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ON CAPUTO TEMPERED FRACTIONAL COUPLED SYSTEMS WITH THREE POINTS BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate the existence, uniqueness, and Ulam stability of solutions to systems of Caputo-tempered fractional differential equations subject to three-point boundary conditions. The analysis is grounded in an extended version of Perov's fixed point theorem and a Krasnoselskii-type approach. By integrating these methods with techniques involving vector-valued metrics and matrix sequences that converge to zero, we establish their primary theoretical results. To further elucidate the findings, illustrative examples are provided in the final section of the paper.

1. Introduction

Fractional differential equations generalize ordinary differential equations to non-integer orders, offering a powerful framework for modeling and understanding a wide range of phenomena in science and engineering, such as electrochemistry, control systems, porous media, and electromagnetism [4, 7, 8, 10, 37, 39, 40, 44]. In recent years, the study of boundary value problems for fractional differential equations has attracted significant attention from mathematicians, leading to substantial advancements in the field [3, 5–7].

Tempered fractional calculus has recently emerged as a significant extension of fractional calculus operators, generalizing various forms and providing analytic kernels that effectively capture the transition between normal and anomalous diffusion. The concept of tempered fractional derivatives extends beyond the conventional framework of fractional derivatives to accommodate functions characterized by exponentially decaying tails. In the pioneering work by Buschman [13], the initial definitions of fractional integration involving weak singular and exponential kernels were introduced. Demonstrated to be a valuable tool, this extension holds particular significance in applications where memory effects are crucial, such as in viscoelastic materials, nonlocal models in physics, and fractional-order control systems [15, 24, 25]. The tempered fractional derivative enables a more precise depiction of underlying dynamics,

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encompassing both long-range memory and rapidly decaying behaviors. For a more comprehensive exploration of this subject, additional insights can be found in references [22, 23, 26, 27, 35, 36].

Ulam-Hyers stability is a concept in functional equation theory, originating from Ulam's 1940 question about the conditions for an approximately additive function to be close to an exact additive function, see [41]. Hyers provided a partial answer in 1941, proving that if a function approximates an additive condition, there is an exact additive function close to the approximate one, see [16]. This concept has been extended to various functional equations and mathematical settings, significantly influencing their study and applications in various fields. Further elaboration on this topic can be found in [1, 32, 33, 43].

Coupled systems involving fractional differential equations are of interest in various scientific and engineering fields. These systems generally comprise multiple equations, which can be interconnected through their derivatives or the variables they describe. For more information, see publications [2, 10–12] and the references therein. Such systems are used to model fractional-order dynamics in contexts like viscoelastic materials, biological processes, and complex networks. The Ulam stability of ordinary and fractional differential equations has recently been studied in [9, 17, 18, 29–33].

In [35], the authors investigated the following class of Caputo tempered fractional differential equation:

$$\begin{cases} \left({}^{C}_{\kappa_1}\mathfrak{D}^{\zeta,\omega}_{t}y\right)(t) = f\left(t, y(t), \left({}^{C}_{\kappa_1}\mathfrak{D}^{\zeta,\omega}_{t}y\right)(t)\right); \ t \in J := [\kappa_1, \kappa_2], \\ iy(\kappa_1) + jy(\kappa_2) = \varsigma y(\eta) + \rho, \end{cases}$$

where $0 < \zeta < 1$, $\omega \geqslant 0$, ${}^{C}_{\kappa_1}\mathfrak{D}^{\zeta,\omega}_{t}$ is the Caputo tempered fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R}$ is a continuous function, $\kappa_1 < \eta < \kappa_2 < +\infty$, i, j, ρ, ς are real constants.

As a continuation of previous results, in this paper, we study existence, uniqueness and Ulam stability results for the fractional differential equation involving the Caputo tempered fractional derivative:

$$\begin{cases} {}^{C}_{0}\mathfrak{D}^{\zeta_1,\omega_1}_{t} y_1(t) = f_1(t, y_1(t), y_2(t)), & t \in J := [0, b], \\ {}^{C}_{0}\mathfrak{D}^{\zeta_2,\omega_2}_{t} y_2(t) = f_2(t, y_1(t), y_2(t)), & t \in J := [0, b], \\ \alpha_1 y_1(0) + \beta_1 y_1(b) = \gamma_1 y_1(\eta_1) + \delta_1, \\ \alpha_2 y_2(0) + \beta_2 y_2(b) = \gamma_2 y_2(\eta_2) + \delta_2, \end{cases} \tag{1}$$

where $0 < \zeta_1, \zeta_2 \leqslant 1$, $\omega_1, \omega_2 > 0$, $f_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for $i = 1, 2$, ${}^{C}_{0}\mathfrak{D}^{\zeta_i,\omega_i}_{t}$, $i = 1, 2$, are the Caputo tempered fractional derivatives of order ζ_i, $i = 1, 2$, α_i, β_i, γ_i, δ_i, for $i = 1, 2$ are positive constants, $\eta_i \in [0, b]$ for $i = 1, 2$.

We begin by presenting some preliminary results and introducing the concept of matrices that converge to zero. In Section 3, we provide sufficient conditions for the existence of solutions to system (1), using an application of the Krasnoselskii fixed point theorem. In Section 4, we employ a Perov-type fixed point theorem to derive existence and uniqueness results. In Section 5, establish the Ulam stability of the system. Both methods rely on the use of convergent matrices and vector norms. In Section 5, several examples are provided to illustrate the main results.

2. Preliminaries

In this section, we recall some basic notations, definitions, and lemmas and the necessary notation needed in the remainder of the paper.

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J to $\mathbb{R}$, with the norm

$$\|y\|_C := \sup_{t\in[0,b]} |y(t)|.$$

Then, $C(J,\mathbb{R})\times C(J,\mathbb{R})$ is the generalized Banach space equipped the norm

$$\|(y_1,y_2)\|_{C\times C} := \left(\|y_1\|_C, \|y_2\|_C\right)^T.$$

DEFINITION 1. (The Riemann-Liouville tempered fractional integral [22, 34, 38]) Suppose that the function $\Psi \in C([0,b],\mathbb{R})$, $\omega > 0$. Then, the Riemann-Liouville tempered fractional integral of order ζ is defined by

$$_0I_t^{\zeta,\omega}\Psi(t) = e^{-\omega t}\ {}_0I_t^{\zeta}\left(e^{\omega t}\Psi(t)\right) = \frac{1}{\Gamma(\zeta)}\int_0^t \frac{e^{-\omega(t-\tau)}\Psi(\tau)}{(t-\tau)^{1-\zeta}}d\tau, \tag{2}$$

where ${}_0I_t^{\zeta}$ denotes the Riemann-Liouville fractional integral [19], defined by

$$_0I_t^{\zeta}\Psi(t) = \frac{1}{\Gamma(\zeta)}\int_0^t \frac{\Psi(\tau)}{(t-\tau)^{1-\zeta}}d\tau. \tag{3}$$

Obviously, the tempered fractional integral (2) reduces to the Riemann-Liouville fractional integral (3) if $\omega = 0$.

DEFINITION 2. (The Riemann-Liouville tempered fractional derivative [22, 34]) For $n-1<\zeta<n$; $n\in\mathbb{N}^+$, $\omega \geqslant 0$. The Riemann-Liouville tempered fractional derivative is defined by

$$_0\mathfrak{D}_t^{\zeta,\omega}\Psi(t) = e^{-\omega t}{}_0\mathfrak{D}_t^{\zeta}\left(e^{\omega t}\Psi(t)\right) = \frac{e^{-\omega t}}{\Gamma(n-\zeta)}\frac{d^n}{dt^n}\int_0^t \frac{e^{\omega\tau}\Psi(\tau)}{(t-\tau)^{\zeta-n+1}}dt,$$

where ${}_0\mathfrak{D}_t^{\zeta}$ denotes the Riemann-Liouville fractional derivative [19], given by

$$_0\mathfrak{D}_t^{\zeta}\Psi(t) = \frac{d^n}{dt^n}\left({}_0\mathscr{I}_t^{n-\zeta}\Psi(t)\right) = \frac{1}{\Gamma(n-\zeta)}\frac{d^n}{dt^n}\int_0^t \frac{\Psi(\tau)}{(t-\tau)^{\zeta-n+1}}d\tau.$$

DEFINITION 3. (The Caputo tempered fractional derivative [22, 38]) For $n-1<\zeta<n$; $n\in\mathbb{N}^+$, $\omega\geqslant 0$. The Caputo tempered fractional derivative is defined as

$${}_0^C\mathfrak{D}_t^{\zeta,\omega}\Psi(t) = e^{-\omega t}\ {}_0^C\mathfrak{D}_t^{\zeta}\left(e^{\omega t}\Psi(t)\right) = \frac{e^{-\omega t}}{\Gamma(n-\zeta)}\int_0^t \frac{1}{(t-\tau)^{\zeta-n+1}}\frac{d^n}{d\tau^n}\left(e^{\omega\tau}\Psi(\tau)\right)d\tau,$$

where ${}_{0}^{C}\mathfrak{D}_t^{\zeta,\omega}$ denotes the Caputo fractional derivative [19], given by

$$ {}_{0}^{C}\mathfrak{D}_t^{\zeta}\Psi(t) = \frac{1}{\Gamma(n-\zeta)}\int_0^t \frac{1}{(t-\tau)^{\zeta-n+1}}\frac{d^n}{d\tau^n}\Psi(\tau)d\tau. $$

LEMMA 1. ([22]) *For a constant C,*

$$ {}_{0}\mathfrak{D}_t^{\zeta,\omega}C = Ce^{-\omega t}{}_{0}\mathfrak{D}_t^{\zeta}e^{\omega t}, \quad {}_{b_1}^{C}\mathfrak{D}_t^{\zeta,\omega}C = Ce^{-\omega t}\,{}_{0}^{C}\mathfrak{D}_t^{\zeta}e^{\omega t}. $$

Obviously, ${}_{0}\mathfrak{D}_t^{\zeta,\omega}(C) \neq {}_{0}^{C}\mathfrak{D}_t^{\zeta,\omega}(C)$. And, ${}_{b_1}^{C}\mathfrak{D}_t^{\zeta,\omega}(C)$ is no longer equal to zero, being different from ${}_{0}^{C}\mathfrak{D}_t^{\alpha}(C) \equiv 0$.

LEMMA 2. ([22,38]) *Let $\Psi \in C^n([0,b],\mathbb{R})$, $\omega \geqslant 0$ and $n-1 < \zeta < n$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the following composite properties:*

$$ {}_{b_1}I_t^{\zeta,\omega}\left[{}_{0}^{C}\mathfrak{D}_t^{\zeta,\omega}\Psi(t)\right] = \Psi(t) - \sum_{k=0}^{n-1} e^{-\omega t}\frac{(t-0)^k}{k!}\left[\left.\frac{d^k\left(e^{\omega t}\Psi(t)\right)}{dt^k}\right|_{t=0}\right], $$

and

$$ {}_{0}^{C}\mathfrak{D}_t^{\zeta,\omega}\left[{}_{a}I_t^{\zeta,\omega}\Psi(t)\right] = \Psi(t), \ \textit{for} \ \zeta \in (0,1). $$

DEFINITION 4. ([14]) Let X be a nonempty set. By a vector-valued metric on X we mean a map $d: X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u,v) \geqslant 0$ for all u, $v \in X$, and if $d(u,v) = 0$, then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all u, $v \in X$;

(iii) $d(u,v) \leqslant d(u,w) + d(w,v)$ for all u, v, $w \in X$.

A set X with a vector-valued metric d is called a *generalized metric space*. In this space, the notions of Cauchy sequence, convergence, completeness, and open and closed sets are similar to those in usual metric spaces. Here, if $x, y \in \mathbb{R}^n$, where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$, by $x \leqslant y$ we mean $x_i \leqslant y_i$ for $i = 1, 2, \cdots, n$. The pair (X,d) is a generalized metric space with

$$ d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}. $$

Notice that d is a generalized metric on X if and only if d_i, $i = 1, 2, \cdots, n$, are metrics on X.

Similarly, a *vector valued norm* on a linear space X is a mapping $\|\cdot\| : X \to \mathbb{R}_+^n$ with $\|x\| = 0$ only for $x = 0$, $\|\lambda x\| = |\lambda|\|x\|$ for $x \in X$ and $\lambda \in \mathbb{R}$, and $\|x+y\| \leqslant \|x\| + \|y\|$ for every x, $y \in X$. Associated to a vector valued norm $\|\cdot\|$ is a *vector valued metric* $d(x,y) := \|x-y\|$, and we say that $(X, \|\cdot\|)$ is a *generalized Banach space* if X is complete with respect to d.

Next, we define what is meant by a matrix that is convergent to zero [28].

DEFINITION 5. ([14]) A square matrix with real entries is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, all the eigenvalues of M are in the open unit disc $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the identity matrix in $\mathscr{M}_{n\times n}(\mathbb{R})$.

The following result gives some characterizations of a matrix that converges to zero.

THEOREM 1. ([14]) *Let $M \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$; the following assertions are equivalent:*

(a) *M is convergent to zero;*

(b) *$M^k \to 0$ as $k \to \infty$;*

(c) *The matrix $(I-M)$ is nonsingular and*

$$(I-M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots.$$

(d) *The matrix $(I-M)$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.*

Some examples of matrices that are convergent to zero include:

(i) $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a,b \in \mathbb{R}_+$ and $a+b<1$;

(ii) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a,b \in \mathbb{R}_+$ and $a+b<1$;

(iii) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a,b,c \in \mathbb{R}_+$ and $|a-b|<1$, $a>1$, $b>0$.

Before we state Perov's fixed point theorem we must define a contractive operator.

DEFINITION 6. ([14]) Let (X,d) be a generalized metric space. An operator $N : X \to X$ is called contractive associated with a generalized metric d on X, if there exists a convergent to zero matrix M such that

$$d(T(x),T(y)) \leqslant Md(x,y), \quad \text{for all } x,y \in X.$$

Next, we give Perov's fixed point theorem and Krasnoselskii's fixed point theorem for a sum of two operators.

THEOREM 2. ([14] (Perov's fixed point theorem)) *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a contractive operator with matrix M. Then T has a unique fixed point u, and for each $u_0 \in X$,*

$$d(T^k(u_0),u) \leqslant M^k(I-M)^{-1}d(u_0,T(u_0)) \quad \textit{where } k \in \mathbb{N}.$$

THEOREM 3. (Krasnoselskii's fixed point theorem [14]) *Let $(X,\|\cdot\|)$ be a generalized Banach space. Suppose that Ψ and Σ are two operators $X \longrightarrow X$ such that*

($\mathscr{A}_1$) *Ψ completely continuous operator,*

($\mathscr{A}_2$) *Σ be a continuous and M-contraction operator,*

($\mathscr{A}_3$) *the matrix $I-M$ has the absolute property if*

$$\mathscr{B} = \left\{ y \in X \mid \lambda\Psi(y) + \lambda\Sigma\left(\frac{y}{\lambda}\right) = y,\ \lambda \in (0,1) \right\},$$

is bounded.

Then the equation

$$y = \Psi(y) + \Sigma(y), \quad y \in X,$$

has at least one fixed point.

For our discussion of the existence and uniqueness of the solutions and stability to our problem, we also need the following concepts and results.

LEMMA 3. *Let $0 < \zeta_1 < 1$, $\Delta_1 = \alpha_1 + \beta_1 e^{-\omega_1 b} - \gamma_1 e^{-\omega_1 \eta} \neq 0$ and $h_1 : J \to \mathbb{R}$ is a continuous functions. Then the problem*

$$ {}_0^C\mathfrak{D}_t^{\zeta_1,\omega_1} y_1 = h_1(t); \quad t \in J := [0,b], \tag{4}$$

$$\alpha_1 y_1(0) + \beta_1 y_1(b) = \gamma_1 y_1(\eta_1) + \delta_1, \tag{5}$$

has a unique solution defined by

$$\begin{aligned} y_1(t) = {} & \frac{\delta_1}{\Delta_1} + \frac{\gamma_1}{\Delta_1\Gamma(\zeta_1)} \int_0^{\eta_1} (\eta_1 - s)^{\zeta_1 - 1} e^{-\omega_1(\eta_1 - s)} h_1(s)ds \\ & + \frac{\beta_1}{\Delta_1\Gamma(\zeta_1)} \int_0^b (t-s)^{\zeta_1 - 1} e^{-\omega_1(b-s)} h_1(s)ds \\ & + \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1-1} e^{-\omega_1(t-s)} h_1(s)ds. \end{aligned}$$

Proof. Applying the Riemann-Liouville tempered fractional integral of order ζ_1 to both sides the equation (4), and by using Lemma 2 and if $t \in J$, we get

$$y_1(t) - e^{-\omega_1 t} y_1(0) = \frac{1}{\Gamma(\zeta_1)} \int_0^t e^{-\omega_1(t-s)} (t-s)^{\zeta_1-1} h(s)ds.$$

From the condition (5), we get

$$\begin{aligned} & \alpha_1 y_1(0) + \beta_1 \left(y_1(0) + \frac{1}{\Gamma(\zeta_1)} \int_0^b (b-s)^{\zeta_1-1} e^{-\omega_1(b-s)} h_1(s)ds \right) \\ & = \gamma_1 y_1(0) + \frac{\gamma_1 \rho_1^{\zeta_1 - 1}}{\Gamma(\zeta_1)} \int_0^{\eta_1} (\eta_1 - s)^{\zeta_1-1} e^{-\omega_1(\eta_1 - s)} h_1(s)ds + \delta_1. \end{aligned}$$

Then

$$(\alpha_1+\beta_1 e^{-\omega_1 b}-\gamma_1 e^{-\omega_1 \eta})y_1(0)=\delta_1+\frac{\gamma_1}{\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}h_1(s)ds \\ -\frac{\beta_1}{\Gamma(\zeta_1)}\int_0^{b}(b-s)^{\zeta_1-1}e^{-\omega_1(b-s)}h_1(s)ds.$$

By dividing, we have

$$\begin{aligned} y_1(0)=\frac{\delta_1}{\Delta_1}&+\frac{\gamma_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}h_1(s)dsds \\ &+\frac{\beta_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{b}(b-s)^{\zeta_1-1}e^{-\omega_1(b-s)}h_1(s)ds. \end{aligned} \tag{6}$$

Substituting the values of $y_1(0)$ in (6), we obtain:

$$\begin{aligned} y_1(t)=\frac{\delta_1}{\Delta_1}&+\frac{\gamma_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}h_1(s)ds \\ &+\frac{\beta_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{b}(t-s)^{\zeta_1-1}e^{-\omega_1(b-s)}h_1(s)ds \\ &+\frac{1}{\Gamma(\zeta_1)}\int_0^{t}(t-s)^{\zeta_1-1}e^{-\omega_1(t-s)}h_1(s)ds. \quad \square \end{aligned}$$

Using Lemma 3, the solution $(y_1,y_2)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$ of the system (1) is given by

$$\begin{aligned} y_1(t)=\frac{\delta_1}{\Delta_1}&+\frac{\gamma_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}f_1(s,y_1(s),y_2(s))ds \\ &+\frac{\beta_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{b}(t-s)^{\zeta_1-1}e^{-\omega_1(b-s)}f_1(s,y_1(s),y_2(s))ds \\ &+\frac{1}{\Gamma(\zeta_1)}\int_0^{t}(t-s)^{\zeta_1-1}e^{-\omega_1(t-s)}f_1(s,y_1(s),y_2(s))ds, \end{aligned}$$

and

$$\begin{aligned} y_2(t)=\frac{\delta_2}{\Delta_2}&+\frac{\gamma_2}{\Delta_2\Gamma(\zeta_2)}\int_0^{\eta_2}(\eta_2-s)^{\zeta_2-1}e^{-\omega_2(\eta_2-s)}f_2(s,y_1(s),y_2(s))ds \\ &+\frac{\beta_2}{\Delta_2\Gamma(\zeta_2)}\int_0^{b}(t-s)^{\zeta_2-1}e^{-\omega_2(b-s)}f_2(s,y_1(s),y_2(s))ds \\ &+\frac{1}{\Gamma(\zeta_2)}\int_0^{t}(t-s)^{\zeta_2-1}e^{-\omega_2(t-s)}f_2(s,y_1(s),y_2(s))ds. \end{aligned}$$

3. Existence result

The existence result is based on Krasnoselskii's fixed point theorem. Let us introduce the following hypotheses:

(H_1) The functions $f_1, f_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist a_1^*, a_2^*, a_3^*, $a_4^* > 0$ such that:

$$|f_1(t,y_1,y_2) - f_1(t,\bar{y}_1,\bar{y}_2)| \leqslant a_1^*|y_1 - \bar{y}_1| + a_2^*|y_2 - \bar{y}_2|,$$

and

$$|f_2(t,y_1,y_2) - f_2(t,\bar{y}_1,\bar{y}_2)| \leqslant a_3^*|y_1 - \bar{y}_1| + a_4^*|y_2 - \bar{y}_2|,$$

for any (y_1,y_2), $(\bar{y}_1,\bar{y}_2) \in \mathbb{R} \times \mathbb{R}$.

(H_2) Define the square matrix with spectral radius is less than one

$$\chi = \begin{pmatrix} a_1^*\Upsilon_1 & a_2^*\Upsilon_1 \\ a_3^*\Upsilon_2 & a_4^*\Upsilon_2 \end{pmatrix},$$

where the entries in χ are defined by

$$\Upsilon_1 = \frac{1}{\Gamma(\zeta_1+1)}\left(b^{\zeta_1}\left(1+\frac{\beta_1}{|\Delta_1|}\right)+\frac{\gamma_1\eta_1^{\zeta_1}}{|\Delta_1|}\right),$$

and

$$\Upsilon_2 = \frac{1}{\Gamma(\zeta_2+1)}\left(b^{\zeta_2}\left(1+\frac{\beta_2}{|\Delta_2|}\right)+\frac{\gamma_2\eta_2^{\zeta_2}}{|\Delta_2|}\right).$$

Also, let

$$\varpi_1 = \frac{1}{|\Delta_1|\Gamma(\zeta_1+1)}\left(\beta_1 b^{\zeta_1} + \gamma_1\eta_1^{\zeta_1}\right),$$

and

$$\varpi_2 = \frac{1}{|\Delta_2|\Gamma(\zeta_2+1)}\left(\beta_2 b^{\zeta_2} + \gamma_2\eta_2^{\zeta_2}\right).$$

THEOREM 4. *Assume that (H_1), (H_2) are satisfied, if the matrix*

$$M = \begin{pmatrix} a_1^*\varpi_1 & a_2^*\varpi_1 \\ a_3^*\varpi_2 & a_4^*\varpi_2 \end{pmatrix} \tag{7}$$

converges to 0, then the problem (1) has at least one solution.

Proof. Let

$$\Phi : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R}) \times C(J,\mathbb{R}),$$

be the operator defined by

$$\Phi(y_1,y_2) = \Psi(y_1,y_2) + \Sigma(y_1,y_2), \quad (y_1,y_2) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}),$$

where

$$\Psi(y_1,y_2) = (\Psi_1(y_1,y_2),\Psi_2(y_1,y_2)),$$
$$\Sigma(y_1,y_2) = (\Sigma_1(y_1,y_2),\Sigma_2(y_1,y_2)),$$
$$\Psi_1(y_1(t),y_2(t)) = \frac{1}{\Gamma(\zeta_1)}\int_0^t (t-s)^{\zeta_1-1}e^{-\omega(t-s)}f_1(s,y_1(s),y_2(s))ds,$$
$$\Psi_2(y_1(t),y_2(t)) = \frac{1}{\Gamma(\zeta_2)}\int_0^t (t-s)^{\zeta_2-1}e^{-\omega(t-s)}f_2(s,y_1(s),y_2(s))ds,$$

$$\begin{aligned}\Sigma_1(y_1(t),y_2(t)) = {} & \frac{\delta_1}{\Delta_1}+\frac{\gamma_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}f_1(s,y_1(s),y_2(s))ds\\ & +\frac{\beta_1}{\Delta_1\Gamma(\zeta_1)}\int_0^b (b-s)^{\zeta_1-1}e^{-\omega_1(b-s)}f_1(s,y_1(s),y_2(s))ds,\end{aligned}$$

and

$$\begin{aligned}\Sigma_2(y_1(t),y_2(t)) = {} & \frac{\delta_1}{\Delta_2}+\frac{\gamma_2}{\Delta_2\Gamma(\zeta_2)}\int_0^{\eta_2}(\eta_2-s)^{\zeta_2-1}e^{-\omega_1(\eta_2-s)}f_2(s,y_1(s),y_2(s))ds\\ & +\frac{\beta_2}{\Delta_2\Gamma(\zeta_2)}\int_0^b (b-s)^{\zeta_2-1}e^{-\omega_2(b-s)}f_2(s,y_1(s),y_2(s))ds.\end{aligned}$$

In order to show that the conditions of Theorem 3 are satisfied, we will proceed in several steps.

Step 1: Σ is a generalized contraction.
Let $(y_1,y_2),(\bar{y_1},\bar{y_2}) \in C(J,\mathbb{R})\times C(J,\mathbb{R})$, using the assumption (H_1), we deduce that

$$\begin{aligned}&|\Sigma_1(y_1(t),y_2(t))-\Sigma_1(\bar{y_1}(t),\bar{y_2}(t))|\\
&\quad\leqslant \frac{\gamma_1}{|\Delta_1|\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y}_1(s),\bar{y}_2(s))|ds\\
&\qquad+\frac{\beta_1}{|\Delta_1|\Gamma(\zeta_1)}\int_0^b(b-s)^{\zeta_1-1}e^{-\omega_1(b-s)}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y}_1(s),\bar{y}_2(s))|ds\\
&\quad\leqslant \frac{\gamma_1}{|\Delta_1|\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y}_1(s),\bar{y}_2(s))|ds\\
&\qquad+\frac{\beta_1}{|\Delta_1|\Gamma(\zeta_1)}\int_0^b(b-s)^{\zeta_1-1}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y}_1(s),\bar{y}_2(s))|ds\\
&\quad\leqslant \frac{1}{\Gamma(\zeta_1)|\Delta_1|}(a_1^*\|y_1-\bar{y_1}\|_C+a_2^*\|y_2-\bar{y_2}\|_C)\\
&\qquad\times\left(\beta_1\int_0^b(b-s)^{\zeta_1-1}ds+\gamma_1\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}ds\right)\\
&\quad\leqslant \frac{a_1^*}{|\Delta_1|\Gamma(\zeta_1+1)}\left(\beta_1 b^{\zeta_1}+\gamma_1\eta_1^{\zeta_1}\right)\|y_1-\bar{y_1}\|_C\\
&\qquad+\frac{a_2^*}{|\Delta_1|\Gamma(\zeta_1+1)}\left(\beta_1 b^{\zeta_1}+\gamma_1\eta_1^{\zeta_1}\right)\|y_2-\bar{y_2}\|_C.\end{aligned}$$

Thus,
$$\|\Sigma_1(y_1,y_2)-\Sigma_1(\bar{y_1},\bar{y_2})\|_C \leqslant \varpi_1\left(a_1^*\|y_1-\bar{y_1}\|_C+a_2^*\|y_2-\bar{y}_2\|_C\right), \tag{8}$$
where
$$\varpi_1=\frac{1}{|\Delta_1|\Gamma(\zeta_1+1)}\left(\beta_1 b^{\zeta_1}+\gamma_1\eta_1^{\zeta_1}\right).$$
Similarly, we have
$$\begin{aligned}|\Sigma_2(y_1(t),y_2(t))-\Sigma_2(\bar{y_1}(t),\bar{y_2}(t))| &\leqslant \frac{a_3^*}{|\Delta_2|\Gamma(\zeta_2+1)}\left(\beta_2 b^{\zeta_2}+\gamma_2\eta_2^{\zeta_2}\right)\|y_1-\bar{y_1}\|_C\\ &\quad+\frac{a_4^*}{|\Delta_2|\Gamma(\zeta_2+1)}\left(\beta_2 b^{\zeta_2}+\gamma_2\eta_2^{\zeta_2}\right)\|y_2-\bar{y_2}\|_C.\end{aligned}$$
Thus,
$$\|\Sigma_1(y_1,y_2)-\Sigma_1(\bar{y_1},\bar{y_2})\|_C \leqslant \varpi_2\left(a_3^*\|y_1-\bar{y_1}\|_C+a_4^*\|y_2-\bar{y}_2\|_C\right), \tag{9}$$
where
$$\varpi_2=\frac{1}{|\Delta_2|\Gamma(\zeta_2+1)}\left(\beta_2 b^{\zeta_2}+\gamma_2\eta_2^{\zeta_2}\right).$$
From (8) and (9), we obtain
$$\begin{pmatrix}\|\Sigma_1(y_1,y_2)-\Sigma_1(\bar{y_1},\bar{y_2})\|_C\\ \|\Sigma_2(y_1,y_2)-\Sigma_2(\bar{y_1},\bar{y_2})\|_C\end{pmatrix}\leqslant M\begin{pmatrix}\|y_1-\bar{y_1}\|_C\\ \|y_2-\bar{y_2}\|_C\end{pmatrix},$$
with
$$M=\begin{pmatrix}a_1^*\varpi_1 & a_2^*\varpi_1\\ a_3^*\varpi_2 & a_4^*\varpi_2\end{pmatrix}.$$
Since M converge to zero, this implies that Σ is a contraction operator.

Step 2: Φ is completely continuous operator.
The continuity of (f_1,f_2) implies that the operator $\Psi(y_1,y_2)=(\Psi_1(y_1,y_2),\Psi_2(y_1,y_2))$ is continuous.

We want to show that Ψ maps bounded sets into relatively compact sets, so let D be a bounded subset of $C(J,\mathbb{R})$. Then there exists $q>0$ such that $\|y_1\|_C\leqslant q_1$ and $\|y_2\|_C\leqslant q_2$ for all $(y_1,y_2)\in D$. Let $(y_1,y_2)\in D$.

Then for each $t\in J$, we have
$$\begin{aligned}|\Psi_1(y_1(t),y_2(t))| &\leqslant \frac{1}{\Gamma(\zeta_1)}\int_0^t(t-s)^{\zeta_1-1}e^{-w_1(t-s)}|f_1(s,y_1(s),y_2(s))|ds\\ &\leqslant \frac{1}{\Gamma(\zeta_1)}\int_0^t(t-s)^{\zeta_1-1}e^{-w_1(t-s)}|f_1(s,y_1(s),y_2(s))-f_1(s,0,0)|ds\\ &\quad+\frac{1}{\Gamma(\zeta_1)}\int_0^t(t-s)^{\zeta_1-1}e^{-w(t-s)}|f_1(s,0,0)|ds\\ &\leqslant \frac{1}{\Gamma(\zeta_1)}\int_0^t(t-s)^{\zeta_1-1}e^{-w_1(t-s)}\left(a_1^*|y_1(s)|+a_2^*|y_2(s)|+b_1^*\right)ds\\ &\leqslant \frac{b^{\zeta}}{\Gamma(\zeta_1+1)}(a_1^*q_1+a_2^*q_2+b_1^*),\end{aligned}$$

which implies that

$$\|\Psi_1(y_1,y_2)\|_C \leqslant \frac{2b^{\zeta_1}}{\Gamma(\zeta_1+1)}(a^*q^*+b_1^*),$$

where $a^* = \max\{a_1^*, a_2^*\}$, $q^* = \max\{q_1, q_2\}$ and $b_1^* = \sup\limits_{t\in J}|f_1(t,0,0)|$.

Similarly, we have that

$$\|\Psi_2(y_1,y_2)\|_C \leqslant \frac{2b^{\zeta_2}}{\Gamma(\zeta_2+1)}(\tilde{a}^*q^*+b_2^*),$$

where $\tilde{a}^* = \max\{a_3, a_4\}$, $q^* = \max\{q_1, q_2\}$ and $b_2^* = \sup\limits_{t\in J}|f_2(t,0,0)|$.

Therefore, Ψ maps bounded sets in $C(J,\mathbb{R})$ into bounded sets in $C(J,\mathbb{R})$. Moreover, for any $t \in J$ and $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&|\Psi_1(y_1,y_2)(\tau_2) - \Psi_1(y_1,y_2)(\tau_1)| \\
&\quad \leqslant \frac{1}{\Gamma(\zeta_1)}\int_0^{\tau_1}(\tau_2-s)^{\zeta_1-1}e^{-w_1(\tau_2-s)} - (\tau_1-s)^{\zeta_1-1}e^{-w_1(\tau_1-s)}|f_1(s,y_1(s),y_2(s))|ds \\
&\qquad + \frac{1}{\Gamma(\zeta_1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\zeta_1-1}e^{-w_1(\tau_2-s)}|f_1(s,y_1(s),y_2(s))|ds \\
&\quad \leqslant (a^*q+b_1^*)\left[\int_0^{\tau_1}\frac{\left((\tau_2-s)^{\zeta_1-1}e^{-w_1(\tau_2-s)} - (\tau_1-s)^{\zeta_1-1}e^{-w_1(\tau_1-s)}\right)}{\Gamma(\zeta_1)}ds \right. \\
&\qquad \left. + \frac{(\tau_2-\tau_1)^{\zeta_1-1}}{\Gamma(\zeta_1+1)}\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&|\Psi_2(y_1,y_2)(\tau_2) - \Psi_2(y_1,y_2)(\tau_1)| \\
&\quad \leqslant \frac{1}{\Gamma(\zeta_1)}\int_0^{\tau_1}\left((\tau_2-s)^{\zeta_2-1}e^{-w_2(\tau_2-s)} - (\tau_1-s)^{\zeta_2-1}e^{-w_2(\tau_1-s)}\right)|f_2(s,y_1(s),y_2(s))|ds \\
&\qquad + \frac{1}{\Gamma(\zeta_2)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\zeta_1-1}e^{-w_2(\tau_2-s)}|f_2(s,y_1(s),y_2(s))|ds \\
&\quad \leqslant (\tilde{a}^*q+b_2^*)\left[\int_0^{\tau_1}\frac{(\tau_2-s)^{\zeta_1-1}e^{-w_2(\tau_2-s)} - (\tau_1-s)^{\zeta_1-1}e^{-w_2(\tau_1-s)}}{\Gamma(\zeta_2)}ds \right. \\
&\qquad \left. + \frac{(\tau_2-\tau_1)^{\zeta_2-1}}{\Gamma(\zeta_2+1)}\right],
\end{aligned}$$

which is not dependent on the pair (y_1, y_2) and the quantity $(\tau_1, \tau_2) \to 0$ which ensures that $\Psi(D)$ is equicontinuous. So, $\Psi(D)$ is relatively compact on $C(J,\mathbb{R})$.

Step 3. To show that the set

$$\mathscr{B}=\left\{(y_1,y_2)\in C(J,\mathbb{R})\times C(J,\mathbb{R}):\lambda\Psi(y_1,y_2)+\lambda\Sigma\left(\frac{y_1}{\lambda},\frac{y_2}{\lambda}\right),\ \lambda\in(0,1)\right\}$$

is bounded, let $(y_1,y_2)\in\mathscr{B}$. Then,

$$\begin{aligned}y_1(t)&=\frac{\delta_1}{\Delta_1}+\frac{\gamma_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}e^{-\omega_1(\eta_1-s)}f_1(s,y_1(s),y_2(s))ds\\&+\frac{\beta_1}{\Delta_1\Gamma(\zeta_1)}\int_0^{b}(t-s)^{\zeta_1-1}e^{-\omega_1(b-s)}f_1(s,y_1(s),y_2(s))ds\\&+\frac{1}{\Gamma(\zeta_1)}\int_0^{t}(t-s)^{\zeta_1-1}e^{-\omega_1(t-s)}f_1(s,y_1(s),y_2(s))ds,\end{aligned}$$

and

$$\begin{aligned}y_2(t)&=\frac{\delta_2}{\Delta_2}+\frac{\gamma_2}{\Delta_2\Gamma(\zeta_2)}\int_0^{\eta_2}(\eta_2-s)^{\zeta_2-1}e^{-\omega_2(\eta_2-s)}f_2(s,y_1(s),y_2(s))ds\\&+\frac{\beta_2}{\Delta_2\Gamma(\zeta_2)}\int_0^{b}(t-s)^{\zeta_2-1}e^{-\omega_2(b-s)}f_2(s,y_1(s),y_2(s))ds\\&+\frac{1}{\Gamma(\zeta_2)}\int_0^{t}(t-s)^{\zeta_2-1}e^{-\omega_2(t-s)}f_2(s,y_1(s),y_2(s))ds.\end{aligned}$$

So,

$$|y_1(t)|\leqslant\frac{\delta_1}{\Gamma(\zeta_1)}+\frac{1}{\Gamma(\zeta_1+1)}\left(b^{\zeta_1}\left(1+\frac{\beta_1}{|\Delta_1|}\right)+\frac{\gamma_1\eta_1^{\zeta_1}}{|\Delta_1|}\right)(a_1^*\|y_1\|_C+a_2^*\|y_2\|_C+b_1^*),$$

and

$$|y_2(t)|\leqslant\frac{\delta_2}{\Gamma(\zeta_2)}+\frac{1}{\Gamma(\zeta_2+1)}\left(b^{\zeta_2}\left(1+\frac{\beta_2}{|\Delta_2|}\right)+\frac{\gamma_2\eta_1^{\rho_2 r_2}}{|\Delta_2|}\right)(a_3^*\|y_1\|_C+a_4^*\|y_2\|_C+b_2^*).$$

This implies

$$\|y_1\|_C\leqslant\frac{\delta_1}{\Gamma(\zeta_1)}+\Upsilon_1b_1^*+a_1\Upsilon_1\|y_1\|_C+a_2\Upsilon_1\|y_1\|_C,$$

and

$$\|y_2\|_C\leqslant\frac{\delta_2}{\Gamma(\zeta_2)}+b_2^*\Upsilon_2+a_3\Upsilon_2\|y_1\|_C+a_4\Upsilon_2\|y_1\|_C.$$

Thus, we have

$$\begin{pmatrix}1-a_1^*\Upsilon_1 & a_2^*\Upsilon_1\\ a_3^*\Upsilon_2 & 1-a_4^*\Upsilon_2\end{pmatrix}\begin{pmatrix}\|y_1\|_C\\ \|y_2\|_C\end{pmatrix}\leqslant\begin{pmatrix}\dfrac{\delta_1}{\Gamma(\zeta_1)}+\Upsilon_1b_1^*\\ \dfrac{\delta_2}{\Gamma(\zeta_2)}+\Upsilon_2b_2^*\end{pmatrix}.$$

Therefore,

$$(I-\chi)\begin{pmatrix}\|y_1\|_C\\ \|y_2\|_C\end{pmatrix} \leqslant \begin{pmatrix}\dfrac{\delta_1}{\Gamma(\zeta_1)}+\Upsilon_1 b_1^*\\ \dfrac{\delta_2}{\Gamma(\zeta_2)}+\Upsilon_2 b_2^*\end{pmatrix}. \tag{10}$$

Since χ is matrix converge then $I-\chi$ is invertible $(I-\chi)^{-1}$ has a positive elements, we have

$$\begin{pmatrix}\|y_1\|_C\\ \|y_2\|_C\end{pmatrix} \leqslant (I-\chi)^{-1}\begin{pmatrix}\dfrac{\delta_1}{\Gamma(\zeta_1)}+\Upsilon_1 b_1^*\\ \dfrac{\delta_2}{\Gamma(\zeta_2)}+\Upsilon_2 b_2^*\end{pmatrix}.$$

Thus, by the Theorem 3, there exists at least one fixed point of Φ which in turn is a solution of problem (1). $\square$

4. Uniqueness result

In this section, we study the uniqueness result for the problem (1). The result is based on Perov's fixed point theorem.

THEOREM 5. *If the assumptions (H_1)–(H_2) and (7) are satisfied, then system (1) has a unique solution (y_1^*, y_2^*).*

Proof. For any $(y_1,y_2),(\bar{y_1},\bar{y_2}) \in C(J,\mathbb{R})\times C(J,\mathbb{R})$, we obtain:

$$\begin{aligned}
&|\Phi_1(y_1,y_2)(t)-\Phi_1(\bar{y_1},\bar{y_2})(t)|\\
&\quad\leqslant \frac{1}{\Gamma(\zeta_1)}\int_0^t (t-s)^{\zeta_1-1}e^{-w_1(t-s)}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y_1}(s),\bar{y_2}(s))|ds\\
&\qquad+\frac{\beta_1}{|\Delta_1|\Gamma(\zeta_1)}\int_0^b (b-s)^{\zeta_1-1}e^{-w_1(b-s)}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y_1}(s),\bar{y_2}(s))|ds\\
&\qquad+\frac{\gamma_1}{|\Delta_1|\Gamma(\zeta_1)}\int_0^{\eta_1} (\eta_1-s)^{\zeta_1-1}e^{-w_1(\eta_1-s)}|f_1(s,y_1(s),y_2(s))-f_1(s,\bar{y_2}(s),\bar{y_2}(s))|ds\\
&\quad\leqslant (a_1^*\|y_1-\bar{y_1}\|_C+a_2^*\|y_2-\bar{y_2}\|_C)\frac{1}{\Gamma(\zeta_1)}\int_0^t (t-s)^{\zeta_1-1}ds\\
&\qquad+(a_1^*\|y_1-\bar{y_1}\|_C+a_2^*\|y_2-\bar{y_2}\|_C)\\
&\qquad\times\frac{1}{\Gamma(\zeta_1)}\left(\beta_1\int_0^b (b-s)^{\zeta_1-1}ds+\gamma_1\int_0^{\eta_1}(\eta_1-s)^{\zeta_1-1}ds\right)\\
&\quad\leqslant \frac{a_1^*}{\Gamma(\zeta_1+1)}\left(b^{\zeta_1}\left(1+\frac{\beta_1}{|\Delta_1|}\right)+\frac{\gamma_1\eta_1^{\zeta_1}}{|\Delta_1|}\right)\|y_1-\bar{y_1}\|_C\\
&\qquad+\frac{a_2^*}{\Gamma(\zeta_1+1)}\left(b^{\zeta_1}\left(1+\frac{\beta_1}{|\Delta_1|}\right)+\frac{\gamma_1\eta_1^{\zeta_1}}{|\Delta_1|}\right)\|y_2-\bar{y_2}\|_C\\
&\quad\leqslant \Upsilon_1(a_1^*\|y_1-\bar{y_1}\|_C+a_2^*\|y_2-\bar{y_2}\|_C),
\end{aligned}$$

where

$$\Upsilon_1 = \frac{1}{\Gamma(\zeta_1+1)}\left(b^{\zeta_1}\left(1+\frac{\beta_1}{|\Delta_1|}\right)+\frac{\gamma_1\eta_1^{\zeta_1}}{|\Delta_1|}\right).$$

Thus,

$$\|\Phi_1(y_1,y_2)-\Phi_1(\bar{y_1},\bar{y_2})\|_C \leqslant \Upsilon_1\left(\Lambda_1\|y_1-\bar{y_1}\|_C+\Lambda_2\|y_2-\bar{y}_2\|_C\right). \tag{11}$$

Similarly, one can find that

$$\|\Phi_2(y_1,y_2)-\Phi_2(\bar{y_1},\bar{y_2})\|_C \leqslant \Upsilon_2\left(\Lambda_3\|y_1-\bar{y_1}\|_C+\Lambda_4\|y_2-\bar{y_2}\|_C\right), \tag{12}$$

where

$$\Upsilon_2 = \frac{1}{\Gamma(\zeta^{\zeta_2+1})}\left(b^{\zeta_2}\left(1+\frac{\beta_2}{|\Delta_2|}\right)+\frac{\gamma_2\eta_2^{\chi_2^{\zeta}}}{|\Delta_2|}\right).$$

Thus it follows from (11) and (12), that

$$\begin{pmatrix}\|\Phi_1(y_1,y_2)-\Phi_1(\bar{y_1},\bar{y_2})\|_C\\ \|\Phi_2(y_1,y_2)-\Phi_2(\bar{y_1},\bar{y_2})\|_C\end{pmatrix} \leqslant \chi\begin{pmatrix}\|y_1-\bar{y_1}\|_C\\ \|y_2-\bar{y_2}\|_C\end{pmatrix},$$

where

$$\chi = \begin{pmatrix}\Lambda_1\Upsilon_1 & \Lambda_2\Upsilon_1\\ \Lambda_3\Upsilon_2 & \Lambda_4\Upsilon_2\end{pmatrix}.$$

By (7), Φ is a contraction operator. As a consequence of Perov's fixed point theorem (Theorem 2), we deduce that Φ has a unique fixed point (y_1^*,y_2^*) which is the unique solution of the problem (1) on $[0,b]$. □

5. Ulam stability result

Now, we consider the stability result of our system.

DEFINITION 7. ([42]) Let (X,d) be a metric space and let $\Phi_1,\ \Phi_2 : X\times X \longrightarrow X$ be two operators. Then the operational equations system

$$\begin{aligned} y_1 &= \Phi_1(y_1,y_2),\\ y_2 &= \Phi_2(y_1,y_2), \end{aligned} \tag{13}$$

is said to be Ulam-Hyers stable if there exist $c_1,c_2,c_3,c_4>0$ such that for each $\varepsilon_1,\varepsilon_2>0$ and each pair $(y_1^*,y_2^*)\in X\times X$

$$\begin{aligned} \|y_1^*-\Phi_1(y_1,y_2)\|_C &\leqslant \varepsilon_1,\\ \|y_2^*-\Phi_2(y_1,y_2)\|_C &\leqslant \varepsilon_2, \end{aligned}$$

and there exists a solution $(y_1^*,y_2^*)\in X\times X$ of (13) such that

$$\begin{aligned} \|y_1^*-y_1\|_C &\leqslant c_1\varepsilon_1+c_2\varepsilon_2,\\ \|y_2^*-y_2\|_C &\leqslant c_3\varepsilon_1+c_2\varepsilon_2. \end{aligned}$$

THEOREM 6. *If the requirements of Theorem 5 are satisfied, then system (1) is Ulam-Hyers stable.*

Proof. Let $(y_1^*, y_2^*) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is a solution of following system:

$$\begin{cases} {}_0^C\mathfrak{D}_t^{\zeta_1,\omega_1} y_1^*(t) = f_1(t, y_1^*(t), y_2^*(t)), & t \in J := [0,b], \\ {}_0^C\mathfrak{D}_t^{\zeta_2,\omega_2} y_2^*(t) = f_2(t, y_1^*(t), y_2^*(t)), & t \in J := [0,b], \\ \alpha_1 y_1^*(0) + \beta_1 y_1^*(b) = \gamma_1 y_1^*(\eta_1) + \delta_1, \\ \alpha_2 y_2^*(0) + \beta_2 y_2^*(b) = \gamma_2 y_2^*(\eta_2) + \delta_2, \end{cases} \tag{14}$$

with $\sigma, \mu \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ two functions depend upon y_1, y_2 two functions such that

$$|\sigma(t)| \leqslant \varepsilon_1 \text{ and } |\mu(t)| \leqslant \varepsilon_2 \text{ for } t \in J.$$

Then, in view of Lemma 3, the solution of (14) is given by

$$\begin{cases} y_1^*(t) = \Phi_1(y_1^*, y_2^*)(t) + \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1 - 1} e^{-\omega_1(t-s)} \sigma(s) ds \\ + \frac{1}{\Delta_1 \Gamma(\zeta_1)} \left(\gamma_1 \int_0^{\eta_1} (\eta_1 - s)^{\zeta_1 - 1} e^{-\omega_1(\eta_1 - s)} \sigma(s) ds + \beta_1 \int_0^b (t-s)^{\zeta_1 - 1} e^{-\omega_1(b-s)} \sigma(s) ds \right), \\ y_2^*(t) = \Phi_2(y_1^*, y_2^*)(t) + \frac{1}{\Gamma(\zeta_2)} \int_0^t (t-s)^{\zeta_2 - 1} e^{-\omega_2(t-s)} \sigma(s) ds \\ + \frac{1}{\Delta_2 \Gamma(\zeta_2)} \left(\gamma_2 \int_0^{\eta_2} (\eta_2 - s)^{\zeta_2 - 1} e^{-\omega_2(\eta_2 - s)} \sigma(s) ds + \beta_2 \int_0^b (t-s)^{\zeta_2 - 1} e^{-\omega_2(b-s)} \mu(s) ds \right). \end{cases} \tag{15}$$

From the first equation of the system (15), we have

$$\begin{aligned} |y_1^*(t) - \Phi_1(y_1^*, y_2^*)(t)| \leqslant & \frac{1}{\Gamma(\zeta_1)} \int_0^t (t-s)^{\zeta_1 - 1} e^{-\omega_1(t-s)} |\sigma(s)| ds \\ & + \frac{1}{\Delta_1 \Gamma(\zeta_1)} \left(\gamma_1 \int_0^{\eta_1} (\eta_1 - s)^{\zeta_1 - 1} e^{-\omega_1(\eta_1 - s)} |\sigma(s)| ds \right. \\ & \left. + \beta_1 \int_0^b (t-s)^{\zeta_1 - 1} e^{-\omega_1(b-s)} |\sigma(s)| ds \right) \\ \leqslant & \frac{1}{\Gamma(\zeta_1 + 1)} \left(b^{\zeta_1} \left(1 + \frac{\beta_1}{|\Delta_1|} \right) + \frac{\gamma_1 \eta_1^{\zeta_1}}{|\Delta_1|} \right). \end{aligned}$$

From which, we have

$$\|y_1^* - \Phi_1(y_1^*, y_2^*)\|_C \leqslant \Upsilon_1 \varepsilon_1, \tag{16}$$

where Υ_1 defined in first part of proof.

Similarly, by the second equation of the system (15), we have

$$\|y_2^* - \Phi_1(y_1^*, y_2^*)\|_C \leqslant \Upsilon_2 \varepsilon_2, \tag{17}$$

where Υ_2 defined in first part of proof.

Using (16), (17), (11) and (12), we obtain

$$
\begin{aligned}
\begin{pmatrix} \|y_1^* - y_1\|_C \\ \|y_2^* - y_2\|_C \end{pmatrix} &= \begin{pmatrix} \|y_1^* - \Phi_1(y_1, y_2)\|_C \\ \|y_2^* - \Phi_2(y_1, y_2)\|_C \end{pmatrix} \\
&\leqslant \begin{pmatrix} \|y_1^* - \Phi_1(y_1^*, y_2^*)\|_C \\ \|y_2^* - \Phi_2(y_1^*, y_2^*)\|_C \end{pmatrix} + \begin{pmatrix} \|\Phi_1(y_1^*, y_2^*) - \Phi_1(y_1, y_2)\|_C \\ \|\Phi_2(y_1^*, y_2^*) - \Phi_2(y_1, y_2\|_C \end{pmatrix} \\
&\leqslant \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} + \chi \begin{pmatrix} \|y_1^* - y_1\|_C \\ \|y_2^* - y_2\|_C \end{pmatrix}.
\end{aligned}
$$

Since χ is matrix converge then $I - \chi$ is invertible $(I - \chi)^{-1}$ has a positive elements, we immediately obtain

$$
\begin{pmatrix} \|y_1^* - y_1\|_C \\ \|y_2^* - y_2\|_C \end{pmatrix} \leqslant (I - \chi)^{-1} \varepsilon,
$$

where $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$. If we denote $(I - \chi)^{-1} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, then we obtain

$$
\|y_1^* - y_1\|_C \leqslant c_1 \varepsilon_1 + c_2 \varepsilon_2,
$$

$$
\|y_2^* - y_2\|_C \leqslant c_3 \varepsilon_1 + c_2 \varepsilon_2.
$$

Then, the system (1) is Ulam-Hyers stable. □

6. Some examples

In this section we give two examples illustrate usefulness of our main results.

EXAMPLE 1. We consider the following system of Caputo tempered fractional differential boundary values problem:

$$
\begin{cases}
{}_0^C\mathfrak{D}_t^{\frac{1}{2},2} y_1(t) = \dfrac{1}{e^{(t+2)}} \left(t^2 + t y_1(t) + y_2(t)\right), & t \in J := [0,1], \\
{}_0^C\mathfrak{D}_t^{\frac{1}{4},3} y_2(t) = \dfrac{e^{-2t+t}\,(y_1(t) + y_2(t))}{(e^t + e^{-t})\,(1 + y_1(t) + y_2(t)))}, & t \in J := [0,1], \\
y_1(0) + y_1(1) = y_1\left(\dfrac{1}{2}\right) + \dfrac{1}{3}, \\
y_2(0) + y_2(1) = y_2\left(\dfrac{2}{3}\right) + \dfrac{1}{4},
\end{cases}
\tag{18}
$$

where $\zeta_1 = \frac{1}{2}$, $\zeta_2 = \frac{1}{4}$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$ and $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{2}{3}$, $\delta_2 = \frac{2}{3}$, $\delta_2 = \frac{1}{4}$. Let

$$
f_1(t, y_1, y_2) = \frac{1}{e^{(t+2)}} \left(t^2 + t y_1(t) + y_2(t)\right),
$$

and

$$f_2(t,y_1,y_2) = \frac{e^{-2t+t}\left(y_1(t)+y_2(t)\right)}{\left(e^t+e^{-t}\right)\left(1+y_1(t)+y_2(t)\right)}.$$

For any (y_1,y_2), $(\bar{y}_1,\bar{y}_2) \in \mathbb{R}\times\mathbb{R}$ and $t\in[0,1]$.

$$|f_1(t,y_1,y_2) - f_1(t,\bar{y}_1,\bar{y}_2)| \leqslant e^{-2}\left(|y_1-\bar{y}_1| + |y_2-\bar{y}_2|\right),$$

and

$$|f_2(t,y_1,y_2) - f_2(t,\bar{y}_1,\bar{y}_2)| \leqslant \left(|y_1-\bar{y}_1| + |y_2-\bar{y}_2|\right).$$

Hence the condition (H_1) is verified with $a_1^* = a_2^* = e^{-2}$ and $a_2^* = a_3^* = 1$.

For this example $\varpi_1 = 1.9263$ and $\varpi_2 = 0.6465$. Then we have

$$M = \begin{pmatrix} 0.2307 & 0.2307 \\ 0.6443 & 0.6443 \end{pmatrix}.$$

Since $c \geqslant 1$ and by the shown examples in the preliminary section, we deduce that the matrix M converge to zero.

Since all conditions of Theorem 4 are satisfied, our problem (18) has at least a solution.

EXAMPLE 2. Consider the following system of Caputo tempered fractional differential Boundary values problem:

$$\begin{cases} {}^{C}_{0}\mathfrak{D}_t^{\frac{2}{5},2} y_1(t) = \dfrac{|y_1(t)|}{|y_1(t)|+1} \times \dfrac{\sin^4(t)}{(10^t+9)^2} + \sin y_2(t) + \dfrac{t^2}{5} + \dfrac{\sqrt{7}}{3}, & t\in J := [0,\pi], \\ {}^{C}_{0}\mathfrak{D}_t^{\frac{2}{3},1} y_2(t) = \dfrac{t\sin y_1(t)}{15(5+3t^2)} + \dfrac{t|y_2(t)|}{100(1+|y_2(t)|)}, & t\in J := [0,\pi], \\ y_1(0) + \dfrac{\pi}{2} y_1(\pi) = \pi y_1(2) + \dfrac{\sqrt{2}}{2}, \\ y_2(0) + y_2(\pi) = \dfrac{\pi}{2} y_2\left(\dfrac{5}{2}\right) + \dfrac{\pi}{3}, \end{cases} \tag{19}$$

where $\alpha_1 = \beta_2 = 1$, $\beta_1 = \gamma_2 = \dfrac{\pi}{2}$, $\gamma_1 = \alpha_1 = \pi$, $f : J\times\mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f_1(t,y_1,y_2) = \frac{|y_1(t)|}{|y_1(t)|+1} \times \frac{\sin^4(t)}{(10^t+9)^2} + \sin y_2(t) + \frac{t^2}{5} + \frac{\sqrt{7}}{3},$$

and

$$f_1(t,y_1,y_2) = \frac{t\sin y_1(t)}{15(5+3t^2)} + \frac{t|y_2(t)|}{100(1+|y_2(t)|)}.$$

It easy to find that

$$|f_1(t,y_1,y_2) - f_1(t,\bar{y}_1,\bar{y}_2)| \leqslant \frac{1}{10^2}\left(|y_1-\bar{y}_1| + \frac{1}{e^{\pi}}|y_2-\bar{y}_2|\right),$$

and

$$|f_2(t,y_1,y_2) - f_2(t,\bar{y}_1,\bar{y}_2)| \leqslant \frac{\pi}{15(5+3\pi^2)}\left(|y_1-\bar{y}_1| + \frac{\pi}{100}|y_2-\bar{y}_2|\right),$$

for all $t \in J$ and $(y_1,y_2),(\bar{y}_1,\bar{y}_2)$. Hence the condition (H_1) is verified with $a_1^* = \frac{1}{10^2}$, $a_2^* = e^{-\pi}$, $a_3^* = \frac{\pi}{15(5+3\pi^2)}$ and $a_4^* = \frac{\pi}{100}$.

For this example $\Upsilon_1 = 11.0964$ and $\Upsilon_2 = 5.2041$, and the matrix

$$\chi = \begin{pmatrix} 0.1109 & 0.4794 \\ 0.0732 & 0.1583 \end{pmatrix},$$

which has two the eigenvalues $|\lambda_1| = 0.0549$ and $\lambda_2 = 0.3249$.

Simple computations show that all conditions of Theorem 5 and Theorem 6 are satisfied, then system (19) has a unique solution (y_1^*, y_2^*) and is Hyers-Ulam-stable.

7. Conclusion

In this paper, we have made a substantial contribution to the study of certain classes of fractional differential systems involving the Caputo tempered fractional derivative. We have investigate qualitative and quantitative results such as the existence, uniqueness, and Ulam stability of solutions to Caputo-tempered fractional systems subject to three-point boundary conditions. The methodologies utilized are primarily grounded in an extended version of Perov's fixed point theorem and a Krasnoselskii-type approach combined with some techniques involving vector-valued metrics and matrix sequences that converge to zero. We have chosen to include illustrative examples demonstrating that the requirements of the theorems are indeed met. We intentionally reserve the application aspect for future papers, as it does not align with the primary focus of this study. We hope for this study to serve as a cornerstone such future endeavors. In future research, we aim to explore additional classes of fractional differential systems and inclusions, including problems with retarded (delayed) and advanced arguments, as well as impulsive problems, focusing on both instantaneous and non-instantaneous impulses.

Declarations

Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

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