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CONTROL OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS IN THE RIEMANN–LIOUVILLE SENSE

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Abstract. In this paper, we prove a bang-bang principle and a maximum principle, analogous to Pontryagin's, for systems of linear fractional differential equations in the Riemann-Liouville sense. We then apply these results to two examples, illustrating how incorporating fractional dynamics can improve optimal arrival times.

1. Introduction

In what follows, we will study the system of differential equations

$$\begin{aligned} (D^{\alpha}_{0+}y)(t) &= Ay(t)+Bu(t), \quad t>0, \\ (I^{1-\alpha}_{0+}y)(0+) &= y^0, \end{aligned} \tag{1}$$

where $\alpha \in (0,1]$ is the fractional index, A is a matrix in $M_{n\times n}(\mathbb{R})$, B is a matrix in $M_{n\times m}(\mathbb{R})$, $y^0 \in \mathbb{R}^n$ and $u:[0,\infty)\to[-1,1]^m$ is a measurable function called control. The operators D^{α}_{0+} and $I^{1-\alpha}_{0+}$ are the left-sided Riemann-Liouville derivative and integral, respectively. We denote by $\mathscr{A}$ the set of all controls u. The elements of $\mathscr{A}$ are referred to as admissible controls.

If the control u is a continuous function, with some degree of singularity at 0, it is known in the literature (see [8] or [9], for example) that the solution to the equation (1) is given by

$$y(t) = e^{tA}_{\alpha}y^0 + \int_0^t e^{(t-s)A}_{\alpha}Bu(s)ds, \quad \text{for all} \quad t>0, \tag{2}$$

where e^{tA}_{α} is the fractional exponential (operator). However, it is well-known that in applications (see [1], [12] or [19]), useful controls are those applied a countable number of times. That is, we are interested in the case where the control is of the bang-bang type. For such controls, we do not know of an explicit reference for the existence of the solution. Thus, in Section 2, we briefly prove that the formula (2) still holds for essentially bounded controls, but now the equality holds for almost all $t>0$.

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We will say that an initial state $y^0 \in \mathbb{R}^n$ is admissible if the dynamic $y(t)$, the solution of (1), takes it to the origin, 0, in finite time. Let G be the set of admissible states. In this paper, we demonstrate (see Lemma 1) that if we start at a point in G, then there exist optimal time and control, τ^* and u^*, respectively. Furthermore, we will see in Theorem 2 that the control u^* is of the bang-bang type. In the context of applications, an important aspect is a method to find the optimal control. This is a consequence of a maximum principle that we prove in Theorem 3. As an application of the results, we study two examples in Section 4 where the advantage of using fractional dynamics is evident.

In the literature review, we found a wide variety of papers that study the topic from a theoretical or applied point of view, see for example [2, 10, 14–16] or the references mentioned in these works. However, we have not found any work that addresses these principles, bang-bang and maximum, from the perspective presented here. It is also worth noting that our approach does not entirely follow the classical case, i.e., $\alpha = 1$, as the operator e_{α}^{tA} is not invertible for $\alpha \neq 1$. This fact, for instance, prevents us from obtaining Kalman-type controllability criteria, see [5] or [21]. In our case, this leads to our maximum principle involving the optimal arrival time, τ^*, see the formula (7), in contrast to the classical case, see for example [4] or [7].

The paper is organized as follows. In Section 2, we show the existence and uniqueness of the solution of (1) for essentially bounded controls. In Section 3, we address the proof of the bang-bang and maximum principles. In Section 4, we present two application examples, and finally, in Section 5, we provide some conclusions and present some problems that may be of interest.

2. Some preliminary results

Let us recall the following concepts. Let $\alpha \in (0,1)$ and $f : [0,\infty) \to \mathbb{R}$ be a function. The Riemann-Liouville fractional integral of order α is defined as

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}}.$$

On the other hand, the Riemann-Liouville fractional derivative of order α is given by

$$(D_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha}},$$

where Γ denotes the usual gamma function. In particular, formulas (2.1.56) and (2.1.53) from [8] provide us with the derivative and integral of the fractional exponential function (operator),

$$(D_{0+}^{\alpha} e_{\alpha}^{tA})(x) = A e_{\alpha}^{xA}, \tag{3}$$

and

$$(I_{0+}^{1-\alpha} e_{\alpha}^{tA})(x) = E_{\alpha}[x^{\alpha} A], \tag{4}$$

where E_{α} is the Mittag-Leffler function (or operator).

THEOREM 1. *Let* $\alpha \in (0,1]$, $A \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times m}(\mathbb{R})$, $b \in \mathbb{R}^n$ *and* $u : [0,\infty) \to [-1,1]^m$ *be a measurable function. Then, equation* (1) *has a unique solution* $y(t)$, *and the equality* (2) *holds almost everywhere in* $t > 0$.

Proof. The uniqueness of the solution follows immediately from the fractional Gronwall's inequality [18]. Let us verify that the expression given in (2) satisfies (1). The linearity of the fractional derivative gives us

$$(D^{\alpha}_{0+}y)(x) = \left(D^{\alpha}_{0+}e^{tA}_{\alpha}y^0\right)(x) + \left(D^{\alpha}_{0+}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)ds\right)(x).$$

Let us calculate each therm in the above expression. From (3) we have for the first

$$\left(D^{\alpha}_{0+}e^{tA}_{\alpha}y^0\right)(x) = Ae^{xA}_{\alpha}y^0.$$

The second term can be written as

$$\left(D^{\alpha}_{0+}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)ds\right)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)dsdt.$$

First, note that

$$\int_0^x \left|(x-t)^{-\alpha}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)ds\right| dt \leqslant \Gamma(1-\alpha)||B||\int_a^x E_{\alpha}\left[||A||(x-s)^{\alpha}\right]ds < \infty.$$

Therefore, we can use the Fubini's theorem to conclude that

$$\int_0^x (x-t)^{-\alpha}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)dsdt = \Gamma(1-\alpha)\int_0^x E_{\alpha}\left[(x-s)^{\alpha}A\right]Bu(s)ds. \tag{5}$$

Using the Lebesgue differentiation theorem, see [6], we can immediately conclude that

$$\frac{d}{dx}\int_0^x E_{\alpha}\left[(x-s)^{\alpha}A\right]Bu(s)ds = \int_0^x Ae^{(x-s)A}_{\alpha}Bu(s)ds + u(x), \quad \text{for almost all} \quad x.$$

From which the first part of (1) immediately follows.

Next, let us verify that the initial condition is also satisfied. From formula (4), we obtain

$$\lim_{x\downarrow 0}\left(I^{1-\alpha}_{0+}e^{tA}_{\alpha}y^0\right)(x) = \lim_{x\downarrow 0}E_{\alpha}[x^{\alpha}A]y^0 = y^0.$$

Furthermore, proceeding as in (5), we get

$$\left|\left(I^{1-\alpha}_{0+}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)ds\right)(x)\right| \leqslant ||B||\int_0^x E_{\alpha}[(x-s)^{\alpha}||A||]ds.$$

Consequently, we can apply the Dominated Convergence Theorem to deduce that

$$\lim_{x\downarrow 0}\left(I^{1-\alpha}_{0+}\int_0^t e^{(t-s)A}_{\alpha}Bu(s)ds\right)(x) = 0.$$

Thus, $(I^{1-\alpha}_{0+}y)(0+) = b$, as required. $\square$

3. Bang-bang and maximum principles

Next, we introduce some notation. We denote the solution of equation (1) as $y(t;y^0,u)$, or simply $y(t)$ when there is no confusion. For each $t \geqslant 0$, we define the set

$$G(t) = \{x \in \mathbb{R}^n : \text{there exists } u \in \mathscr{A},\ y(t;x,u) = 0\}.$$

Furthermore, for each $y^0 \in \mathbb{R}^n$, we define

$$\tau^* = \inf\{t \geqslant 0 : y^0 \in G(t)\}.$$

Let $G := \bigcup_{t \geqslant 0} G(t)$ be the set of admissible initial values.

LEMMA 1. *If $y^0 \in G$, then there exists $u^* \in \mathscr{A}$ such that $y(\tau^*;y^0,u^*) = 0$, i.e., u^* is an admissible optimal control.*

Proof. Let $t_1 \geqslant 0$ such that $y^0 \in G(t_1)$. Let (t_n) be a decreasing sequence such that $y^0 \in G(t_n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \tau^*$. Then, for each $n \in \mathbb{N}$, there exists $u_n \in \mathscr{A}$ such that

$$e_\alpha^{t_n A} y^0 = -\int_0^{t_n} e_\alpha^{(t_n - s)A} B u_n(s) ds. \tag{6}$$

By the Banach-Alaoglu theorem (see [3]), there exists $u^* \in L^\infty([0,t_1])$ and a subsequence (u_{n_k}) of (u_n) that converges in the weak-$*$ topology to u^*.

First, we verify that $\tau^* > 0$, if $y^0 \neq 0$. Suppose $\tau^* = 0$. Then,

$$\frac{1}{\Gamma(\alpha)} y^0 = \lim_{n\to\infty} (t_n)^{1-\alpha} e_\alpha^{t_n A} y^0 = -\lim_{n\to\infty} (t_n)^{1-\alpha} \int_0^{t_n} e_\alpha^{(t_n - s)A} B u_n(s) ds = 0.$$

This is absurd since $y^0 \neq 0$.

The continuity of the function $t \mapsto e_\alpha^{At}$ on $(0,\infty)$ implies ($\lim_{n\to\infty} t_n = \tau^* > 0$)

$$\lim_{n\to\infty} e_\alpha^{t_n A} y^0 = e_\alpha^{\tau^* A} y^0.$$

Furthermore, applying the Dominated Convergence Theorem and the weak-$*$ convergence of (u_{n_k}) to u^*, we obtain

$$\lim_{n\to\infty} \int_0^{t_{n_k}} e_\alpha^{(t_{n_k} - s)A} B u_{n_k}(s) ds = \int_0^{\tau^*} e_\alpha^{(\tau^* - s)A} B u^*(s) ds.$$

From (6), it immediately follows that $y^0 \in G(\tau^*)$. $\square$

Recall that a control $u = (u^1, \ldots, u^m)^T \in \mathscr{A}$ is of bang-bang type if, for each $i \in \{1, \ldots, m\}$, $|u^i(t)| = 1$, for almost every $t > 0$.

THEOREM 2. (Bang-bang principle) *If $y^0 \in G$, then there exists an optimal control u^* of bang-bang type.*

Proof. Consider the set

$$K = \{u \in \mathscr{A} : y(y^0;\tau^*,u) = 0\}.$$

From Lemma 1, we have $K \neq \emptyset$. Moreover, the convexity of the set $[-1,1]^m$ implies that K is also a convex set.

Using the Banach-Alaoglu theorem and the continuity of the function $t \mapsto e_\alpha^{At}$ on $(0,\infty)$, it follows that K is closed in the weak-$*$ topology. By the Krein-Milman theorem (see [17]), K has at least one extreme point, u^*. If we assume u^* is not a bang-bang control, then u^* can be expressed as a convex combination of two elements in K. Therefore, u^* is an optimal bang-bang type control (see, for instance, [20] or [4]). □

Now let us address a maximum principle, analogous to Pontryagin's maximum principle in the classical case. Before that, a bit more notation. Let $y^0 \in \mathbb{R}^n$ and $t > 0$,

$$F(t) = \{y^1 \in \mathbb{R}^n : \text{there exist } u \in \mathscr{A},\ y(y^0;t,u) = y^1\}.$$

As in the classical case, it holds that $F(t) \subset \mathbb{R}^n$ is a convex and compact set. Moreover, if $y^0 \in G$ and τ^* is the optimal time, given by Lemma 1, then $0 \in \partial F(\tau^*)$ (see, for example, [20]).

THEOREM 3. (Maximum principle) *If $y^0 \in G$, then there exists a vector $h \in \mathbb{R}^n$, $h \neq 0$, such that*

$$h^T e_\alpha^{(\tau^*-s)A} Bu^*(s) = \max_{a\in[-1,1]^m} \left\{ h^T e_\alpha^{(\tau^*-s)A} Ba \right\}, \tag{7}$$

for almost every $s \in [0,\tau^*]$.

Proof. Since $F(\tau^*)$ is a convex set and $0 \in F(\tau^*)$, there exists a hyperplane that supports $F(\tau^*)$ at 0, see Theorem 2 in Appendix B.3 of [11]. This implies that there exists a vector $h \neq 0$ such that

$$h \cdot y^1 \leqslant 0, \quad \text{for all} \quad y^1 \in F(\tau^*), \tag{8}$$

where the dot represents the usual inner product in $\mathbb{R}^n$.

Furthermore, note that for each $u \in \mathscr{A}$, the vector

$$y^1 := e_\alpha^{\tau^* A} y^0 + \int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu(s)ds$$

is in $F(\tau^*)$, see Theorem 1. Moreover, $0 \in F(\tau^*)$ implies

$$0 = e_\alpha^{\tau^* A} y^0 + \int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu^*(s)ds.$$

From these two expressions and (8), we deduce that

$$h^T e_\alpha^{\tau^* A} y^0 + h^T \int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu(s)ds \leqslant h^T e_\alpha^{\tau^* A} y^0 + h^T \int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu^*(s)ds.$$

Therefore,

$$\int_0^{\tau^*} h^T e_\alpha^{(\tau^*-s)A} B\left[u^*(s)-u(s)\right] ds \geqslant 0, \quad \text{for all} \quad u \in \mathscr{A}. \tag{9}$$

If equality (7) does not hold, then there exists an admissible control $\hat{u}$ such that (see [20] or [4])

$$\int_0^{\tau^*} h^T e_\alpha^{(\tau^*-s)A} B\left[u^*(s)-\hat{u}(s)\right] ds < 0.$$

This contradicts inequality (9). Therefore, equality (7) holds true. □

4. Examples

In this section, we present two examples to illustrate the results discussed in the preceding section. These applications underscore key aspects of fractional calculus that are essential for modeling real-world phenomena.

EXAMPLE 1. Consider the following fractional differential equation, which could be used to model the walk of a tightrope walker (see [7]),

$$\begin{aligned} (D^\alpha 0{+}y)(t) &= y(t)+u(t), \quad t>0, \\ (I_{0+}^{1-\alpha} y)(0+) &= y^0, \end{aligned} \tag{10}$$

where $\alpha \in (0,1]$, $u:[0,\infty] \to [-1,1]$ is a measurable function, and $y^0 \in \mathbb{R}$. The solution to the above equation is, see (2),

$$y(t) = e_\alpha^t y^0 + \int_0^t e_\alpha^{t-s} u(s) ds.$$

Note that

$$F(t) = \{y^1 \in \mathbb{R} : \text{there exists } u \in \mathscr{A} \text{ such that } y(y^0;t,u) = y^1\}.$$

Since $-1 \leqslant u \leqslant 1$, it follows that

$$F(t) = \left[e_\alpha^t y^0 - E_\alpha[t^\alpha]+1, e_\alpha^t y^0 + E_\alpha[t^\alpha]-1\right].$$

Therefore, the dynamic $y(t)$, starting at y^0, can reach the origin 0 at time t if

$$\frac{1-E_\alpha[t^\alpha]}{e^t\alpha} \leqslant y^0 \leqslant \frac{E\alpha[t^\alpha]-1}{e_\alpha^t}.$$

In this way,

$$G(t) = \left[-\frac{E_\alpha[t^\alpha]-1}{e_\alpha^t}, \frac{E_\alpha[t^\alpha]-1}{e_\alpha^t}\right].$$

This implies, that

$$G = \left(-\sup_{t>0} \frac{E_\alpha[t^\alpha]-1}{e_\alpha^t}, \sup_{t>0} \frac{E_\alpha[t^\alpha]-1}{e_\alpha^t}\right).$$

Let $y^0 \in G$. By Theorem 3, there exists $h \neq 0$ such that

$$u^*(s) = \operatorname{sign}(h), \quad 0 \leqslant s \leqslant \tau^*.$$

Thus, the optimal control u^* is constant. Suppose, for example, that $y^0 > 0$. If $u^* = 1$, the solution to the initial problem (10) is

$$y(t) = e^t_\alpha y^0 + E_\alpha[t^\alpha] - 1.$$

Therefore, $y(t) > 0$ for each $t > 0$, which means 0 is never reached. On the other hand, if $u^* = -1$, then

$$y(t) = e^t_\alpha y^0 - E_\alpha[t^\alpha] + 1,$$

and 0 is reached exactly at time τ^* (the optimal time, see (1)). Therefore,

$$y^0 = \frac{E_\alpha[(\tau^*)^\alpha] - 1}{e^{\tau^*}_\alpha}. \tag{11}$$

Given y^0 and α, we numerically solve equation (11) to find τ^*. The results are presented in Table 1.

$y_0 \setminus \alpha$	0.25	0.5	0.75	1
0.25	0.137	0.189	0.237	0.287
0.999	0.912	1.852	2.968	9.136
0.9999	0.913	1.856	2.980	15.299
1	0.914	1.857	2.981	∞
1.5	1.646	4.813	19.414	∞
1.7	1.976	6.446	32.955	∞
1.8	2.149	7.355	41.783	∞

Table 1: *Optimal time τ^* for different values of the parameters y_0 and α.*

From Table 1, we observe that, given a fixed initial condition, the optimal time decreases with respect to the fractional index. Furthermore, it is noted that for relatively small initial values, the classical dynamics have a finite optimal time. However, for slightly larger initial values, the optimal time tends to infinity. It is worth highlighting that in these cases, the fractional dynamics always produce a finite optimal time.

EXAMPLE 2. The following example is a simplified model of the problem of parking a car that has two engines, one at the rear and one at the front, and can only move in a straight line. Let us consider the following variables:

- Let $q(t)$ be the position of the car at time $t \geqslant 0$.
- Let $D^\alpha_{0+} q(t)$ be the fractional velocity at time $t \geqslant 0$.
- Let $u(t)$ be the thrust of the motors, $-1 \leqslant u(t) \leqslant 1$, for $t \geqslant 0$.

The rear engine (left) will be denoted by $+1$ and the front engine (right) by -1. The problem is to find the optimal way to turn the $+1$ and -1 engines on or off, not simultaneously, so that the vehicle parks at 0 in the shortest possible time, thus saving fuel. If we assume that the car has a mass of one, then a generalized law of motion will tell us that

$$D_{0+}^{2\alpha} q(t) = u(t),$$

where $D_{0+}^{2\alpha} = D_{0+}^{\alpha} D_{0+}^{\alpha}$ is the composition operator. Let us define $y(t) := (q(t), v(t))^T$, where $v(t) := D_{0+}^{\alpha} q(t)$. Therefore, we obtain

$$D_{0+}^{\alpha} y(t) = Ay(t) + Bu(t),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the initial condition is

$$(I_{0+}^{1-\alpha} y)(0+) = \begin{pmatrix} q_0 \\ v_0 \end{pmatrix}.$$

Let τ^* be the optimal time of arrival at the origin. Since $A^k = 0$ for all $k \geqslant 2$, therefore

$$e_{\alpha}^{(\tau^*-s)A} = (\tau^* - s)^{\alpha-1} \begin{pmatrix} 1/\Gamma(\alpha) & (\tau^*-s)^{\alpha}/\Gamma(2\alpha) \\ 0 & 1/\Gamma(\alpha) \end{pmatrix}. \tag{12}$$

By Theorem 3, there exists a vector $h = (h_1, h_2)^T \in \mathbb{R}^2 \backslash \{(0,0)\}$ such that

$$u^*(s) = \operatorname{sign}\left(\frac{h_1(\tau^*-s)^{\alpha}}{\Gamma(2\alpha)} + \frac{h_2}{\Gamma(\alpha)} \right), \quad s \in [0, \tau^*].$$

Suppose the vehicle is moving to the right of the origin, hence it is using the $+1$ engine. At this moment, which we will take as the starting point, $t = 0$, we apply the engine change, that is, we immediately switch to the -1 engine. The equation of motion will be

$$D_{0+}^{2\alpha} q(t) = -1,$$

for a certain period of time until the $+1$ engine is used again. In this case $q_0 > 0$, therefore $h_1 \neq 0$ and $h_2 \neq 0$, otherwise we would obtain a constant control and with a single engine change, it is not possible to bring the car to the origin. Thus, the next change will be to the $+1$ engine, therefore

$$\frac{h_1(\tau^*-s)^{\alpha}}{\Gamma(2\alpha)} + \frac{h_2}{\Gamma(\alpha)} > 0.$$

The function

$$S(s) := \frac{h_1(\tau^*-s)^{\alpha}}{\Gamma(2\alpha)} + \frac{h_2}{\Gamma(\alpha)}, \quad s \in [0, \tau^*],$$

is monotonic; hence, it changes sign only once. Let $t_c \in (0,\tau^*)$ denote this change of sign, so that $S(t_c)=0$. Therefore, the optimal control will be of bang-bang type (see Theorem 2)

$$u^*(t)=\begin{cases} -1, & 0<t<t_c,\\ 1, & t_c<t<\tau^*.\end{cases}$$

Next, let us find the switch time t_c. From (2) and (12), we conclude that

$$\begin{pmatrix} q(t)\\ v(t)\end{pmatrix}=\begin{cases}\begin{pmatrix}\frac{q_0}{\Gamma(\alpha)}t^{\alpha-1}+\frac{v_0}{\Gamma(2\alpha)}t^{2\alpha-1}\\ \frac{v_0}{\Gamma(\alpha)}t^{\alpha-1}\end{pmatrix}-\begin{pmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ \frac{t^{\alpha}}{\Gamma(\alpha+1)}\end{pmatrix}, & 0<t<t_c,\\ \begin{pmatrix}\frac{q_0}{\Gamma(\alpha)}t^{\alpha-1}+\frac{v_0}{\Gamma(2\alpha)}t^{2\alpha-1}\\ \frac{v_0}{\Gamma(\alpha)}t^{\alpha-1}\end{pmatrix}-\begin{pmatrix}\frac{t^{2\alpha}-(t-t_c)^{2\alpha}}{\Gamma(2\alpha+1)}\\ \frac{t^{\alpha}-(t-t_c)^{\alpha}}{\Gamma(\alpha+1)}\end{pmatrix}+\begin{pmatrix}\frac{(t-t_c)^{2\alpha}}{\Gamma(2\alpha+1)}\\ \frac{(t-t_c)^{\alpha}}{\Gamma(\alpha+1)}\end{pmatrix}, & t_c<t<\tau^*.\end{cases}$$

Since we know that $(q(\tau^*),v(\tau^*))=(0,0)$, then

$$0=\frac{q_0(\tau^*)^{\alpha-1}}{\Gamma(\alpha)}+\frac{v_0(\tau^*)^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{(\tau^*)^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2(\tau^*-t_c)^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{13}$$

$$0=\frac{v_0(\tau^*)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\tau^*)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2(\tau^*-t_c)^{\alpha}}{\Gamma(\alpha+1)}. \tag{14}$$

From equation (13), we deduce that

$$t_c=\tau^*\left\{1-\left(\frac{1}{2}-\frac{\alpha v_0}{2\tau^*}\right)^{1/\alpha}\right\}. \tag{15}$$

Substituting this value of t_c into (14), we obtain

$$0=\frac{q_0(\tau^*)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\tau^*)^{2\alpha}}{\Gamma(2\alpha)}\left\{\frac{v_0}{\tau^*}-\frac{1}{2\alpha}+\frac{1}{4\alpha}\left(1-\frac{\alpha v_0}{\tau^*}\right)^2\right\}. \tag{16}$$

We numerically solve equation (16) to find τ^*, and then substitute this value into (15) to determine the switch time t_c of the motor. Our numerical experiments are summarized in Table 2.

(q_0,v_0)	α	0.25	0.5	0.75	1
$(1,2)$	t_c	1.570	2.656	3.312	3.732
	τ^*	1.592	2.987	4.259	5.464
$(2,1)$	t_c	1.394	2.115	2.431	2.581
	τ^*	1.436	2.520	3.398	4.162
$(2,2)$	t_c	1.916	3.048	3.647	4
	τ^*	1.954	3.493	4.805	6

Table 2: *Values of τ^* and t_c for different initial conditions (q_0,v_0).*

From Table 2, we observe that fractional dynamics provide a better optimal time, which decreases with the fractional index.

5. Conclusions

In this paper, we have demonstrated the existence of solutions for linear fractional differential equations in the sense of Riemann-Liouville when perturbed by an essentially bounded function, i.e., when the control is essentially bounded. For these equations, we have shown that there exists an optimal time and an optimal bang-bang type control. Furthermore, we have established the validity of a maximum principle, similar to Pontryagin's maximum principle in the classical case. We have also provided two application examples of the results, in which we observed that fractional dynamics offer the best optimal times and that these times improve for small values of the fractional parameter.

In our study, we considered that the dynamics start at a point y^0, meaning $(I_{0+}^{1-\alpha}y)(0+) = y^0$, and reach a point y^1 at a certain time $t > 0$, $y(t) = y^1$. Among the various lines of future research, potential areas include developing controllability criteria or redefining the concept of reaching a state and studying the basic principles of control, see for example [13]. One possible concept of reaching a state is: we reach a state y^1 at time t if $(I_{0+}^{1-\alpha}y)(t+) = y^1$. This concept would be in harmony with the initial condition concept.

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