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ANALYSIS OF A CLASS OF Δ–FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS INVOLVING THE ϕ–CAPUTO DERIVATIVE ON TIME SCALES

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Abstract. This paper explores the existence of solutions for fractional hybrid differential equations that involve the ϕ-Caputo derivative, defined on time scales. The ϕ-Caputo derivative extends the classical Caputo derivative by adapting it to time scales, making it possible to model systems that exhibit both continuous and discrete behavior. This unique characteristic allows the ϕ-Caputo derivative to capture dynamics that occur across varying time intervals, providing a more versatile framework for mathematical modeling.

To demonstrate the existence of solutions, the study leverages Dhage's fixed point theorem is a powerful and widely recognized tool for establishing the existence of fixed points in Banach algebras. By applying this theorem to fractional hybrid differential equations on time scales, the research introduces an innovative approach that bridges theoretical concepts with practical application.

To validate the correctness and real-world relevance of the theoretical findings, the study includes a practical example involving uncertainty modeling in physical systems. This example illustrates how the proposed method can be applied in scenarios where both deterministic and uncertain behaviors are present.

The outcomes of this research offer promising potential for various fields, including biology, engineering, and control theory, where dynamic systems often require flexible mathematical frameworks to address complex behavior.

1. Introduction

In recent years, fractional differential equations have gained significant attention as powerful tools for modeling complex systems across various fields, including complex systems in control theory [3], diffusion problems in physics [5], economic applications such as interest rates, commodity prices, and market fluctuations [4], as well as precise dimensional analysis in image processing [2]. These equations are particularly effective in capturing phenomena that involve memory effects and uncertainty, making them highly valuable for real-world applications where such factors play a crucial role.

One notable advancement in this area is time scales theory, which unifies continuous and discrete dynamic systems under a single framework. This unified approach

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expands the potential for more versatile and comprehensive mathematical models, especially for systems that exhibit both continuous and discrete behavior.

A key concept in this framework is the ϕ-Caputo derivative, an extension of the classical Caputo derivative designed specifically for time scales. By combining the flexibility of the Caputo derivative with the adaptability of time scales theory, the ϕ-Caputo derivative becomes a powerful tool for analyzing hybrid systems that operate across diffrent time domains [7, 14].

While previous studies have explored the ϕ-Caputo derivative within the context of classical fractional differential equations, its application to fractional hybrid differential equations remains relatively limited. This study seeks to address this gap by employing Dhage's fixed point theorem to demonstrate the existence of solutions for such equations. Dhage's theorem is particularly well-suited for this purpose, as it efficiently handles conditions involving Lipschitz continuity and compactness – key properties that are critical when analyzing complex dynamic systems [8–10, 15].

The primary contribution of this study lies in the introduction of a novel solution method that applies the ϕ-Caputo derivative to hybrid differential equations on time scales. To validate the theoretical results, the study includes a practical example focused on uncertainty modeling in physical systems, demonstrating the method's applicability in real-world scenarios.

The findings of this research hold promising potential for a wide range of applications, particularly in dynamic systems found in biology, engineering, and control theory. By extending the capabilities of the ϕ-Caputo derivative to hybrid systems, this work paves the way for improved modeling techniques in fields where both deterministic and uncertain behaviors must be accounted for.

In [13], Otrocol particularly explored the following problem:

$$\begin{cases} v'(\lambda) = \mathscr{F}(\lambda, v(\lambda)) + \mathscr{G}\left(\lambda, \max\limits_{\ell \in [0,\lambda]} v(\ell)\right), \\ v(0) = \varphi, \end{cases}$$

where $\lambda \in [0,d]$, $d \in \mathbb{R}$, $\varphi \in \mathbb{R}^p$, and $\mathscr{F}, \mathscr{G} \in [0,d] \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$.

In [12], the authors investigated the existence and uniqueness of solutions to initial value problems for Caputo Δ-fractional differential equations with maxima on the time scales $\mathbb{T}_{\mathbf{S}}$ of the form:

$$\begin{cases} {}^{C}\Delta_c^{\gamma} v(\lambda) = \xi\left(\lambda, v(\lambda), V(\lambda)\right), \quad \lambda \in \Sigma = [c,d]_{\mathbb{T}_{\mathbf{S}}} := [c,d] \cap \mathbb{T}_{\mathbf{S}}, \quad 0 < \gamma < 1, \\ v(c) = \varphi, \end{cases}$$

where $v : [c,d]_{\mathbb{T}_{\mathbf{S}}} \longrightarrow \mathbb{R}$, $V(\lambda) = \max\limits_{\ell \in [c,\lambda]} v(\ell)$, $d > c$, ${}^{C}\Delta_c^{\gamma}$ is the Caputo Δ-fractional derivative operator of order γ, $\xi : [c,d]_{\mathbb{T}_{\mathbf{S}}} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function, and φ represents a real number.

In this article, we investigate the existence of a solution for ϕ-Caputo's Δ-fractional nonlinear hybrid differential equations with maxima on the time scales:

$$\begin{cases} {}^{C}\Delta_{c}^{\phi,\gamma}\left(\frac{v(\lambda)}{\mathscr{F}(\lambda,v(\lambda))}\right)=\xi\left(\lambda,v(\lambda),V(\lambda)\right), \\ \qquad\qquad\qquad \lambda\in\Sigma=[c,d]_{\mathbb{T}_{\mathbf{S}}}:=\mathbb{T}_{\mathbf{S}}\cap[c,d],\;\; 0<\gamma<1, \\ \frac{v(c)}{\mathscr{F}(c,v(c))}=\varphi, \end{cases} \tag{1}$$

where ${}^{C}\Delta_{c}^{\phi,\gamma}$ is ϕ-Caputo Δ-fractional derivative operator of order γ, $c<d$, $\varphi\in\mathbb{R}$, $v:[c,d]_{\mathbb{T}_{\mathbf{S}}}\longrightarrow\mathbb{R}$, $V(\lambda)=\max_{\ell\in[c,\lambda]} v(\ell)$, $\xi:[c,d]_{\mathbb{T}_{\mathbf{S}}}\times\mathbb{R}^{2}\longrightarrow\mathbb{R}$ is a function.

The structure of this paper unfolds as: in the section 2, we present some preliminary concepts related to fractional calculus. In section 3, we establish criteria for the existence of solutions to the problem above. In section 4, provides an example to illustrate the practical applications of these results.

2. Preliminaries

This section introduces the key mathematical concepts and methods that form the foundation of this study. The provided explanations aim to offer both technical clarity and intuitive understanding.

Time scales theory

Time scales theory bridges continuous and discrete time models, offering a unified framework for analyzing dynamic systems. By blending these two perspectives, time scales theory enables researchers to describe systems that evolve continuously, in discrete steps, or in a combination of both. For example:

- $\mathbb{T}_{\mathbf{S}}=\mathbb{R}\longrightarrow$ Represents continuous systems, such as temperature changes over time.
- $\mathbb{T}_{\mathbf{S}}=\mathbb{Z}\longrightarrow$ Represents discrete systems, like population counts recorded at regular intervals.
- $\mathbb{T}_{\mathbf{S}}=q^{k}$ (where $q>1$ and $k\in\mathbb{N}$) $\longrightarrow$ Describes specialized time scales used in areas like quantum mechanics or dynamic systems with exponential growth patterns.

EXAMPLE 1. Let

(1) $\mathbb{T}_{\mathbf{S}2}=\{\sqrt{r}:\ r\in\mathbb{N}_{0}\}$,

(2) $\mathbb{T}_{\mathbf{S}1}=\{2^{r}:\ r\in\mathbb{Z}\}\cup\{0\}$,

$\mathbb{T}_{\mathbf{S}1}$ and $\mathbb{T}_{\mathbf{S}2}$ are both time scales.

DEFINITION 1. Let $\mathbb{T}_{\mathbf{S}}$ be a time scale. For each $\varsigma \in \mathbb{T}_{\mathbf{S}}$, we define two operators $\alpha : \mathbb{T}_{\mathbf{S}} \to \mathbb{T}_{\mathbf{S}}$ and $\theta : \mathbb{T}_{\mathbf{S}} \to \mathbb{T}_{\mathbf{S}}$, by the following formulas:

$$\theta(\lambda) = \sup\{\varsigma \in \mathbb{T}_{\mathbf{S}} : \varsigma < \lambda\},$$

and

$$\alpha(\lambda) = \inf\{\varsigma \in \mathbb{T}_{\mathbf{S}} : \varsigma > \lambda\}.$$

The operators θ and α, are called backward jump and forward jump, respectively.

In the previous definition, we specify that

1. $\sup\emptyset = \inf\mathbb{T}_{\mathbf{S}}$ (which means $\theta(\lambda) = \lambda$ if the set $\mathbb{T}_{\mathbf{S}}$ contains a minimum element λ),
2. $\inf\emptyset = \sup\mathbb{T}_{\mathbf{S}}$ (which means $\alpha(\lambda) = \lambda$ if the set $\mathbb{T}_{\mathbf{S}}$ contains a maximum element λ),

where $\emptyset$ represents the empty set.

EXAMPLE 2. Let's briefly examine some examples: $\mathbb{T}_{\mathbf{S}} = \mathbb{R}$, and $\mathbb{T}_{\mathbf{S}} = \mathbb{Z}$.

(1) If $\mathbb{T}_{\mathbf{S}} = \mathbb{Z}$, for any number ς in the set $\mathbb{Z}$, we have

$$\begin{cases} \alpha(\lambda) = \inf\{\varsigma \in \mathbb{Z} : \varsigma > \lambda\} = \inf\{\lambda+1, \lambda+2, \lambda+3, \lambda+4, \lambda+5, \cdots\} = \lambda+1, \\ \theta(\lambda) = \lambda - 1. \end{cases}$$

(2) If $\mathbb{T}_{\mathbf{S}} = \mathbb{R}$, then for any number ς in the set $\mathbb{R}$, we have

$$\begin{cases} \alpha(\lambda) = \inf\{\varsigma \in \mathbb{R} : \varsigma > \lambda\} = \inf(\lambda, \infty) = \lambda, \\ \theta(\lambda) = \lambda. \end{cases}$$

DEFINITION 2. Here are some other definitions that we need in this paper:

1. *Left-Scattered*: A point ς is said to be left-scattered if $\theta(\varsigma) < \varsigma$.
2. *Right-Scattered*: A point ς is said to be right-scattered if $\alpha(\varsigma) > \varsigma$.
3. *Isolated*: A point ς is called isolated if it is both left-scattered and right-scattered simultaneously.
4. *Left-Dense*: A point ς is said to be left-dense if $\theta(\varsigma) = \varsigma$ and $\varsigma > \inf\mathbb{T}_{\mathbf{S}}$.
5. *Right-Dense*: A point ς is said to be right-dense if $\alpha(\varsigma) = \varsigma$ and $\varsigma < \sup\mathbb{T}_{\mathbf{S}}$.
6. *Dense*: A point ς is called dense if it is both left-dense and right-dense simultaneously.

DEFINITION 3. Let $g : \mathbb{T}_{\mathbf{S}} \longrightarrow \mathbb{R}$ be a function. The function g is termed rd-continuous if it satisfies two conditions:

1. It is continuous at all dense points in $\mathbb{T}_{\mathbf{S}}$ when approaching from the right.

2. It has finite left-side limits at all left-dense points in $\mathbb{T}_{\mathbf{S}}$.

Let $\mathscr{C}_{rd}$ denote the set of all functions $g : \mathbb{T}_{\mathbf{S}} \longrightarrow \mathbb{R}$ that are rd-continuous.

ϕ-Caputo derivative

The ϕ-Caputo derivative generalizes the classical Caputo derivative to time scales, providing a flexible tool for modeling systems that exhibit memory effects or uncertainty. By incorporating a ϕ function, this derivative adapts to dynamic systems that combine continuous evolution with discrete changes a feature particularly useful in hybrid models. This adaptability makes it ideal for analyzing complex processes such as biological rhythms, financial cycles, or engineering systems with periodic updates.

DEFINITION 4. [6] (Delta derivative) Let $\Psi : \mathbb{T}_{\mathbf{S}} \longrightarrow \mathbb{R}$ be a function and λ be an element of $\mathbb{T}_{\mathbf{S}}$. The Δ-derivative of the function Ψ at the point λ, denoted $\Psi^{\Delta}(\lambda)$ (if it exists), is given such that for all $\kappa > 0$, there exists a neighborhood Υ of $\lambda \in \mathbb{T}_{\mathbf{S}}$ satisfying:

$$\Big|\Psi(\eta(\lambda)) - \Psi(\ell) - \Psi^{\Delta}(\lambda)[\eta(\lambda) - \ell]\Big| \leqslant \kappa|\eta(\lambda) - \ell|, \quad \text{for each } \ell \in \Upsilon.$$

DEFINITION 5. [6] Given a function Ψ defined from $\mathscr{I}$ to $\mathbb{R}$, where $\mathscr{I}$ is a bounded closed interval of $\mathbb{T}_{\mathbf{S}}$. We say that Ψ is a Δ-antiderivative of the function $\psi : [d,r) \longrightarrow \mathbb{R}$ if the following conditions are satisfied:

1. Ψ is continuous on $[d,r]$,

2. $\Psi^{\Delta}(\lambda) = g(\lambda)$ for all $\lambda \in [d,r)$,

3. Ψ is delta differentiable on $[d,r)$.

The Δ-integral of ψ from d to r is defined as follows

$$\int_d^r \psi(\lambda)\Delta\lambda = \Psi(r) - \Psi(d).$$

LEMMA 1. [6] *Given that* $\mathbb{T}_{\mathbf{S}}$ *denotes a time scale and* ψ *is an increasing and continuous function on* $[d,r]$ *within this time scale. Define* ϖ *as the extension of the function* ψ *to the real interval* $[d,r]$ *using the following expression*

$$\varpi(\lambda) = \begin{cases} \psi(\lambda) & \text{if } \lambda \in \mathbb{T}_{\mathbf{S}}, \\ \psi(\ell) & \text{if } \lambda \in (\ell, \eta(\ell)) \notin \mathbb{T}_{\mathbf{S}}. \end{cases}$$

Then,

$$\int_d^r \psi(\lambda)\Delta\lambda \leqslant \int_d^r \varpi(\lambda)d\lambda.$$

DEFINITION 6. (ϕ-Caputo Δ-fractional integral operator on the time scales) Given that $\mathbb{T}_{\mathbf{S}}$ denotes a time scale such that $[c,d]$ is an interval of $\mathbb{T}_{\mathbf{S}}$ and v is an integrable function on the interval $[c,d]$. Let $\phi \in \mathscr{C}^n([c,d],\mathbb{R})$ with $\phi'(t) > 0$ for each t in $[c,d]$. Consider $\gamma > 0$. The Δ-fractional integral at order γ in the sense of ϕ-Caputo for the function v is given by the following expression

$$^{\mathbb{T}_{\mathbf{S}}}J_c^{\gamma,\phi} v(\lambda) := \frac{1}{\Gamma(\gamma)} \int_c^{\lambda} \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1} v(\ell)\Delta\ell.$$

DEFINITION 7. (ϕ-Caputo Δ-fractional derivative operator on the time scales) Given that $\mathbb{T}_{\mathbf{S}}$ denotes a time scale such that $[c,d]$ is an interval of $\mathbb{T}_{\mathbf{S}}$ and $v : \mathbb{T}_{\mathbf{S}} \longrightarrow \mathbb{R}$ is a continuous function. Let $\phi \in \mathscr{C}^n([c,d],\mathbb{R})$ with $\phi'(t) > 0$ for every t in $[c,d]$. The Δ-fractional derivative of order γ in the sens of ϕ-Caputo for the function v is defined by de following expression

$$^{C}\Delta_{c^+}^{\gamma,\phi} v(\lambda) = \frac{1}{\Gamma(n-\gamma)} \int_c^{\lambda} \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{n-\gamma-1} v_{\phi}^{\Delta^{[n]}}(\ell)\Delta\ell, \tag{2}$$

where

$$v_{\phi}^{\Delta^{[n]}}(\ell) = \left(\frac{1}{\phi'(\ell)}\frac{d}{d\ell}\right)^{\Delta^n} v(\ell) \quad \text{and} \quad n = [\gamma]+1.$$

The symbol $[\cdot]$ denotes the integer part.

THEOREM 1. (semigroup property) *Suppose that the function ξ is integrable on the interval $[c,d]$, and that β and α are two positive real constants. Then,*

$$^{\mathbb{T}_{\mathbf{S}}}J_c^{\alpha,\phi}\, {}^{\mathbb{T}_{\mathbf{S}}}J_c^{\beta,\phi} \xi(\lambda) = {}^{\mathbb{T}_{\mathbf{S}}}J_c^{\alpha+\beta,\phi} \xi(\lambda).$$

Dhage's fixed point theorem

Dhage's fixed point theorem is a powerful theoretical tool used to establish the existence of solutions for fractional hybrid differential equations. Its strength lies in its ability to efficiently combine Lipschitz continuity and compactness conditions, both of which are crucial when dealing with dynamic systems that involve uncertainty or non-linear behavior. By leveraging these properties, Dhage's theorem ensures the existence of solutions even in complex systems where traditional methods may fall short.

THEOREM 2. (Dhage theorem) [11] *we consider S as a bounded, closed, convex, and non-empty subset of Ξ, the Banach algebra. Let $\mathscr{G}_1 : \Xi \longrightarrow \Xi$ and $\mathscr{G}_2 : S \longrightarrow \Xi$ be two operators which satisfy the following properties:*

(a) $\mathscr{G}_2$ is completely continuous,

(b) $\mathscr{G}_1$ is Lipschitzian with a Lipschitz constant κ,

(c) $u = \mathscr{G}_1(u)\mathscr{G}_2(v)$ implies that u in S for every v in S, and

(d) $\eta\kappa < 1$, where $\eta = \|\mathscr{G}_2(S)\| = \sup\{\|\mathscr{G}_2(u)\| : u \in S\}$.

Then, the operator $\Psi u = \mathscr{G}_1(u)\mathscr{G}_2(u)$ has a fixed point.

3. Results

This section outlines the proof of solution existence for fractional hybrid differential equations involving the ϕ-Caputo derivative, followed by an illustrative example that demonstrates the practical application of the theoretical results. The following assumptions are required:

$(H1)$ The function $\mathscr{F} \in \mathscr{C}(\Sigma \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ where it satisfies the conditions that:

(i) $|\mathscr{F}(\lambda,v) - \mathscr{F}(\lambda,u)| \leqslant L|v-u|, \quad L > 0$ for all $u,\ v \in \mathscr{C}_{rd}(\Sigma \times \mathbb{R})$.

(ii) The mapping $v \longrightarrow \frac{v}{\mathscr{F}(\lambda,v)}$ is increasing in $\mathbb{R}$ a.e., for $\lambda \in \Sigma$.

$(H2)$ $\xi \in \mathscr{C}(\Sigma \times \mathbb{R}^2, \mathbb{R})$ is a function such that $|\xi(\lambda, v(\lambda), u(\lambda)| \leqslant h(\lambda)$ a.e., $\lambda \in \Sigma$, $h \in \mathscr{C}(\Sigma, \mathbb{R}^+)$.

DEFINITION 8. A function $v \in \mathscr{C}^1_{rd}(\Sigma, \mathbb{R})$ is a solution to the Δ-fractional hybrid problem (1), if it fulfills the initial condition $v(c) = \varphi$ and satisfies the fractional equations ${}^C\Delta_c^{\phi,\gamma} v(\lambda) = \xi\left(\lambda, v(\lambda), V(\lambda)\right)$ on Σ.

LEMMA 2. *Let $\xi : \Sigma \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a rd-continuous and $0 < \gamma < 1$. Then, the function $v \in \mathscr{C}^1_{rd}(\Sigma, \mathbb{R})$ serves as a solution to the Δ-fractional hybrid problem (1) if and only if it verifies the following integral equation:*

$$v(\lambda) = \mathscr{F}\left(\lambda, v(\lambda)\right) \left(\varphi + \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \xi\left(\ell, v(\ell), V(\ell)\right) \Delta\ell\right), \quad (3)$$

where $V(\ell) = \max_{\ell \in [c,\lambda]} v(\ell)$.

Proof. From (2), we have

$$\begin{aligned} {}^C\Delta_{c^+}^{\gamma,\phi}\left(\frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)}\right) &= \frac{1}{\Gamma(1-\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{-\gamma} \left(\frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)}\right)_\phi^\Delta (\ell)\Delta\ell \\ &= {}^{\mathbb{T}}_{\mathbf{S}}J_c^{1-\gamma,\phi}\left(\frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)}\right)_\phi^\Delta. \end{aligned}$$

Since $\gamma \in (0,1)$. Then, the proof can be concluded from the relations

$$\begin{aligned} {}^{\mathbb{T}}\mathrm{s}J_c^{\gamma,\phi}\, {}^C\Delta_c^{\phi,\gamma}\left(\frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)}\right) &= {}^{\mathbb{T}}\mathrm{s}J_c^{\gamma,\phi}\, {}^{\mathbb{T}}\mathrm{s}J_c^{1-\gamma,\phi}\left(\frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)}\right)_\phi^\Delta \\ &= \frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)} - \frac{v(c)}{\mathscr{F}\left(c, v(c)\right)} \\ &= \frac{v(\lambda)}{\mathscr{F}\left(\lambda, v(\lambda)\right)} - \varphi, \end{aligned}$$

it follows that

$$v(\lambda) = \mathscr{F}(\lambda, v(\lambda))\left(\varphi + \frac{1}{\Gamma(\gamma)}\int_c^\lambda \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1}\xi(\ell, v(\ell), V(\ell))\Delta\ell\right),$$

where $V(\ell) = \max_{\ell\in[c,\lambda]} v(\ell)$. $\square$

THEOREM 3. *Assuming that the conditions* $(H1)$ *and* $(H2)$ *are met. Then, under the following condition*

$$L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(1+\gamma)}(\phi(d)-\phi(c))^\gamma\right) < 1, \tag{4}$$

the Δ*-fractional hybrid problem* (1) *has a solution.*

Proof. Let $M_\varphi = |\varphi|$ and $\Xi = (\mathscr{C}(\Sigma,\mathbb{R}), \|.\|)$, where $\|v\| = \sup_{\lambda\in\Sigma}|v(\lambda)|$. It is evident that Ξ forms a Banach algebra, where multiplication is defined as follows

$$(uv)(\lambda) = u(\lambda)v(\lambda), \quad \lambda\in\Sigma, \quad u, v\in\Xi.$$

We define a subset S of Ξ as

$$S = \{v\in\Xi : \|v\| \leqslant r\},$$

where

$$r = \frac{M_{\mathscr{F}}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(1+\gamma)}(\phi(d)-\phi(c))^\gamma\right)}{1 - L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma\right)}.$$

It is easy to prove that S is a bounded, closed and convex subset of Ξ, the Banach algebra. Let's consider the operators $\mathscr{G}_1 : \Xi \longrightarrow \Xi$ and $\mathscr{G}_2 : S \longrightarrow \Xi$, which are defined as follows

$$\mathscr{G}_1 v(\lambda) = \mathscr{F}(\lambda, v(\lambda)), \tag{5}$$

$$\mathscr{G}_2 v(\lambda) = \varphi + \frac{1}{\Gamma(\gamma)}\int_c^\lambda \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1}\xi(\ell, v(\ell), V(\ell))\Delta\ell. \tag{6}$$

The operator equation form representing the equivalent integral equation Eq. (3) corresponding to the fractional hybrid problem (1) is expressed as

$$v = \mathscr{G}_1 v\, \mathscr{G}_2 v, \quad v\in\Xi.$$

We establish that the operators $\mathscr{G}1$ and $\mathscr{G}2$ verify the conditions stated in Theorem 2.

The proof is structured in four key steps to ensure clarity and logical progression:

1. *Verifying the Lipschitz condition*: The proof begins by confirming that the given function satisfies the Lipschitz condition.

We prove that $\mathscr{G}_1$ is Lipschitz.

Applying the Lipschitz condition to $\mathscr{F}$, with $v,u \in \Xi$ and $\lambda \in \Sigma$, we obtain

$$\begin{aligned} |\mathscr{G}_1 v(\lambda) - \mathscr{G}_1 u(\lambda)| &= |\mathscr{F}(\lambda, v(\lambda)) - \mathscr{F}(\lambda, u(\lambda))| \\ &\leqslant L|v(\lambda) - u(\lambda)|. \end{aligned}$$

This gives,

$$\|\mathscr{G}_1 v - \mathscr{G}_1 u\| \leqslant L\|v - u\|, \quad u,v \in \Xi.$$

2. *Demonstrating compactness*: Next, the Ascoli-Arzelà theorem is applied to establish compactness. This step ensures that the function is contained within a bounded and closed set, a key requirement for proving the existence of a solution.

In this step, we will show the complete continuity of the operator $\mathscr{G}_2$. We demonstrate that $\mathscr{G}_2 : S \longrightarrow \Xi$ is a continuous and compact operator on S into Ξ.

To begin, we establish the continuity of $\mathscr{G}_2$ on S. Let $v_n \in S$ converging to $v \in S$. For each $\lambda \in \Sigma$, we have

$$\begin{aligned} &|\mathscr{G}_2 v_n(\lambda) - \mathscr{G}_2 v(\lambda)| \\ &\qquad \leqslant \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} |\xi(\ell, v_n(\ell), V_n(\ell)) - \xi(\ell, v(\ell), V(\ell))| \Delta\ell, \end{aligned}$$

using Lemma 1, we get

$$\begin{aligned} &|\mathscr{G}_2 v_n(\lambda) - \mathscr{G}_2 v(\lambda)| \\ &\qquad \leqslant \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} |\xi(\ell, v_n(\ell), V_n(\ell)) - \xi(\ell, v(\ell), V(\ell))| d\ell. \end{aligned}$$

On the other hand, from $(H2)$, we deduce that

$$\begin{aligned} &\phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} |\xi(\ell, v_n(\ell), V_n(\ell)) - \xi(\ell, v(\ell), V(\ell))| \\ &\qquad \leqslant 2\phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \|h\|_\infty, \end{aligned}$$

and

$$\lim_{n\to\infty} \xi(\ell, v_n(\ell), V_n(\ell)) = \xi(\ell, v(\ell), V(\ell)).$$

Applying Lebesgue's dominated convergence theorem leads us to the following limit

$$\lim_{n\to\infty} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} |\xi(\ell, v_n(\ell), V_n(\ell)) - \xi(\ell, v(\ell), V(\ell))| d\ell = 0, \quad \text{for all} \quad \lambda \in \Sigma.$$

This prove that $\mathscr{G}_2$ is a continuous on S.

Based on assumption $(H2)$, for every $v \in S$ and $\lambda \in \Sigma$, we obtain

$$\begin{aligned}|\mathscr{G}_2 v(\lambda)| &\leqslant |\varphi| + \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1} |\xi(\ell, v(\ell), V(\ell))| \Delta\ell \\ &\leqslant M_\varphi + \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1} |h(\ell)| \Delta\ell \\ &\leqslant M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1} \Delta\ell.\end{aligned}$$

From Lemma 1, we derive that

$$\begin{aligned}|\mathscr{G}_2 v(\lambda)| &\leqslant M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma)} \int_c^b \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1} d\ell \\ &\leqslant M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)} (\phi(d)-\phi(c))^\gamma.\end{aligned}$$

This gives,

$$\|\mathscr{G}_2 v\| \leqslant M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)} (\phi(d)-\phi(c))^\gamma. \tag{7}$$

This demonstrates that the operator $\mathscr{G}_2$ is uniformly bounded on Σ.

Furthermore, we establish that the operator $\mathscr{G}_2(S)$ is an equicontinuous set in Ξ. Let $v \in S$ and λ_1, $\lambda_2 \in \Sigma$ such that $\lambda_1 < \lambda_2$. Then, we have

$$\begin{aligned}&|\mathscr{G}_2 v(\lambda_2) - \mathscr{G}_2 v(\lambda_1)| \\ &\leqslant \frac{1}{\Gamma(\gamma)} \int_c^{\lambda_1} \phi'(\ell) \left((\phi(\lambda_2)-\phi(\ell))^{\gamma-1} - (\phi(\lambda_1)-\phi(\ell))^{\gamma-1} \right) |\xi(\ell, v(\ell), V(\ell))| \Delta\ell \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_{\lambda_1}^{\lambda_2} \phi'(\ell)(\phi(\lambda_2)-\phi(\ell))^{\gamma-1} |\xi(\ell, v(\ell), V(\ell))| \Delta\ell.\end{aligned}$$

According to Lemma 1, we can obtain the following

$$\begin{aligned}&|\mathscr{G}_2 v(\lambda_2) - \mathscr{G}_2 v(\lambda_1)| \\ &\qquad \leqslant \frac{1}{\Gamma(\gamma)} \int_c^{\lambda_1} \phi'(\ell) \left((\phi(\lambda_2)-\phi(\ell))^{\gamma-1} - (\phi(\lambda_1)-\phi(\ell))^{\gamma-1} \right) |h(\ell)| d\ell \\ &\qquad\quad + \frac{1}{\Gamma(\gamma)} \int_{\lambda_1}^{\lambda_2} \phi'(\ell)(\phi(\lambda_2)-\phi(\ell))^{\gamma-1} |h(\ell)| d\ell \\ &\qquad \leqslant \frac{\|h\|_\infty}{\Gamma(\gamma+1)} \left((\phi(\lambda_2)-\phi(c))^\gamma - (\phi(\lambda_1)-\phi(c))^\gamma \right).\end{aligned}$$

This demonstrates that $\mathscr{G}_2(S)$ is equicontinious set in Ξ. Since $\mathscr{G}_2(S)$ is equicontinious and uniformly bounded set in Ξ. Then, by applying the Ascoli-Arzela theorem, we get $\mathscr{G}_2$ is completely continuous.

3. *Establishing subset invariance*: The proof then demonstrates that the operator remains within the specified subset, effectively defining the boundaries of the solution set.

Let $v \in \Xi$ and $u \in S$ such that $v = \mathscr{G}_1 v\, \mathscr{G}_2 u$. Our objective is to demonstrate that $v \in S$. Utilizing hypothesis $(H1)$ and condition (7), we obtain

$$\begin{aligned}|v(\lambda)| &= |\mathscr{G}_1 v(\lambda)\mathscr{G}_2 u(\lambda)|\\ &\leqslant \Big\{|\mathscr{F}(\lambda, v(\lambda)) - \mathscr{F}(\lambda,0)| + |\mathscr{F}(\lambda,0)|\Big\}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma\right)\\ &\leqslant \{L|v(\lambda)| + M_{\mathscr{F}}\}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma\right).\end{aligned}$$

This gives,

$$\begin{aligned}|v(\lambda)| &\leqslant \frac{M_{\mathscr{F}}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma\right)}{1 - L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma\right)}\\ &= r, \qquad \text{for all } \lambda \in \Sigma.\end{aligned}$$

Therefore,

$$\|v\| \leqslant r.$$

This establishes that $v \in S$.

4. *Linking the Lipschitz constant to the maximum function*: Now, we show that $\kappa\eta < 1$.

We have

$$\kappa = L \quad \text{and} \quad \eta = M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma.$$

By condition (4), we can get the following

$$\eta\kappa = L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^\gamma\right) < 1.$$

By proceeding through steps 1 through 4, we can conclude that all the conditions outlined in Theorem 2 are met. Thus, the following equation

$$v = \mathscr{G}_1 v \mathscr{G}_2 v$$

possesses a solution in S, which is a solution to the Δ-fractional hybrid problem (1). This concludes the proof of the theorem. □

4. Example application

To illustrate the practical relevance of the theoretical results, the study presents an example that models uncertainty in physical systems. In this example, the functions presented satisfy the Lipschitz condition, attain a maximum value within the defined interval, and meet the conditions outlined by Dhage's Fixed Point Theorem.

By demonstrating these properties, the example highlights how the proposed method can be applied in various real-world scenarios. For instance, the findings may prove useful in modeling biological processes involving irregular growth patterns, analyzing temperature fluctuations in engineering systems, or predicting volatility in financial markets. These potential applications underscore the study's contribution to advancing mathematical modeling techniques for complex dynamic systems. Let us consider the following hybrid problem:

$$\begin{cases} {}^{C}\Delta_{c}^{\phi,\gamma}\left(\frac{2|v(t)|+2|v(t)|^2}{e^{t-1}}\right) = \dfrac{\exp\{-\max_{\ell\in[0,t]}|v(\ell)|\}}{|v(t)|+1+t^2}, \quad t \in \Sigma = [0,1]_{\mathbb{T}_{\mathbf{S}}}, \\ v(0) = 0, \end{cases} \tag{8}$$

where $\mathbb{T}_{\mathbf{S}}$ represent any time scale that includes both 0 and 1. Here $c = 0$, $d = 1$, $\phi = t$, $\gamma = \frac{1}{2}$, $M_{\varphi} = 0$ and

$$\mathscr{F}(t,v) = \frac{e^{t-1}}{2(1+v)},$$

$$\xi(t,v,u) = \frac{e^{-u}}{|v|+t^2+1}.$$

Let $t \in [0,1]$ and u, v in $\mathbb{R}$, we have

$$\begin{aligned} |\mathscr{F}(t,v) - \mathscr{F}(t,u)| &\leqslant \frac{1}{2}\left|\frac{|u|-|v|}{(|v|+1)(1+|u|)}\right| \\ &\leqslant \frac{1}{2}|v-u|. \end{aligned}$$

Hence, the condition $(H1)$ holds, with $L = \frac{1}{2}$. We also have the following inequality

$$|\xi(t,v,u)| \leqslant h(t),$$

where $h(t) = \frac{1}{t^2+1}$ and thus

$$\int_0^1 h(t)dt = \int_0^1 \frac{1}{t^2+1}dt = \frac{\pi}{4}.$$

Then, condition $(H2)$ holds.

Now, we can show that

$$L\left(M_{\varphi} + \frac{\|h\|_{\infty}}{\Gamma(\gamma+1)}(\phi(d)-\phi(c))^{\gamma}\right) = 0.4431134627 < 1.$$

Conclusion

This study successfully establishes the existence of solutions for fractional hybrid differential equations involving the ϕ-Caputo derivative. By employing Dhage's Fixed Point Theorem, a solid theoretical framework has been developed, ensuring a rigorous foundation for the proposed method. The validity of this framework was further confirmed through a carefully constructed practical example, demonstrating the method's effectiveness in addressing real-world problems.

The primary contribution of this research lies in addressing a notable gap in the literature – the application of the ϕ-Caputo derivative on time scales to analyze fractional hybrid differential equations. This innovative approach extends the capabilities of existing methods and introduces new possibilities for modeling dynamic systems that exhibit both continuous and discrete behavior.

Beyond its theoretical advancements, this study holds significant practical potential. The proposed method offers promising applications in diverse fields such as biological systems, engineering processes, and financial forecasting – particularly in scenarios where uncertainty plays a critical role.

However, it is important to note that this study primarily focuses on establishing the existence of solutions. To build upon this foundation, future research should explore numerical validation techniques to test the method's performance under various conditions. Additionally, applying the proposed method to real-world problems could provide valuable insights into its practical utility.

Further investigations could also examine alternative fixed-point theorems, such as Krasnoselskii's or Schauder's theorem, to assess their suitability for more complex systems. Exploring these alternative approaches may reveal new strategies for improving solution methods in nonlinear and hybrid differential equations.

By expanding the theoretical groundwork established in this study, future research can help bridge the gap between mathematical theory and practical application, unlocking new possibilities for solving complex dynamic problems.

Future works

The findings of this study open several promising avenues for future research. Expanding upon the theoretical framework and practical implications outlined in this work can help address unanswered questions and explore new applications for the ϕ-Caputo derivative in dynamic systems. Below are some key directions that warrant further investigation:

1. **Development of Numerical Methods:**
   One important direction for future research is the development of efficient numerical methods for solving fractional hybrid differential equations involving the ϕ-Caputo derivative. Developing iterative algorithms specifically designed for these equations could provide practical tools for approximating solutions in complex systems. Additionally, incorporating error analysis techniques would be crucial to evaluate the accuracy, stability, and convergence properties of these numerical

methods. Such advancements would strengthen the practical implementation of the theoretical findings.

2. **Expanding Application Areas:**
The versatility of the proposed method suggests that it could have meaningful applications in various scientific fields. For example, in engineering, the ϕ-Caputo derivative may improve heat transfer models by accounting for memory effects and non-uniform thermal behavior. In biology, this derivative could help model cell growth cycles or population dynamics in environments with variable conditions. Likewise, in economics, it may enhance models for analyzing financial market volatility, particularly in systems influenced by irregular fluctuations or unexpected shocks.

3. **Stability and Asymptotic Behavior:**
Investigating the long-term behavior and stability of fractional hybrid differential equations involving the ϕ-Caputo derivative present another valuable direction for future studies. Analyzing stability properties is essential for ensuring that solutions remain well-behaved over extended time periods, especially in dynamic systems prone to unpredictable changes. Understanding asymptotic behavior would also offer insights into system equilibrium points and long-term trends, further enhancing the reliability of such models.

4. **Generalized Fixed Point Theorems:**
Exploring alternative fixed-point theorems could expand the theoretical foundation established in this study. Investigating methods such as Krasnoselskii's or Schauder's theorem may uncover new solution criteria, particularly for highly nonlinear or multi-dimensional problems. Identifying the strengths and limitations of diffrent fixed-point theorems in this context could provide valuable insights for improving solution techniques for complex systems.

5. **Control Systems on Time Scales:**
The ϕ-Caputo derivative also shows potential for contributing to the development of innovative control strategies for dynamic systems. Future research may explore its role in designing control mechanisms for robotics, biomechanics, and environmental modeling. By accounting for both continuous and discrete dynamics, the ϕ-Caputo derivative may offer improved precision and stability in systems requiring adaptive control strategies.

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REFERENCES

[1] A. AHMADKHANLU AND M. JAHANSHAHI, *On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales*, Bull. Iranian Math. Soc. **38**, 1 (2012), 241–252.

[2] S. AHMED, A. AHMED, I. MANSOOR, F. JUNEJO, A. SAEED, *Output feedback adaptive fractional-order super-twisting sliding mode control of robotic manipulator*, Iran. J. Sci. Technol. Trans. Elect. Eng. **45**, (2021), 335–347.

[3] S. Í. ARAZ, *Analysis of a Covid-19 model: optimal control, stability, and simulations*, Alexandria Eng. J. **60**, 1 (2021), 647–658.

[4] A. ATANGANA, S. ÍǴRET ARAZ, *Mathematical model of COVID-19 spread in Turkey and South Africa: theory, methods, and applications*, Adv. Dif. Equ. **2020**, 1(2020), 1–89.

[5] M. AWADALLA, Y. YAMENI, *Modeling exponential growth and exponential decay real phenomena by ϕ-Caputo fractional derivative*, J. Adv. Mathe. Comput. Sci. **28**, 2 (2018), 1–13.

[6] A. CERNEA, *On a fractional differential inclusion with Maxima*, Frac. Calc. Appl. Anal. **19**, 5 (2016), 1292–1305.

[7] N. CHEFNAJ, K. HILAL AND A. KAJOUNI, *Existence and uniqueness of solutions for ϕ-Caputo fractional neutral sequential differential equations on time scales*, Journal of Applied Mathematics and Computing **70**, 6 (2024), 5251–5268.

[8] N. CHEFNAJ, K. HILAL AND A. KAJOUNI, *Impulsive ϕ-Caputo hybrid fractional differential equations with non-local conditions*, J. Math. Sci. (2023), 1–12.

[9] N. CHEFNAJ, K. HILAL AND A. KAJOUNI, *The existence, uniqueness and Ulam-Hyers stability results of a hybrid coupled system with ϕ-Caputo fractional derivatives*, Journal of Applied Mathematics and Computing **70**, 3 (2024), 2209–2224.

[10] N. CHEFNAJ, A. TAQBIBT, K. HILAL AND S. MELLIANI, *Study of nonlocal boundary value problems for hybrid differential equations involving ϕ-Caputo Fractional Derivative with measures of noncompactness*, J. Math. Sci. (2023), 1–10.

[11] B. C. DHAGE, *On a fixed point theorem in Banach algebras with applications*, Appl. Math. Lett. 18 (2005) 273-280.

[12] E. KARAPINAR, E. BENKHETTOUD, J. E. LAZREGD AND M. BENCHOHRAD, *Fractional differential equations with maxima on time scale via Picard operators*, Filomat **37**, 2 (2023), 393–402.

[13] D. OTROCOL, *Hybrid differential equations with maxima via Picard operators theory*, Stud. Univ. Babes. Bolyai. Math. **61**, (2016), 421–428.

[14] A. TAQBIBT, N. CHEFNAJ, K. HILAL, S. MELLIANI, *ϕ-Caputo fractional differential equations with maxima on time scales*, Journal of Mathematical Sciences, (2024), 1–13.

[15] A. TAQBIBT, M. ELOMARI AND S. MELLIANI, *Nonlocal semilinear ϕ-Caputo fractional evolution equation with a measure of noncompactness in Banach space*, Filomat **37**, 20 (2023), 6877–6890.

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