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EXISTENCE AND UNIQUENESS ANALYSIS FOR FRACTIONAL ∇–DIFFERENCE TWO–POINT BOUNDARY VALUE PROBLEMS WITH FULLY DIFFERENCE BOUNDARY CONDITIONS

YOUSEF GHOLAMI* AND ELNAZ POORTAGHI

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Abstract. In this article a fractional-order ∇-difference two-point boundary value problem of order $[2,3)$ is investigated. One of the interesting parts of this boundary value problem is its boundary conditions that includes all of the possible fractional orders $\alpha \in [0,1)$, $\alpha = 1$, and $\alpha \in [1,2)$. Thanks to the Green's function approach, the main problem is transfered into an appropriate functional space and then, making use of fixed point techniques such as nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems existence of at least one solution is approved for the boundary value problem under study. Next, we choose the Banach contraction principle to make a uniqueness criterion. At the and, we examine our theoretical findings with some numerical prototypes to show applicability of the solvability results in practice.

1. Introduction

Fractional differential equations as widest branch of fractional calculus is known as leading mathematical theory to study the history dependent phenomena. Indeed, generalizing the integer-order integration and consequently differentiation to arbitrary orders, provides this opportunity to compose the time-line of the phenomena under investigation with these interval sliding orders. This unique property of the fractional differential equations makes them more reliable mathematical tools in the modeling of real-life phenomena as well as their engineering approaches. Between variety of the continuous and discrete fractional-order operators, the Riemann-Liouville fractional operators as the prime generation have a very special data preserving property. Actually, if we instantly apply these operators on constant functions, not only constants are kept but also they are composed with the order of applied Riemann-Liouville fractional operators. This nice property may lead us to use Riemann-Liouville fractional operators in both of the continuous and discrete cases to study the qualitative dynamics of real-life phenomena in frame of fractional-order differential equations. In order

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* Corresponding author.

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of find more details on the importance of fractional differential equations and particularly, Riemann-Liouville fractional differential equations we suggest the followers to see [1], [22], [32], [35], [37], [38] and related bibliography cited therein.

Paying attention on the discrete versions of the Riemann-Liouville fractional operators, we have forward and backward difference operators that are known as the Riemann-Liouville fractional Δ-difference and ∇-difference operators, respectively. During the past two decades the literature has witnessed considerable attention to investigate on the discrete fractional differential equations. One of the most interesting research fields of fractional difference equations is Green's function technique on an appropriate fixed point scheme for solvability of discrete fractional boundary value problems. The interested follower can find huge number of research works on this topic. As instance, we suggest the following collection for more consultation on this concept, [2]–[21], [25]–[24], [33], [34], [36], [39] and [40]. In this position we give some motivations to this article. The authors in [18] considered the following system of fractional ∇-difference two-point boundary value problems:

$$\begin{aligned} &\begin{pmatrix} \nabla^{\alpha}_{a^+}u(t) \\ \nabla^{\beta}_{a^+}v(t) \end{pmatrix} + \begin{pmatrix} f(t,v(t)) \\ g(t,u(t)) \end{pmatrix} = 0, \\ &\begin{pmatrix} u(a+1) \\ u(b+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v(a+1) \\ v(b+1) \end{pmatrix}, \end{aligned} \tag{1.1}$$

for which $1 < \alpha,\beta \leqslant 2,\ t \in [a+2,b+1]_{\mathbb{N}} = \{a+2,a+3,\dots,b,b+1\}$, and $a,b \in \mathbb{Z}$ with $a \geqslant 0,\ b \geqslant 3$. Also, $f,g : [a+2,b+1]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$. Using Green's function approach for some fixed point techniques, two existence criteria has presented for the fractional ∇-difference two-point boundary value sysytem (1.1).

In [23], the authors studied the following fractional ∇-difference boundary value problem

$$\begin{cases} -\left(\nabla^{\xi}_{\rho(e)}u\right)(z) = p(z,u(z)), & z \in \mathbb{N}^{f}_{e+2}, \\ u(e) = g(u), \quad u(f) = 0, \end{cases} \tag{1.2}$$

where $e,f \in \mathbb{R}$, with $f-e \in \mathbb{N}_3$, $\xi \in (1,2)$, $p : \mathbb{N}^{f}_{e+2} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function, and $g \in C\left(\mathbb{N}^{f}_{e},\mathbb{R}\right)$. As in the previous case, the Green's function technique on desired functional space has chosen to obtain some existence, multiplicity and uniqueness of positive solutions for the fractional ∇-difference boundary value problem (1.2).

Motivating by the above works, we are interested in the following two-point fractional ∇-difference boundary value problem:

$$\begin{cases} \left(\nabla^{\alpha}_{a^+}x\right)(z) + \Theta(z,x) = 0, \quad z \in \mathbb{N}^{b+1}_{a+2} = [a+2,a+3,\dots,b,b+1]_{\mathbb{N}}, \quad 2 \leqslant \alpha < 3, \\ \left(\nabla^{\alpha-1}_{a^+}x\right)(a+1) = (\nabla x)(b+1) = \left(\nabla^{\alpha-2}_{a^+}x\right)(b+1) = 0, \end{cases} \tag{1.3}$$

having the setting $a,b \in \mathbb{Z}_+,\ b-a \in \mathbb{N}_2$. Also, $\Theta : \mathbb{N}^{b+1}_{a+2} \times \mathbb{R} \longrightarrow \mathbb{R}$. Traditionally, the appeared fractional-order operators in related problems have to be identified. Accordinglly, $\nabla^{\gamma}_{t_0^+}$ stands for the fractional ∇-difference operator of order γ with lower terminal t_0, that will be defined clearly in the next section. Considering the special boundary conditions that cover all orders belong to the interval $(0,2)$, opens up a new window into the garden of Green's functions.

Here, we state the organization of rest of the paper. Section 2 includes all preliminary requirements to reach the main resluts of this investigation. In this way, it contains some essentials of discrete fractional calculus. Besides, all needed elements of the fixed point theory such as statements of nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems that are chosen be to applied for existence of solutions of the two-point fractional ∇-difference boundary value problem (1.3). Furthermore, the Banach contraction principle is recalled to lead us to a uniqueness criterion for our main problem. Next, we have Section 3 where all of the theoretical existence and uniqueness criteria will be obtained by the use of some detailed analysis linking fractional calculus, Green's function approach, and functional analysis. In Section 4 we consider some test problems to examine validity of the theoretical solvability findings in practice. At the end, we have Section 5 to summarize research line of this investigation.

2. Preliminaries

As organized, in this section we provide everything is needed to obtain some existence and uniqueness criteria for boundary value problem (1.3). To this aim, we divide this section into two parts. The first part contains all of necessary definitions and technical lemmas of ∇-difference fractional calculus that will be needed in the theoretical analysis in next section. In the section part, we have some fixed point techniques that in combination with the Green's function approach in an appropriate functional frame will lead us to the expected solvability tools. So, at the first step we start with presenting the foundation of ∇-difference operators as follows.

DEFINITION 2.1. [2, 22] Suppose p is a positive integer. Then, by the rising function of z we mean the following

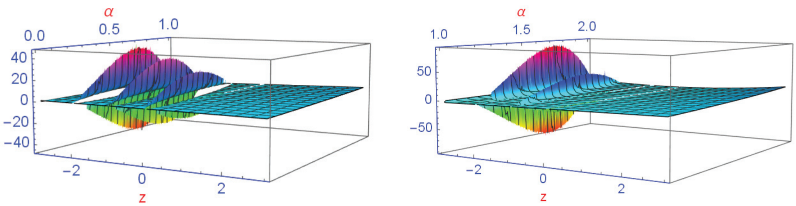
$$z^{\overline{p}} = \prod_{j=0}^{p-1}(z+j), \quad z^{\overline{0}} = 1.$$

In the case that we generalize the positive integer p to arbitrary real number α, hence, we have so called fractional rising function given by

$$z^{\overline{\alpha}} = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \quad z \in \mathbb{R} - \mathbb{Z}_{-}, \quad 0^{\overline{\alpha}} = 0. \tag{2.1}$$

In order to illustrate graphical structure of the fractional rising functions, we have Figure 1, below.

As will be seen, the fractional rising functions are kernels of the fractional ∇-difference operators, indeed. Relying on them, we define now the fractional ∇-sum operators as follows.



(a) $z^{\overline{\alpha}},\ \alpha\in(0,1)$ (b) $z^{\overline{\alpha}},\ \alpha\in(1,2)$

Figure 1: *Fractional rising function for* $\alpha\in(0,2)$

DEFINITION 2.2. [2,22] Left and right sided fractional ∇-sum operators of order $\alpha>0$ are defined as

$$\nabla_{a^+}^{-\alpha}\phi(z)=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{z}(z-\delta(s))^{\overline{\alpha-1}}\phi(s),\quad \phi:\mathbb{N}_a\longrightarrow\mathbb{R}, \tag{2.2}$$

$$\nabla_{b_-}^{-\alpha}\phi(z)=\frac{1}{\Gamma(\alpha)}\sum_{s=z}^{b}(s-\delta(z)^{\overline{\alpha-1}}\phi(s),\quad \phi:\mathbb{N}^b\longrightarrow\mathbb{R}, \tag{2.3}$$

respectively, where $\delta(s)=s-1$, $\mathbb{N}_a=\{a,a+1,a+2,\dots\}_{\mathbb{N}}$ and $\mathbb{N}^b=\{\dots,b-2,b-1,b\}_{\mathbb{N}}$.

Based on definitions of the fractional ∇-sum operators, we can now define corresponding fractional ∇-difference operators as follows.

DEFINITION 2.3. [2, 22] Left and right sided fractional ∇-difference operators of order $0\leqslant n-1<\alpha\leqslant n$, $n\in\mathbb{N}$ are defined as

$$\nabla_{a^+}^{\alpha}\phi(z)=\frac{1}{\Gamma(n-\alpha)}\nabla_z^n\left(\sum_{s=a}^{z}(z-\delta(s))^{\overline{n-\alpha-1}}\phi(s)\right), \tag{2.4}$$

$$\nabla_{b_-}^{\alpha}\phi(z)=\frac{(-1)^n}{\Gamma(n-\alpha)}\Delta_z^n\left(\sum_{s=z}^{b}(s-\delta(z)^{\overline{n-\alpha-1}}\phi(s)\right), \tag{2.5}$$

for which, $\alpha>0$, $n=[\alpha]+1$ and Δ_z traditionally demonstrates the forward difference operator acting on the variable z.

We finalize discrete fractional calculus by presenting some of essential properties of fractional ∇-sum and ∇-difference operators including power rules and mutual inversion interactions, as follows.

LEMMA 2.4. [12, 18] *Suppose* ϕ *is a real-valued function and* $\xi>0$, $0\leqslant n-1<\eta\leqslant n$. *Then*

(P_1) $\nabla_{a^+}^{-\xi}\nabla_{a^+}^{-\eta}\phi(z)=\nabla_{a^+}^{-(\xi+\eta)}\phi(z)=\nabla_{a^+}^{-\eta}\nabla_{a^+}^{-\xi}\phi(z)$,

(P_2) $\nabla_{a^+}^{-\eta}\nabla_{a^+}^{\eta}\phi(z)=\phi(z)+c_1(z-a)^{\overline{\eta-1}}+c_2(z-a)^{\overline{\eta-2}}+\ldots+c_n(z-a)^{\overline{\eta-n}}$, $c_i\in\mathbb{R},\ i=1,2,\ldots,n.$

(P_3) $\nabla_{a^+}^{\nu}\nabla_{a^+}^{-\eta}\phi(z)=\phi(z).$

(P_4) $\nabla_{a^+}^{\eta}(z-a)^{\overline{\xi}}=\dfrac{\Gamma(\xi+1)}{\Gamma(\xi-\eta+1)}(z-a)^{\overline{\xi-\eta}},\ \xi-\eta+1\notin\mathbb{Z}.$

We are now at the beginning of the second part, where we state the nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorem for existence of at least one solution for farctional boundary value problem (1.3). In addition, the Banach contraction principle is chosen to establish a uniqueness criterion for (1.3). Finally, our desired normed functional space is introduced.

THEOREM 2.5. ([18]) Nonlinear alternative of Leray-Schauder. *If Λ be a convex subset of Banach space $\mathscr{B}$ and Ψ is an open subset of Λ with $0\in\Psi$, then, each completely continuous mapping $T:\overline{\Psi}\to\Lambda$ satisfies at least one of the two following properties:*

(L_1) *There exist an $\psi\in\overline{\Psi}$ such that $T\psi=\psi$.*

(L_2) *There exist an $\phi\in\partial\Psi$ and $\theta\in(0,1)$ such that $\phi=\theta T\phi$.*

THEOREM 2.6. ([18]) Krasnoselskii-Zabreiko fixed point theorem. *Suppose $\mathscr{B}$ is a Banach space. Let $T:\mathscr{B}\to\mathscr{B}$ is a completely continuous mapping. If $\mathscr{L}:\mathscr{B}\to\mathscr{B}$ be a linear bounded mapping does not admit 1 as an eigenvalue. If*

$$\lim_{\|\phi\|\to\infty}\frac{\|T\phi-\mathscr{L}\phi\|}{\|\phi\|}=0, \tag{2.6}$$

then, T has a fixed point in $\mathscr{B}$.

Next, we have the Banach fixed point theorem that will lead us to make a uniqueness criterion for (1.3).

THEOREM 2.7. ([31]) Banach fixed point theorem. *Suppose $\mathscr{B}$ is a complete metric space with a contraction mapping $T:\mathscr{B}\longrightarrow\mathscr{B}$. Then, T admits a unique fixed point in $\mathscr{B}$.*

As the last preparatory tool, we introduce here the Banach space $\mathscr{B}$ as follows.

$$\mathscr{B}=\left(\left\{w\,\middle|\,w:\mathbb{N}_{a+2}^{b+1}\longrightarrow\mathbb{R}\right\};\|.\|_{\mathscr{B}}\right),\quad \|w\|_{\mathscr{B}}=\max_{z\in\mathbb{N}_{a+2}^{b+1}}|w(z)|.$$

3. Main results: existence and uniqueness analysis

This section contains theoretical aspect of this investigation, where it will prove that under which conditions the fractional ∇-difference two-point boundary value problem (1.3) has at least one solution, as well as under which one it admits a unique solution. In order to reach the main conclusion, we are going to divide this section into three subsections. In the first subsection, we work on the Green's function approach to transform two-point boundary value problem (1.3) into corresponding ∇-sum equation. In this case, we are ready to migrate into the Banach space $\mathscr{B}$ that allows us to apply our analysis on the mentioned fixed point theorems in the previous section. Second subsection is devoted to the existence results. In this subsection we apply the nonlinear alternative of Leray-Schauder fixed point theorem as well as the Krasnoselskii-Zabreiko fixed point theorem on Banach space $\mathscr{B}$ to prove that under certain conditions the two-point boundary value problem (1.3) has at least one solution. In the third subsection and in the light of the Banach fixed point theorem, we need some hypothesis that enables us to have a contraction mapping in the Banach space $\mathscr{B}$. Thanks to the existence of contraction mappings, the Banach fixed point theorem directly guarantees existence of a unique solution for two-point boundary value problem (1.3). So, we start with the Green's function analysis as follows.

3.1. Green's function analysis

This subsection includes two technical lemmas in general. In frame of these technical lemmas detailed structure of the Green's function corresponding to the fractional ∇-difference two-point boundary value problem (1.3) is identified.

LEMMA 3.1. *Suppose $\Xi : \mathbb{N}_{a+2}^{b+1} \longrightarrow \mathbb{R}$. Then, the solution $x(z)$ of the fractional ∇-difference two-point boundary value problem*

$$\begin{cases} \left(\nabla_{a^+}^{\alpha} x\right)(z) + \Xi(z) = 0, \quad z \in \mathbb{N}_{a+2}^{b+1}, \quad 2 \leqslant \alpha < 3, \\ \left(\nabla_{a^+}^{\alpha-1} x\right)(a+1) = (\nabla x)(b+1) = \left(\nabla_{a^+}^{\alpha-2} x\right)(b+1) = 0, \end{cases} \tag{3.1}$$

uniquely solves the fractional ∇-sum equation

$$x(z) = \sum_{s=a+2}^{b+1} G(z,s)\Xi(s), \quad z \in \mathbb{N}_{a+2}^{b+1}, \tag{3.2}$$

for which, $G(z,s)$ denotes the Green's function of the boundary value problem (3.1), *and is given by*

$$G(z,s) = \begin{cases} (\alpha-1)\Bigg\{\dfrac{\Gamma(z-a+\alpha-3)}{(z-a-1)!}\Bigg[(z-a+\alpha-3)(b-s+2)+\dfrac{(b-a)!}{\alpha-3}\cdot\dfrac{\Gamma(b-s+\alpha)}{(b-s1)!} \\ \cdot\dfrac{1}{\Gamma(b-a+\alpha-3)}\Bigg]-\dfrac{\alpha-2}{\alpha-3}\cdot\dfrac{(b-a)!}{\Gamma(b-a+\alpha-3)}\cdot(b-s+2)\Bigg\}-(z-s+1)^{\overline{\alpha-1}}; \\ \qquad a+2\leqslant s\leqslant z\leqslant b+1, \\ (\alpha-1)\Bigg\{\dfrac{\Gamma(z-a+\alpha-3)}{(z-a-1)!}\Bigg[(z-a+\alpha-3)(b-s+2)+\dfrac{(b-a)!}{\alpha-3}\cdot\dfrac{\Gamma(b-s+\alpha)}{(b-s1)!} \\ \cdot\dfrac{1}{\Gamma(b-a+\alpha-3)}\Bigg]-\dfrac{\alpha-2}{\alpha-3}\cdot\dfrac{(b-a)!}{\Gamma(b-a+\alpha-3)}\cdot(b-s+2)\Bigg\}; \\ \qquad a+2\leqslant z+1\leqslant s\leqslant b+1. \end{cases} \tag{3.3}$$

Proof. Let us start the proof with

$$\left(\nabla^{\alpha}_{a^+}x\right)(z) = -\Xi(z), \quad z\in\mathbb{N}^{b+1}_{a+2}, \quad 2\leqslant\alpha<3. \tag{3.4}$$

Applying the inversion rule (P_2) in Lemma 2.4, we arrive at the following:

$$x(z) = c_1(z-a)^{\overline{\alpha-1}} + c_2(z-a)^{\overline{\alpha-2}} + c_3(z-a)^{\overline{\alpha-3}} - \left(\nabla^{-\alpha}_{a^+}\Xi\right)(z). \tag{3.5}$$

Based on (3.5), and making use of the power rule (P_4) in Lemma 2.4, we get the following results:

$$\left(\nabla^{\alpha-1}_{a^+}x\right)(z) = c_1\Gamma(\alpha) - \sum_{s=a+2}^{z}\Xi(s), \tag{3.6}$$

$$\left(\nabla^{\alpha-2}_{a^+}x\right)(z) = c_1\Gamma(\alpha) + c_2\Gamma(\alpha-1) - \sum_{s=a+2}^{z}(z-s+1)\Xi(s), \tag{3.7}$$

$$\begin{aligned}(\nabla x)(z) &= c_1(\alpha-1)(z-a)^{\overline{\alpha-2}} + c_2(\alpha-2)(z-a)^{\overline{\alpha-3}} + c_3(\alpha-3)(z-a)^{\overline{\alpha-4}} \\ &\quad - \frac{1}{\Gamma(\alpha-1)}\sum_{s=a+2}^{z}(z-s+1)^{\overline{\alpha-2}}\Xi(s).\end{aligned} \tag{3.8}$$

Next step is to apply the boundary conditions. In this way, we have

$$\left(\nabla^{\alpha-1}_{a^+}x\right)(a+1) = 0 \Longrightarrow c_1 = 0, \tag{3.9}$$

$$\left(\nabla^{\alpha-2}_{a^+}x\right)(b+1) = 0 \Longrightarrow c_2 = \frac{\sum_{s=a+2}^{b+1}(b-s+2)\Xi(s)}{\Gamma(\alpha-1)}, \tag{3.10}$$

$$(\nabla x)(b+1)=0 \Longrightarrow$$
$$c_3=\frac{\sum_{s=a+2}^{b+1}\left[(b-s+2)^{\overline{\alpha-2}}-(\alpha-2)(b-s+2)(b-a+1)^{\overline{\alpha-3}}\right]\Xi(s)}{(\alpha-3).\Gamma(\alpha-2).(b-a+1)^{\overline{\alpha-4}}}. \tag{3.11}$$

If we substitute (3.9)–(3.11) into (3.5), we get that

$$\begin{aligned} x(z)&=\frac{(z-a)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\sum_{s=a+2}^{b+1}(b-s+2)\Xi(s)\\ &+(z-a)^{\overline{\alpha-3}}\frac{\sum_{s=a+2}^{b+1}\left[(b-s+2)^{\overline{\alpha-2}}-(\alpha-2)(b-s+2)(b-a+1)^{\overline{\alpha-3}}\right]\Xi(s)}{(\alpha-3).\Gamma(\alpha-2).(b-a+1)^{\overline{\alpha-4}}}\\ &-\frac{1}{\Gamma(\alpha)}\sum_{s=a+2}^{z}(z-s+1)^{\overline{\alpha-1}}\Xi(s). \end{aligned} \tag{3.12}$$

Appropriately splitting the domain of s in the appearing finite sums, one has

$$x(z)=\frac{1}{\Gamma(\alpha)}\sum_{s=a+2}^{b+1}H(z,s)\Xi(s), \tag{3.13}$$

where,

$$H(z,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\alpha-1)\Bigg\{(z-a)^{\overline{\alpha-2}}(b-s+2)+\frac{1}{(\alpha-3)(b-a+1)^{\overline{\alpha-4}}}\\ \quad\cdot\left[(z-a)^{\overline{\alpha-3}}(b-s+2)^{\overline{\alpha-2}}-(\alpha-2)(b-s+2)\right]\Bigg\}-(z-s+1)^{\overline{\alpha-1}};\\ \qquad a+2\leqslant s\leqslant z\leqslant b+1,\\ (\alpha-1)\Bigg\{(z-a)^{\overline{\alpha-2}}(b-s+2)+\frac{1}{(\alpha-3)(b-a+1)^{\overline{\alpha-4}}}\\ \quad\cdot\left[(z-a)^{\overline{\alpha-3}}(b-s+2)^{\overline{\alpha-2}}-(\alpha-2)(b-s+2)\right]\Bigg\};\\ \qquad a+2\leqslant z+1\leqslant s\leqslant b+1.\end{cases} \tag{3.14}$$

If we use definition of the fractional rising function $(z-a)^{\overline{\beta}}$, given by (2.1), the fact that for each positive integer m, $\Gamma(m)=(m-1)!$, and this key point that

$$(z-a)^{\overline{\beta}}=(z-a+\beta-1)(z-a)^{\overline{\beta-1}}, \tag{3.15}$$

then, having some simplification on the second part of (3.14) we come to the conclusion that the prime Green's function $H(z,s)$ changes its form into the Green's function $G(z,s)$ given by (3.3). To see happening of this process, we follow the forthcoming sequence of calculations:

$$\begin{aligned}
&(\alpha-1)(z-a)^{\overline{\alpha-2}}(b-s+2)+(\alpha-1)\frac{(z-a)^{\overline{\alpha-3}}(z-a)^{\overline{\alpha-2}}}{(\alpha-3)(b-a+1)^{\overline{\alpha-4}}}\\
&\qquad\qquad -(\alpha-1)\frac{(\alpha-2)(b-s+2)}{(\alpha-3)(b-a+1)^{\overline{\alpha-4}}}\\
=&(\alpha-1)\left\{(z-a)^{\overline{\alpha-2}}(b-s+2)+\frac{(z-a)^{\overline{\alpha-3}}(z-a)^{\overline{\alpha-2}}}{(\alpha-3)(b-a+1)^{\overline{\alpha-4}}}-\frac{(\alpha-2)(b-s+2)}{(\alpha-3)(b-a+1)^{\overline{\alpha-4}}}\right\}\\
=&(\alpha-1)\left\{\frac{\Gamma(z-a+\alpha-2)}{\Gamma(z-a)}(b-s+2)+\frac{\dfrac{\Gamma(z-a+\alpha-3)\Gamma(b-s+\alpha)}{\Gamma(z-a)\Gamma(b-s+2)}}{(\alpha-3)\dfrac{\Gamma(b-a+\alpha-3)}{\Gamma(b-a+1)}}\right.\\
&\qquad\qquad\left.-\frac{(\alpha-2)(b-s+2)}{(\alpha-3)\dfrac{\Gamma(b-a+\alpha-3)}{\Gamma(b-a+1)}}\right\}\\
=&(\alpha-1)\left\{\frac{\Gamma(z-a+\alpha-2)}{(z-a-1)!}(b-s+2)\right.\\
&\qquad\qquad+\frac{(b-a)!\Gamma(z-a+\alpha-3)\Gamma(b-s+\alpha)}{(\alpha-3)(z-a-1)!(b-s+1)!\Gamma(b-a+\alpha-3)}\\
&\qquad\qquad\left.-\frac{\alpha-2}{\alpha-3}(b-a)!\frac{b-s+2}{\Gamma(b-a+\alpha-3)}\right\}\\
=&(\alpha-1)\left\{\frac{\Gamma(z-a+\alpha-3)}{(z-a-1)!}\left[(z-a+\alpha-3)(b-s+2)\right.\right.\\
&\qquad\qquad\left.+\frac{(b-a)!}{\alpha-3}\frac{\Gamma(b-s+\alpha)}{(b-s+1)!}\frac{1}{\Gamma(b-a+\alpha-3)}\right]\\
&\qquad\qquad\left.-\frac{\alpha-2}{\alpha-3}\frac{(b-a)!}{\Gamma(b-a+\alpha-3)}(b-s+2)\right\}.
\end{aligned}\tag{3.16}$$

The clarifying sequence of calculations in (3.16) guarantees that $x(z)$ uniquely characterizes solution of the fractional ∇-difference two-point boundary value problem (3.1) in the form of fractional ∇-sum equation

$$x(z)=\sum_{s=a+2}^{b+1}G(z,s)\Xi(s),\quad z\in\mathbb{N}_{a+2}^{b+1}.$$

So, the conclusion of this technical lemma is approved now. □

In the next technical lemma we obtain maximum of the Green's function $G(z,s)$.

LEMMA 3.2. *Suppose $a,b\in\mathbb{Z}_+$ with $b-a\in\mathbb{N}_2$. Then,*

$$\max_{z,s\in\mathbb{N}_{a+2}^{b+1}} G(z,s)=\frac{1}{\Gamma(\alpha-1)}\left\{\frac{\Gamma(b-a+\alpha-2)}{(b-a-1)!}\left[(b-a+\alpha-3)\frac{\alpha-2}{\alpha-3}+1\right]\right.$$
$$\left.-\frac{\alpha-2}{\alpha-3}\frac{(b-a)(b-a)!}{\Gamma(b-a+\alpha-3)}\right\}. \tag{3.17}$$

Proof. Frist part of the proof is devoted to find out which one of two pieces in Green's function $G(z,s)$ gives us its maximum vlaue at $\mathbb{N}_{a+2}^{b+1}$. To this aim, we need to use the following key point:

$$(z-s+1)^{\overline{\alpha-1}}=\frac{\Gamma(z-s+\alpha)}{\Gamma(z-s+1)}\geqslant 0,\quad a+2\leqslant s\leqslant z\leqslant b+1,\ \alpha\in[2,3). \tag{3.18}$$

So, it is clear now that for each $a+2\leqslant z+1\leqslant s\leqslant b+1$,

$$\max_{z,s\in\mathbb{N}_{a+2}^{b+1}} G(z,s)=G_{\max}(z,s)=\frac{1}{\Gamma(\alpha-1)}\left\{\frac{\Gamma(z-a+\alpha-3)}{(z-a-1)!}\left[(z-a+\alpha-3)(b-s+2)\right.\right.$$
$$\left.+\frac{(b-a)!}{\alpha-3}\cdot\frac{\Gamma(b-s+\alpha)}{(b-s1)!}\cdot\frac{1}{\Gamma(b-a+\alpha-3)}\right] \tag{3.19}$$
$$\left.-\frac{\alpha-2}{\alpha-3}\cdot\frac{(b-a)!}{\Gamma(b-a+\alpha-3)}\cdot(b-s+2)\right\}.$$

With a direct calculation it follows that for each $z,s\in\mathbb{N}_{a+2}^{b+1}$,

$$\Delta_z(z-a)^{\overline{\alpha-2}}\geqslant 0,\quad \Delta_z(z-a)^{\overline{\alpha-3}}\leqslant 0. \tag{3.20}$$

Also, for each $s\in\mathbb{N}_{a+2}^{b+1}$,

$$\Delta_s(b-s+2)^{\overline{\alpha-2}}\leqslant 0. \tag{3.21}$$

Considering (3.19) and making use of the monotonicity results (3.20) and (3.21), yields the following:

$$\max_{z,s\in\mathbb{N}_{a+2}^{b+1}} G(z,s)=G_{\max}(b+1,a+2). \tag{3.22}$$

Equivalently, we get that

$$\max_{z,s\in\mathbb{N}_{a+2}^{b+1}} G(z,s)=\frac{1}{\Gamma(\alpha-1)}\left\{\frac{\Gamma(b-a+\alpha-2)}{(b-a-1)!}\left[(b-a+\alpha-3)\frac{\alpha-2}{\alpha-3}+1\right]\right.$$
$$\left.-\frac{\alpha-2}{\alpha-3}\frac{(b-a)(b-a)!}{\Gamma(b-a+\alpha-3)}\right\}.$$

This completes the proof. □

Completing the analysis of Green's function corresponding to the fractional ∇-difference two-point boundary value problem (1.3), its time to attempt for the solvability analysis. The first part of solvability analysis is devoted to the existence of solutions for (1.3) and is presented in the next subsection.

3.2. Existence analysis

In this part of main results, we shall make use of two fixed point theorems nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko to obtain two citeria for existence of at least one solution for the fractional ∇-difference two-point boundary value problem (1.3). To this aim, we need at first some hypotheses on the nonlinearity of (1.3). Prior to presenting existence analysis, assume that the following hypotheses are satisfied.

(E_1) There exist positive real-valued function $\mu : \mathbb{N}_{a+2}^{b+1} \longrightarrow \mathbb{R}^+$ and positive real-valued increasing function $\nu : \mathbb{R} \longrightarrow \mathbb{R}^+$, satsifying the following property:

$$|\Theta(z,x)| \leqslant \mu(z)\nu(|x|).$$

(E_2) There exists real-valued function $\sigma : \mathbb{N}_{a+2}^{b+1} \longrightarrow \mathbb{R}$, for which the following limit holds:

$$\lim_{\|x\|_{\mathscr{B}}\to\infty} \frac{\Theta(z,x)}{x} = \sigma(z).$$

In the followig theorem, using the nonlinear alternative of Leray-Schauder fixed point theorem stated by Theorem 2.5, we prove the first existence result for the fractional ∇-difference two-point boundary value problem (1.3).

THEOREM 3.3. *Suppose that the hypothesis (E_1) is satisfied. Then, the fractional ∇-difference two-point boundary value problem (1.3) has at least one solution in the Banach space $\mathscr{B}$, provided that*

$$\sum_{s=a+2}^{b+1} |\mu(z)| < \left|G_{\max}\right|^{-1} \varepsilon \nu(\varepsilon)^{-1}. \tag{3.23}$$

Proof. Since we are going to use fixed point theory to reach an existence result, so, we need some change on the fractional ∇-difference boundary value problem (1.3). Our strategy to solve (1.3) is to prove that second part of Theorem 2.5, namely (L_2) is not satisfied and consequently the first part must hold. So, as the first step we define the operator $T : \mathscr{B} \longrightarrow \mathscr{B}$ is follows:

$$(Tx)(z) = \sum_{s=a+2}^{b+1} G(z,s)\Theta(z,s), \quad z \in \mathbb{N}_{a+2}^{b+1}, \tag{3.24}$$

for which the Green's function $G(z,s)$ is defined by (3.3). Checking out the statement of Theorem 2.5, we need to solve the following changed boundary value problem:

$$\begin{cases} \left(\nabla_{a^+}^{\alpha} x\right)(z) + \theta\Theta(z,x) = 0, \quad z \in \mathbb{N}_{a+2}^{b+1} = [a+2, a+3, \dots, b, b+1]_{\mathbb{N}}, \quad 2 \leqslant \alpha < 3, \\ \left(\nabla_{a^+}^{\alpha-1} x\right)(a+1) = (\nabla x)(b+1) = \left(\nabla_{a^+}^{\alpha-2} x\right)(b+1) = 0, \end{cases} \tag{3.25}$$

with $\theta \in (0,1)$. Thus, according to (3.24) and (3.25), we find out that in order to solve the changed fractional ∇-difference boundary value problem (3.25), we have to solve the following fixed point problem

$$\theta T x = x. \tag{3.26}$$

It is necessary to characterize exactly $\Psi \subset \mathscr{B}$ in Theorem 2.5. To this end, we define it as follows:

$$\Psi = \left\{ x \in \mathscr{B} \middle| \; \|x\|_{\mathscr{B}} < \varepsilon \right\}, \quad \varepsilon \in \mathbb{R}^+. \tag{3.27}$$

In this case, this is obvious that for each $x \in \partial\Psi$, we have $\|x\|_{\mathscr{B}} = \varepsilon$. Now, we are ready to prove that second part of Theorem 2.5 is not satisfied. To this aim, we suppose on the contrary that the second part (L_2) is satisfied. So, there exist $x \in \partial\Psi$ and $\theta \in (0,1)$ such that

$$\theta T x = x. \tag{3.28}$$

Then, we get that

$$\theta \|Tx\|_{\mathscr{B}} = \|x\|_{\mathscr{B}}. \tag{3.29}$$

Applying (3.24), on (3.29), yields the following reslut:

$$\begin{aligned} \theta \sum_{s=a+2}^{b+1} G(z,s)\Theta(z,s) &= \varepsilon. \\ \Longrightarrow \theta |G_{\max}| \sum_{s=a+2}^{b+1} \Theta(z,s) &\geqslant \varepsilon. \\ \Longrightarrow \theta |G_{\max}| \nu(\varepsilon) \sum_{s=a+2}^{b+1} |\mu(z)| &\geqslant \varepsilon. \end{aligned} \tag{3.30}$$

Based on the above analysis, we come to the conclusion that

$$\sum_{s=a+2}^{b+1} |\mu(z)| \geqslant \left|G_{\max}^{-1}\right| \varepsilon \nu^{-1}(\varepsilon), \tag{3.31}$$

which makes a contradiction with the necessary condition (3.23). So, it has proven that the second part of Theorem 2.5, (L_2), is not satisfied and consequently its first part has to be satisfied, that is the fixed point problem

$$Tx = x, \tag{3.32}$$

where the opertaor T is defined by (3.24), has at least one fixed point in the Banach space $\mathscr{B}$. It equivalently means that, the fractional ∇-difference two-point boundary vlaue problem (1.3) has at least one solution in the Banach space $\mathscr{B}$. So, the proof is completed. □

Having the first existence criterion in hand, it is time to make second existence criterion. This purpose is done using the Krasnoselskii-Zabreiko fixed point theorem as follows.

THEOREM 3.4. *Suppose the hypothesis* (E_2) *is satisfied. Then, the fractional* ∇*-difference two-point boundary value problem* (1.3) *admits at least one solution in* $\mathscr{B}$ *provided that the following condition holds:*

$$\sum_{s=a+2}^{b+1} \max_{z} |G(z,s)| < \|\sigma\|_{\mathscr{B}}^{-1}. \tag{3.33}$$

Proof. We start the proof by chracterizing the bounded linear mapping $\mathscr{L}$ as follows:

$$\mathscr{L} : \mathscr{B} \longrightarrow \mathscr{B}, \quad (\mathscr{L}x)(z) = \sum_{s=a+2}^{b+1} G(z,s)x(s)\sigma(s), \quad z \in \mathbb{N}_{a+2}^{b+1}. \tag{3.34}$$

As one of the conditions of the Krasnoselskii-Zabreiko fixed point theorem, we have to show that 1 is not an eigenvalue of the mapping $\mathscr{L}$. Indeed, in the light of hypothesis (E_2) and the necessary condition (3.33), one may conclude that for each $z \in \mathbb{N}_{a+2}^{b+1}$,

$$\|\mathscr{L}x\|_{\mathscr{B}} \leqslant \sum_{s=a+2}^{b+1} \max_{z} |G(z,s)| \|x\|_{\mathscr{B}} \|\sigma\|_{\mathscr{B}}, \tag{3.35}$$

and consequently,

$$\|\mathscr{L}x\|_{\mathscr{B}} < \|x\|_{\mathscr{B}}. \tag{3.36}$$

So, $\mathscr{L}$ is a blunded linear mapping and does not admit 1 as its eigenvalue. Next, we have to analyze $(T-\mathscr{L})x$ as follows. For an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} \|Tx - \mathscr{L}x\|_{\mathscr{B}} &= \left\| \sum_{s=a+2}^{b+1} G(z,s) \Big\{ \Theta(s) - x(z)\sigma(s) \Big\} \right\|_{\mathscr{B}} \\ &\leqslant \sum_{s=a+2}^{b+1} \max_{z} G(z,s) \|\Theta(s) - x(z)\sigma(s)\|_{\mathscr{B}} \\ &< G_{\max}(b-a)\varepsilon \|x\|_{\mathscr{B}}. \end{aligned} \tag{3.37}$$

The limit approach (3.37) directly leads us to this fact that

$$\lim_{\|x\|_{\mathscr{B}} \to \infty} \frac{\|Tx - \mathscr{L}x\|_{\mathscr{B}}}{\|x\|_{\mathscr{B}}} = 0. \tag{3.38}$$

Since all conditions of the Krasnoselskii-Zabreiko fixed point Theorem 2.6 hold, we come to the conclusion that the fractional ∇-difference two-point boundary value problem (1.3) has at least one solution in $\mathscr{B}$. This completes the proof. □

As observed above, during the existence analysis we made use of the nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems and some appropriate hypotheses to prove that the fractional ∇-difference two-point boundary value problem (1.3) admits at least one solution in the Banach space $\mathscr{B}$. So, as planned we are done with the existence analysis and continue with the uniqueness analysis. To this aim, we shall apply the Banach fixed point theorem to show that under appropriate conditions, the boundary value problem (1.3) has a unique solution in $\mathscr{B}$.

3.3. Uniqueness analysis

This is the last part of our solvability analysis, where we have to prove that under which conditions the fractional ∇-difference two-point boundary value problem (1.3) has exactly one solution in $\mathscr{B}$. To this aim, we state and prove the following theorem.

THEOREM 3.5. *Suppose that* $\Theta(z,x) : \mathbb{N}_{a+2}^{b+1} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Lipschitzian with respect to* x*, that is there exist positive parameter* ρ *such that*

$$|\Theta(z,x) - \Theta(z,y)| \leqslant \rho|x-y|, \quad x,y \in \mathscr{B}. \tag{3.39}$$

Then, the fractional ∇*-difference two-point boundary value problem* (1.3) *has a unique solution in* $\mathscr{B}$*, in the case that*

$$G_{\max}\rho(b-a) < 1. \tag{3.40}$$

Proof. Let us consider once again the operator T defined by (3.24) as follows:

$$(Tx)(z) = \sum_{s=a+2}^{b+1} G(z,s)\Theta(z,s), \quad z \in \mathbb{N}_{a+2}^{b+1}. \tag{3.41}$$

Thanks to the Lipschitz nature of the nonlinearity $\Theta(z,s)$ with respect to the second variable as well as the necessary condition (3.40), we get that for each $x,y \in \mathscr{B}$,

$$\begin{aligned} \|Tx - Ty\|_{\mathscr{B}} &= \left\| \sum_{s=a+2}^{b+1} G(z,s)(\Theta(z,x) - \Theta(z,y)) \right\|_{\mathscr{B}} \\ &\leqslant \sum_{s=a+2}^{b+1} \rho \max_{z} |G(z,s)| \|x-y\|_{\mathscr{B}} \\ &= \rho(b-a)G_{\max}\|x-y\|_{\mathscr{B}} \\ &< \|x-y\|_{\mathscr{B}}. \end{aligned} \tag{3.42}$$

So, it has proven that the operator T given by (3.41) is a contraction operator in the Banach space $\mathscr{B}$. So, according to the Banach fixed point Theorem 2.7, T has a unique fixed point in $\mathscr{B}$. Equivalently, the Banach fixed point theorem guarantees that the fractional ∇-difference two-point boundary value problem (1.3) admits exactly one solution in the Banach space $\mathscr{B}$. This completes the proof. $\square$

REMARK 3.6. Having a general overview on the structure of solvability analysis, it is straightforward that in all of the existence and uniqueness analysis, the Green's function approach is the most fundamental factor. Indeed, the role of Green's function as discrete kernels of fractional ∇-sum equations enables us to investigate fractional ∇-sum equations in appropriate functional spaces instead of the original fractional ∇-difference boundary value problems. This key point indicates the importance of Green's function approach in the investigation on the solvability of boundary value problems in general and fractional ∇-difference boundary value problems in particular.

4. Test problems

EXAMPLE 4.1. Consider the following fractional ∇-difference two-point boundary value problem:

$$\begin{cases} \left(\nabla_{0^+}^{2.5}x\right)(z) + \dfrac{(1-\sin(z))\exp(1+0.01x)}{40} = 0, \quad z \in \mathbb{N}_2^6, \\ \left(\nabla_{0^+}^{1.5}x\right)(2) = \left(\nabla x\right)(6) = \left(\nabla_{0^+}^{0.5}x\right)(6) = 0, \end{cases} \tag{4.1}$$

Comparing the test model (4.1) by (1.3), it figures out that the following setting are chosen:

$$\alpha = 2.5, \tag{4.2}$$
$$a = 0, \quad b = 5, \tag{4.3}$$
$$\mu(z) = 0.25(1-\sin(z)), \tag{4.4}$$
$$\nu = 0.01\exp\left(1+\frac{x}{100}\right), \tag{4.5}$$
$$\varepsilon = 10. \tag{4.6}$$

In the following, we graphically illustrate the functions $\mu(z)$ and $\nu(x)$ in Figure 2.

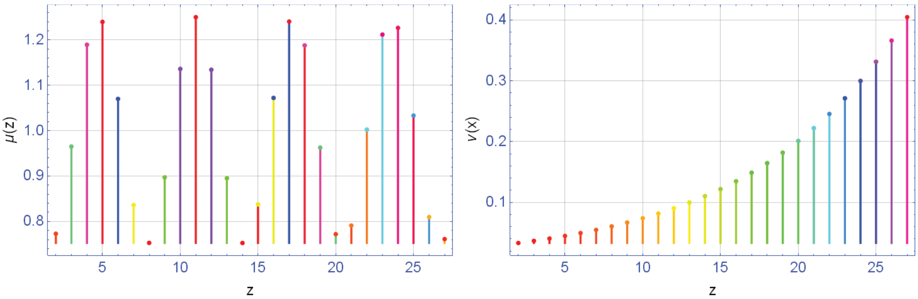


Figure 2: *Left:* $\mu(z) = 0.25(1-\sin(z))$. *Right:* $\nu(x) = 0.01\exp\left(1+\frac{x}{100}\right)$

An immediate evaluation shows that the hypothesis (E_1) holds. Also, with a direct calculation, we arrive at the following numerical results:

$$G_{\max} \approx 55.733, \tag{4.7}$$

$$\sum_{2}^{6} |\mu(z)| \approx 3.0709, \tag{4.8}$$

$$\nu(\varepsilon) \approx 0.03, \tag{4.9}$$

$$G_{\max}^{-1} \varepsilon \nu(\varepsilon)^{-1} \approx 5.9809. \tag{4.10}$$

Now, if we compare (4.8) with (4.10), we come to the conclusion that

$$\sum_{2}^{6} |\mu(z)| < G_{\max}^{-1} \varepsilon \nu(\varepsilon)^{-1}. \tag{4.11}$$

Since all conditions of Theorem 3.3 are satisfied, so, the fractional ∇-difference two-point boundary value problem (4.1) has at least one solution in the Banach space $\mathscr{B}$.

EXAMPLE 4.2. As second prototype, we are going to examine validity of the existence criterion given by Theorem 3.4. To this aim, we consider the fractional ∇-difference two-point boundary value problem

$$\begin{cases} \left(\nabla_{0^+}^{2.75} x\right)(z) + x\exp(-7z^2) = 0, \quad z \in \mathbb{N}_3^8, \\ \left(\nabla_{0^+}^{1.75} x\right)(3) = \left(\nabla x\right)(8) = \left(\nabla_{0^+}^{0.75} x\right)(8) = 0. \end{cases} \tag{4.12}$$

It is clear that, we have the following setting:

$$\alpha = 2.75, \tag{4.13}$$

$$a = 1, \quad b = 7, \tag{4.14}$$

$$\sigma(z) = \exp\left(-7z^2\right), \tag{4.15}$$

$$\Theta(z,x) = x\sigma(z). \tag{4.16}$$

We visualize the functions $\sigma(z)$ and $\Theta(z,x)$ by the Figure 3, as follows.

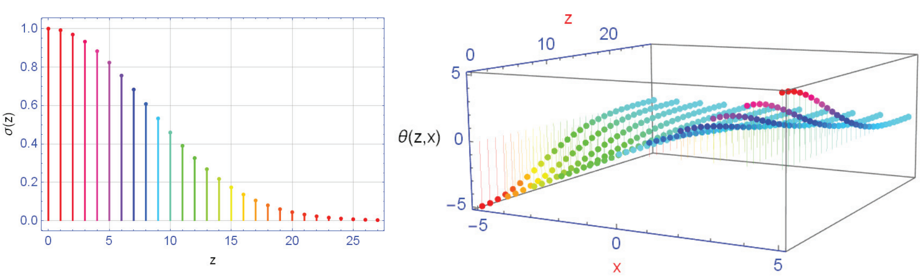


Figure 3: *Left:* $\sigma(z) = \exp(-7z^2)$. *Right:* $\Theta(z,x) = x\sigma(z)$

Considering the above setting, it is obvious that the hypothesis (E_2) is satisfied. Furthermore, according to the above numerical setting we have

$$G_{\max} \approx 112.238, \tag{4.17}$$

$$\|\sigma\|_{\mathscr{B}} \approx 4.3596 \times 10^{-28}. \tag{4.18}$$

So, we get that

$$\sum_{a+2}^{b+1} \max_z G(z,x)\|\sigma\|_{\mathscr{B}} = (b-a)G_{\max}\|\sigma\|_{\mathscr{B}} \approx 2.9359 \times 10^{-25} < 1. \tag{4.19}$$

This shows that the necessary condition (3.33) in Theorem 3.4 is also satisfied. Since all conditions of Theorem 3.4 hold, it has shown that the fractional ∇-difference two-point boundary value proble (4.12) admits at least one solution in the Banach space $\mathscr{B}$.

EXAMPLE 4.3. This is the third prototype, where we have to test practical applicability of the uniqueness criterion given by Theorem 3.5. So, let us consider the following boundary value problem:

$$\begin{cases} \left(\nabla_{0^+}^{2.25}x\right)(z) + \dfrac{x\arctan\left(z^3-1\right)}{2024(1+x)} = 0, \quad z \in \mathbb{N}_4^{10}, \\ \left(\nabla_{0^+}^{1.25}x\right)(4) = \left(\nabla x\right)(10) = \left(\nabla_{0^+}^{0.25}x\right)(10) = 0. \end{cases} \tag{4.20}$$

Taking a look at the boundary value problem (4.20), we find out the following setting:

$$\alpha = 2.25, \tag{4.21}$$

$$a = 2, \quad b = 9, \tag{4.22}$$

$$\beta(z) = \arctan\left(z^3-1\right), \tag{4.23}$$

$$\Theta(z,x) = \frac{x\beta(z)}{2024(1+x)}. \tag{4.24}$$

In the Figure 4, the functions $\beta(z)$ and $\Theta(z,x)$ are illustrated.

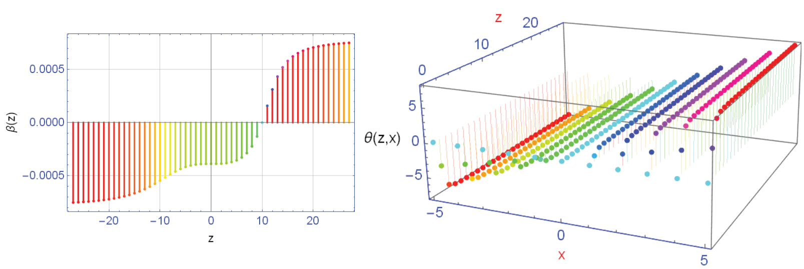


Figure 4: *Left:* $\beta(z) = \arctan\left(z^3-1\right)$. *Right:* $\Theta(z,x) = \dfrac{x\beta(z)}{2024(1+x)}$

A simple calculation gives us the following numerical outcomes:

$$G_{\max} \approx 208.641, \tag{4.25}$$

$$\rho = \max_{z} \beta(z) \approx 0.0007. \tag{4.26}$$

Based on the recent numerical findings, it is easy to check that the necessary condition (3.39) is satisfied. In addition, according to (4.25) and (4.26), one has

$$G_{\max}(b-a)\rho \approx 0.9801 < 1. \tag{4.27}$$

Since all conditions of Theorem 3.5 are fulfilled, so, it has proven that the fractional ∇-difference two-point boundary value problem (4.20) has a unique solution in the Banach space $\mathscr{B}$.

5. Concluding remarks

In this article, a special class of fractional ∇-difference two-point boundary value problems have chosen to be investigated. Considering the literature, most of the fractional-order boundary value problems have an order $\alpha \in [1,2)$ and are subjected (at least partly) to the Dirichlet boundary condition. The proposed boundary value problem in this paper has an order $\alpha \in [2,3)$ and is subjected to the fully difference boundary conditions including the sub-orders $\alpha - 2 \in [0,1)$, $\alpha - 1 \in [1,2)$ and order $\gamma = 1$. This collection of sub-orders in the boundary conditions, predict a new structure for the corresponding Green's function. The fundamental role of the extracted Green's function, enables us link to the fixed point theory, particularly some interesting fixed point techniques such as the nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems, that in frame of some appropriate Banach spaces and under some hypotheses essentially imposed on the nonlinearity $\Theta(z,x)$, led to existence of at least one solution within the considered Banach space. This is another one of differences of this ivestigation in comparision with the existing literature. Since, the Krasnoselskii fixed point theorem is commonly used to reach an existence criterion for considered boundary value problems. In continuation, using the Banach fixed point theorem, a uniqueness criterion for the main problem has obtained. Validity of all the theoretical solvability criteria have examined in Section 4, and it has proven that all of the theoretical findings are valid in practice.

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Yousef Gholami
Khayyam Institute of Mathematics
West Azerbaijan, Miandoab, Iran
and
Department of Applied Mathematics
Sahand University of Technology
P. O. Box: 51335-1996, Tabriz, Iran
e-mail: y_gholami@sut.ac.ir

Elnaz Poortaghi
Division of Mathematics
Ministry of Education
West Azerbaijan, Miandoab Branch, Iran
e-mail: alnazportagi1359@gmail.com