathtt{J}$ and $\mathfrak{u}_1, \mathfrak{u}_2 \in X$.

Take $H = \max_{t\in\mathtt{J}} |\mathfrak{h}(t,0)|$.

THEOREM 6. *Assume that* (H3) *and* (H4) *hold. Then the problem* (7)–(8) *has at least one solution on* $\mathtt{J}$.

Proof. Choose $\mathfrak{r} \geqslant \dfrac{|\mathfrak{u}_0| + |\mathfrak{h}(t,\mathfrak{u}_0)| + H + \frac{\mathfrak{L}_3}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b}) - \ln\psi(0))^{\alpha}}{1-\mathfrak{L}_4}$.

Define the operators $\mathsf{A}_1, \mathsf{A}_2 : \mathscr{C}(\mathtt{J}, B_{\mathfrak{r}}) \to \mathscr{C}(\mathtt{J}, B_{\mathfrak{r}})$ by

$$\begin{aligned}\mathsf{A}_1\mathfrak{u}(t) &= \mathfrak{u}_0 + \mathfrak{h}(t,\mathfrak{u}_0) - \mathfrak{h}(t,\mathfrak{u}(\lambda t)) \\ \mathsf{A}_2\mathfrak{u}(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.\end{aligned}$$

For any $\mathfrak{u},\mathfrak{v} \in B_{\mathfrak{r}}$, we obtain

$$\begin{aligned}|\mathsf{A}_1\mathfrak{u}(t) + \mathsf{A}_2\mathfrak{v}(t)| &\leqslant |\mathfrak{u}_0| + |\mathfrak{h}(t,\mathfrak{u}_0)| + |\mathfrak{h}(t,\mathfrak{u}(\lambda t)) - \mathfrak{h}(t,0)| + |\mathfrak{h}(t,0)| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}|\mathfrak{f}(\tau,\mathfrak{v}(\tau),\mathfrak{v}(\lambda\tau))|d\tau \\ &\leqslant |\mathfrak{u}_0| + |\mathfrak{h}(t,\mathfrak{u}_0)| + \mathfrak{L}_4|\mathfrak{u}(\lambda t)| + |\mathfrak{h}(t,0)| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_3 d\tau \\ \|\mathsf{A}_1\mathfrak{u} + \mathsf{A}_2\mathfrak{v}\| &\leqslant |\mathfrak{u}_0| + |\mathfrak{h}(t,\mathfrak{u}_0)| + \mathfrak{L}_4\|\mathfrak{u}\| + H + \frac{\mathfrak{L}_3}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b}) - \ln\psi(0))^{\alpha} \\ &\leqslant |\mathfrak{u}_0| + |\mathfrak{h}(t,\mathfrak{u}_0)| + \mathfrak{L}_4\mathfrak{r} + H + \frac{\mathfrak{L}_3}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b}) - \ln\psi(0))^{\alpha} \\ &\leqslant (1-\mathfrak{L}_4)\mathfrak{r} + \mathfrak{L}_4\mathfrak{r} = \mathfrak{r}.\end{aligned}$$

Therefore, we can conclude that $\mathsf{A}_1\mathfrak{u}+\mathsf{A}_2\mathfrak{v}\in B_{\mathfrak{r}}$. Next, we show that the operator A_1 is a contraction. For any $t\in\mathsf{J}$ and $\mathfrak{u},\mathfrak{v}\in B_{\mathfrak{r}}$, we get

$$\begin{aligned}|\mathsf{A}_1\mathfrak{u}(t)-\mathsf{A}_1\mathfrak{v}(t)| &\leqslant |\mathfrak{h}(t,\mathfrak{u}(\lambda t))-\mathfrak{h}(t,\mathfrak{v}(\lambda t))|\\ &\leqslant \mathfrak{L}_4|\mathfrak{u}(\lambda t)-\mathfrak{v}(\lambda t)|\|\mathsf{A}_1\mathfrak{u}-\mathsf{A}_1\mathfrak{v}\|\\ &\leqslant \mathfrak{L}_4\|\mathfrak{u}-\mathfrak{v}\|,\end{aligned}$$

and as $\mathfrak{L}_4<1$, thus A_1 is a contraction mapping. Next, we demonstrate that A_2 is uniformly bounded. We observe that

$$\begin{aligned}|\mathsf{A}_2\mathfrak{u}(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t\frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}|\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))|d\tau\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t\frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_3 d\tau\\ &\leqslant \frac{\mathfrak{L}_3}{\Gamma(\alpha+1)}(\ln\psi(\mathfrak{b})-\ln\psi(0))^{\alpha}.\end{aligned}$$

Therefore, A_2 is uniformly bounded. Let $\{\mathfrak{u}_n\}$ be any sequence in $B_{\mathfrak{r}}$ such that $\mathfrak{u}_n\to\mathfrak{u}$. Then

$$\mathfrak{f}(\tau,\mathfrak{u}_n(\tau),\mathfrak{u}_n(\lambda\tau))\to\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))\quad\text{as } n\to\infty,$$

since $\mathfrak{f}$ is continuous. Now, for every $t\in\mathsf{J}$, we get

$$\begin{aligned}|\mathsf{A}_2\mathfrak{u}_n(t)-\mathsf{A}_2\mathfrak{u}(t)| \leqslant \frac{1}{\Gamma(\alpha)}\int_0^t\frac{\psi'(\tau)}{\psi(\tau)}&(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\\ &|\mathfrak{f}(\tau,\mathfrak{u}_n(\tau),\mathfrak{u}_n(\lambda\tau))-\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))|d\tau\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus we see that, A_2 is continuous. Next we demonstrate that the set $\{\mathsf{A}_2\mathfrak{u}(t):\mathfrak{u}\in B_{\mathfrak{r}}\}$ is relatively compact in X, for all $t\in\mathsf{J}$. Clearly $\{\mathsf{A}_2\mathfrak{u}(0):\mathfrak{u}\in B_{\mathfrak{r}}\}$ is compact. Fix $t>0$ and for each $\varepsilon\in(0,t)$ and $u\in B_{\mathfrak{r}}$, define the operator $\mathsf{A}_2^{\varepsilon}$ by

$$\mathsf{A}_2^{\varepsilon}\mathfrak{u}(t)=\frac{1}{\Gamma(\alpha)}\int_0^{t-\varepsilon}\frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{f}(\tau,\mathfrak{u}(\tau),\mathfrak{u}(\lambda\tau))d\tau.$$

Since $\mathfrak{f}$ is completely continuous, clearly the set $\{\mathsf{A}_2^{\varepsilon}\mathfrak{u}(t):\mathfrak{u}\in B_{\mathfrak{r}}\}$ is precompact in X for each ε, we have relative compactness in X for all $t\in\mathsf{J}$ since it is compact for $t=0$ and $0<\varepsilon<t$. In addition, for each $\mathfrak{u}\in B_{\mathfrak{r}}$, we get

$$\begin{aligned}|\mathsf{A}_2\mathfrak{u}(t)-\mathsf{A}_2^{\varepsilon}\mathfrak{u}(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_{t-\varepsilon}^t\frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}|\mathfrak{f}(\tau,u(\tau),u(\lambda\tau))|d\tau\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_{t-\varepsilon}^t\frac{\psi'(\tau)}{\psi(\tau)}(\ln\psi(t)-\ln\psi(\tau))^{\alpha-1}\mathfrak{L}_3 d\tau\\ &\leqslant \frac{\mathfrak{L}_3}{\Gamma(\alpha+1)}(\psi(b)-\psi(b-\varepsilon))^{\alpha},\end{aligned}$$

which tends to zero as $\varepsilon \to 0$. Therefore, the set $\{\mathsf{A}_2(\mathfrak{u}(t)) : \mathfrak{u} \in B_{\mathfrak{r}}\}$ is precompact in X. Let us demonstrate that the operator A_2 is equicontinuous. For any $t_1, t_2 \in \mathrm{J}$ with $t_1 < t_2$, we obtain

$$
\begin{aligned}
&|\mathsf{A}_2(\mathfrak{u}(t_2)) - \mathsf{A}_2(\mathfrak{u}(t_1))| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1} |\mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau))| d\tau \\
& - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t_1) - \ln\psi(\tau))^{\alpha-1} |\mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau))| d\tau \\
\leqslant\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{\psi'(\tau)}{\psi(\tau)} ((\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1} - (\ln\psi(t_1) - \ln\psi(\tau))^{\alpha-1}) \\
& |\mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau))| d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1} \\
& |\mathfrak{f}(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\lambda\tau))| d\tau \\
\leqslant\ & \frac{\mathfrak{L}_3}{\Gamma(\alpha)} \int_0^{t_1} \frac{\psi'(\tau)}{\psi(\tau)} ((\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1} - (\ln\psi(t_1) - \ln\psi(\tau))^{\alpha-1}) d\tau \\
& + \frac{\mathfrak{L}_3}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\psi'(\tau)}{\psi(\tau)} (\ln\psi(t_2) - \ln\psi(\tau))^{\alpha-1} d\tau,
\end{aligned}
$$

as $t_2 \to t_1$, the right hand side of the above inequality tends to zero. Thus, we have demonstrated that $\mathsf{A}_2(B_{\mathfrak{r}})$ is relatively compact. Now we can conclude that A_2 is compact using the Arzela-Ascoli theorem. According to the Krasnoselskii fixed point theorem, there is a fixed point $\mathfrak{u} \in B_{\mathfrak{r}}$ such that $\mathsf{A}_1\mathfrak{u} + \mathsf{A}_2\mathfrak{u} = \mathfrak{u}$, which is a solution of the problem (7)–(8). □

4. Examples

EXAMPLE 1. Consider the following fractional pantograph differential equations

$$
{}^{H}D_{0^+}^{\alpha,\psi}\mathfrak{u}(t) = \frac{1}{10} + \frac{1}{8}\mathfrak{u}(t) + \frac{1}{8}\cos\mathfrak{u}\left(\frac{t}{2}\right), \quad t \in \mathrm{J} = [0,1], \tag{9}
$$

$$
\mathfrak{u}(0) = 0. \tag{10}
$$

Here we take $X = R$, $\alpha = \frac{1}{4}$, $\psi(t) = 2^t$, $\lambda = \frac{1}{2}$ and

$$
\mathfrak{f}(t, \mathfrak{u}, \mathfrak{v}) = \frac{1}{10} + \frac{1}{8}\mathfrak{u}(t) + \frac{1}{8}\cos\mathfrak{v}\left(\frac{t}{2}\right), \quad \mathfrak{u}, \mathfrak{v} \in R, \quad t \in \mathrm{J}.
$$

Let $\mathfrak{u}_i, \mathfrak{v}_i \in R$, $i = 1,2$ and $t \in \mathrm{J}$. Then we have

$$
|\mathfrak{f}(t, \mathfrak{u}_1, \mathfrak{v}_1) - \mathfrak{f}(t, \mathfrak{u}_2, \mathfrak{v}_2)| \leqslant \frac{1}{8}\left(|\mathfrak{u}_1(t) - \mathfrak{u}_2(t)| + |\mathfrak{v}_1(t) - \mathfrak{v}_2(t)|\right).
$$

Thus, the condition (H1) is satisfied with $\mathfrak{L}_1 = \frac{1}{8}$. By simple calculation, we obtain

$$
2\left(\frac{2\mathfrak{L}_1\,(\ln\psi(\mathfrak{b}) - \ln\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right) = 2\left(\frac{\frac{1}{4}\,(\ln 2 - \ln 1)^{\alpha}}{\Gamma(\frac{1}{4}+1)}\right) = 2(0.1912) = 0.3824 < 1.
$$

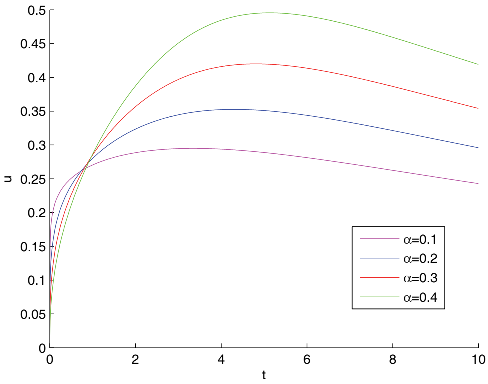

Figure 1: *Graph of* $\mathfrak{u}$ *for different values of* $\alpha = 0.1, 0.2, 0.3, 0.4$

Therefore, by Theorem 3, the problem (9)–(10) has a unique solution on J.

EXAMPLE 2. Consider the following fractional pantograph differential equations with nonlocal conditions

$$ {}^{H}D_{0^+}^{\alpha,\psi}\mathfrak{u}(t) = \frac{1}{5} + \frac{e^{-t}}{30}\mathfrak{u}(t) + \frac{1}{30}\mathfrak{u}\left(\frac{t}{2}\right), \quad t \in \mathrm{J} = [0,1], \tag{11} $$

$$ \mathfrak{u}(0) + \sum_{k=0}^{n} \mathfrak{a}_{\mathfrak{i}}\mathfrak{u}(t_{\mathfrak{i}}) = 0, \quad 0 < t_1 < \cdots < t_n < 1. \tag{12} $$

Here we take $X = R$, $\alpha = \frac{1}{3}$, $\psi(t) = 2^t$, $\mathfrak{a}_{\mathfrak{i}} > 0$, $\mathfrak{i} = 0, 1, \cdots, n$, $\lambda = \frac{1}{2}$ and

$$ \mathfrak{f}(t,\mathfrak{u},\mathfrak{v}) = \frac{1}{5} + \frac{e^{-t}}{30}\mathfrak{u}(t) + \frac{1}{30}\mathfrak{v}\left(\frac{t}{2}\right), \quad \mathfrak{u},\mathfrak{v} \in R $$

$$ g(\mathfrak{u}) = \sum_{k=0}^{n} \mathfrak{a}_{\mathfrak{i}}\mathfrak{u}(t_{\mathfrak{i}}), \quad \mathfrak{u},\mathfrak{v} \in \mathscr{E} = C(J,R), \ t \in \mathrm{J}. $$

Let $\mathfrak{u}_i, \mathfrak{v}_i \in R$, $i = 1, 2$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{E}$ and $t \in \mathrm{J}$. Then we obtain

$$ |\mathfrak{f}(t,\mathfrak{u}_1,\mathfrak{v}_1) - \mathfrak{f}(t,\mathfrak{u}_2,\mathfrak{v}_2)| \leqslant \frac{1}{30}\left(|\mathfrak{u}_1(t) - \mathfrak{u}_2(t)| + |\mathfrak{v}_1(t) - \mathfrak{v}_2(t)|\right) $$

$$ |g(\mathfrak{u}) - g(\mathfrak{v})| \leqslant \sum_{k=0}^{n} \mathfrak{a}_{\mathfrak{i}}|\mathfrak{u}(t_{\mathfrak{i}}) - \mathfrak{v}(t_{\mathfrak{i}})|. $$

Thus, the conditions (H1) and (H2) are satisfied with $\mathfrak{L}_1 = \frac{1}{8}$ and $\mathfrak{L}_2 = \frac{1}{10}$. By simple calculation, we obtain

$$ 2\left(\mathfrak{L}_2 + \frac{2\mathfrak{L}_1\left(\ln\psi(\mathfrak{b}) - \ln\psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\right) = 2\left(\frac{1}{10} + \frac{\frac{1}{15}\left(\ln 2 - \ln 1\right)^{\frac{1}{3}}}{\Gamma(\frac{1}{3}+1)}\right) = 2(0.1613) $$
$$ = 0.3226 < 1. $$

Therefore, by Theorem 4, the problem (11)–(12) has a unique solution on J.

5. Conclusion

This study has contributed to the advancement of modeling dynamical systems with memory effects through the exploration of fractional pantograph differential equations using the Hadamard functional fractional derivative. The establishment of existence and uniqueness results, via the Banach and Krasnoselskii fixed point theorems, has provided a solid foundation for further research in this area. The validation of theoretical findings through illustrative examples underscores the accuracy and efficiency of the proposed methodology. Building on the study's findings, future research directions can focus on many critical areas to enhance our understanding and practical applications:

- Numerical Methods: Further investigation into numerical techniques for solving fractional pantograph differential equations with Hadamard functional fractional derivative can improve computational efficiency and accuracy for tackling real-world problems.

- Stability Analysis: A comprehensive stability analysis of solution under various perturbations and dynamic conditions will provide deeper insights into the robustness of the proposed model.

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