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MULTIPLICITY OF SOLUTIONS FOR HOMOGENEOUS FRACTIONAL HAMILTONIAN SYSTEMS

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Abstract. This paper investigates the multiplicity of solutions for a class of fractional Hamiltonian systems defined by the system:

$$\begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u)(t) + L(t)u(t) = -a(t)\nabla G(u(t)) + b(t)\nabla H(u(t)) + h(t), & t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R}), \end{cases}$$

where ${}_tD^{\alpha}_{\infty}$ and ${}_{-\infty}D^{\alpha}_t$ denote the Liouville-Weyl fractional derivatives with $\frac{1}{2} < \alpha < 1$, $L(t)$ is a symmetric and positive definite matrix in $\mathbb{R}^{N\times N}$, $a(t)$ and $b(t)$ are positive bounded functions, $G(u)$ and $H(u)$ are homogeneous functions on $\mathbb{R}^N$, and $h(t)$ is a given function in $\mathbb{R}^N$. Using variational techniques and the Pohozaev fibering method, we establish the existence of infinitely many solutions when $h(t) = 0$, and at least three solutions when $h(t)$ is non-trivial but sufficiently small. These results are novel and extend previous findings in the literature.

1. Introduction

Fractional differential systems have garnered significant attention due to advancements in fractional calculus theory and their application across various fields like physics, chemistry, biology, and control theory, among others, see [2, 7, 10, 15, 20] for example. Very recently, papers [2] and [23] explore advanced mathematical models involving fractional derivatives and non-linear operators. The first paper investigates a generalized singular capillarity system with the $\mathcal{J}$-Hilfer fractional derivative, while the second analyzes a singular generalized Kirchhoff-double-phase problem with the p-Laplacian operator. Both works extend classical models to study complex systems with singularities and non-linear dynamics.

Recent studies have focused on systems incorporating both left and right fractional derivatives, with significant results on the existence and multiplicity of solutions using nonlinear analysis techniques, see [3, 4, 8, 11, 34, 35]. Critical point theory has also proven to be a powerful tool in this context, aiding in the study of differential systems with variational structures, see [13, 22, 24]. For instance in 2013, Torres [29] investigated the fractional Hamiltonian system

$$\begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u)(t) + L(t)u(t) = \nabla W(t, u(t)), & t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R}), \end{cases} \qquad (\mathcal{FHS})$$

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where ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$ denote left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$ respectively, defined over the entire real axis. Here, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix-valued function, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$ represents the gradient of W with respect to its second variable. Torres established the existence of at least one nontrivial solution to $(\mathcal{FHS})$ using the Mountain Pass Theorem, under the following hypotheses on L and W

(L_0) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R},]0, \infty[)$ such that $l(t) \to \infty$ as $|t| \to \infty$ and

$$L(t)x \cdot x \geqslant l(t)\,|x|^2, \ \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(V_1) $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and there exists a constant $\mu > 2$ such that

$$0 < \mu W(t,x) \leqslant \nabla W(t,x) \cdot x, \ \ \forall (t,x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\});$$

(V_2) $|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly with respect to $t \in \mathbb{R}$;
(V_3) there exists $\overline{W} \in C(\mathbb{R}^N, \mathbb{R})$ such that

$$|W(t,x)| + |\nabla W(t,x)| \leqslant \left|\overline{W}(x)\right|, \ \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Here, "$\cdot$" denotes the standard inner product in $\mathbb{R}^N$ and the associated norm is denoted by $|.|$. In the study of the fractional Hamiltonian system $(\mathcal{FHS})$, various authors have investigated the existence and multiplicity of solutions for fractional Hamiltonian system $(\mathcal{FHS})$ using critical point theory and variational methods. Key references include [6, 14, 16, 17, 25–33, 36, 37, 38], along with their cited sources. Previously, the potential W was assumed to satisfy conditions such as superquadratic [25–33], subquadratic [14, 16, 37, 38], or a combination thereof [6, 17, 36]. Recently, in 2019, Chai and Liu [5] examined the following fractional Hamiltonian system:

$$\begin{cases} {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}u)(t) + L(t)u(t) = \lambda u(t) + b(t)\,|u(t)|^{q-2}\,u(t) + \mu h(t), \ \ t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha < 1$, $q > 2$, $b \in C(\mathbb{R},]0, \infty[)$, $h \in C(\mathbb{R}, \mathbb{R}^N)$, λ, μ are parameters. The operator L satisfies condition (L_0). They introduce the assumption

(H_1) $b \in C(\mathbb{R},]0, \infty[) \cap L^{\infty}(\mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R}^N) \cap L^1(\mathbb{R})$ and $q > 2$.

By employing variational methods and the Nehari manifold, they establish the following results:

THEOREM A. *Under hypotheses* (L_0) *and* (H_1), *if* $h = 0$, *there exists a positive constant* λ_0 *such that problem* (1.1) *has at least one nontrivial weak solution for* $-\infty < \lambda < \lambda_0$.

THEOREM B. *Under hypotheses* (L_0) *and* (H_1), *if* $h \neq 0$, *there exists a positive constant* λ_0 *such that problem* (1.1) *has at least two nontrivial weak solutions for* $-\infty < \lambda < \lambda_0$ *and* $0 < \mu < \frac{\mu_\lambda}{2}$, *where* μ_λ *is a constant depending on* λ.

Note that in problem (1.1), the potential $W(t,x)$ takes the form $W(t,x) = G(x) + b(t)H(x) + h(t) \cdot x$ where the maps $G : u \mapsto \frac{\lambda}{2}|u|^2$ and $H : u \mapsto \frac{1}{q}|u|^q$ are positively homogeneous. The objective of this paper is to generalize the previous results by Chai and Liu to cases where G and H are arbitrary homogeneous maps. Specifically, we consider the nonlinearity W in the form:

$$W(t,x) = -a(t)G(x) + b(t)H(x) + h(t)\cdot x.$$

We adopt the following assumptions on a, b, G, H, L

(L) There exists a positive constant r_0 such that $\inf_{t\in\mathbb{R}} \inf_{|\xi|=1} L(t)\xi\cdot\xi > 0$ and

$$\lim_{|s|\to\infty} meas\Big(\{t \in]s-r_0, s+r_0[/L(t) \leqslant MI_N\}\Big) = 0, \ \ \forall M > 0;$$

(W_1) $G, H \in C^1(\mathbb{R}^N,\mathbb{R})$ and there exist two constants ν,μ with $1 < \nu \leqslant \max\{2,\nu\} < \mu$ such that

$$G(sx) = |s|^{\nu}G(x) \ \text{ and } \ H(sx) = |s|^{\mu}H(x), \ \ \forall (s,x)\in\mathbb{R}\times\mathbb{R}^N;$$

(W_2) $G(x) > 0$ and $H(x) > 0$ for all $|x| = 1$;

(W_3) $a \in C(\mathbb{R},\mathbb{R}^+)$ is bounded if $\nu \geqslant 2$ and $a \in L^{\frac{2}{2-\nu}}(\mathbb{R})$ if $1 < \nu < 2$;

(W_4) $b \in C(\mathbb{R},\mathbb{R}^+)$ is bounded.

With these assumptions, we are prepared to state the main results of this paper.

THEOREM 1.1. (Symmetric case) *Let $h = 0$. Assume that (L) and (W_1)–(W_4) hold. Then system $(\mathcal{FHS})$ admits infinitely many nontrivial solutions.*

THEOREM 1.2. (Non symmetric case) *Let $h \in L^{\mu'}(\mathbb{R},\mathbb{R}^N)$ where $\frac{1}{\mu}+\frac{1}{\mu'} = 1$. Suppose that (L) and (W_1)–(W_4) hold. Then there exists a positive constant δ such that, if $\|h\|_{L^{\mu'}} < \delta$, system $(\mathcal{FHS})$ possesses at least three nontrivial solutions.*

2. Preliminaries

2.1. Liouville-Weyl fractional calculus

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as (see [9, 10, 22])

$$_{-\infty}I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-x)^{\alpha-1}u(x)dx, \tag{2.1}$$

and

$$_{t}I_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(x-t)^{\alpha-1}u(x)dx. \tag{2.2}$$

The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [9, 10, 22])

$$_{-\infty}D_t^{\alpha}u(t) = \frac{d}{dt}(_{-\infty}I_t^{1-\alpha}u)(t) \tag{2.3}$$

and

$$ {}_tD_\infty^\alpha u(t) = -\frac{d}{dt}({}_tI_\infty^{1-\alpha}u)(t). \tag{2.4} $$

The Fourier transform of the Liouville-Weyl differential operators satisfies (see [9, 10])

$$ \widehat{{}_{-\infty}D_t^\alpha u}(s) = (is)^\alpha \widehat{u}(s), \tag{2.5} $$

$$ \widehat{{}_tD_\infty^\alpha u}(s) = (-is)^\alpha \widehat{u}(s). \tag{2.6} $$

Here, $\hat{u}$ denotes the Fourier transform of u.

2.2. Fractional derivative spaces

For $0 < \alpha < 1$, define the semi-norm

$$ |u|_\alpha = \left\| |s|^\alpha \widehat{u} \right\|_{L^2} $$

and the norm

$$ \|u\|_\alpha = (\|u\|_{L^2}^2 + |u|_\alpha^2)^{\frac{1}{2}}, $$

and let

$$ H^\alpha(\mathbb{R},\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^N)}^{\|.\|_\alpha}, $$

where

$$ C_0^\infty(\mathbb{R},\mathbb{R}^N) = \left\{ u \in C^\infty(\mathbb{R},\mathbb{R}^N) / \lim_{|t|\to\infty} u(t) = 0 \right\}. $$

It is well known that $H^\alpha(\mathbb{R},\mathbb{R}^N)$ is a reflexive Banach space. Denoting by $C(\mathbb{R},\mathbb{R}^N)$ the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$, we obtain the following Sobolev lemma.

LEMMA 2.1. [16, Theorem 2.1] *If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R},\mathbb{R}^N) \subset C(\mathbb{R},\mathbb{R}^N)$, and there exists a constant $C = C_\alpha$ such that*

$$ \|u\|_{L^\infty} = \sup_{t\in\mathbb{R}} |u(t)| \leqslant C_\alpha \|u\|_\alpha, \quad \forall u \in H^\alpha(\mathbb{R},\mathbb{R}^N). \tag{2.7} $$

REMARK 2.1. From Lemma 2.1, we know that if $u \in H^\alpha(\mathbb{R},\mathbb{R}^N)$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R},\mathbb{R}^N)$ for all $p \in [2,\infty]$, because

$$ \int_{\mathbb{R}} |u(t)|^p \, dt \leqslant \|u\|_{L^\infty}^{p-2} \|u\|_{L^2}^2. $$

Now, we introduce the following fractional space

$$ X^\alpha = \left\{ u \in H^\alpha(\mathbb{R},\mathbb{R}^N) / \int_{\mathbb{R}} L(t)u(t)\cdot u(t)dt < \infty \right\}. $$

Then X^α is a Hilbert space with the inner product

$$ \langle u,v \rangle = \int_{\mathbb{R}} [{}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha v(t) + L(t)u(t)\cdot v(t)]dt $$

and the corresponding norm

$$\|u\|^2 = \langle u, u \rangle .$$

Since $X^\alpha \subset H^\alpha(\mathbb{R}, \mathbb{R}^N)$, then it is easy to see that X^α is continuously embedded into $L^s(\mathbb{R})$ for $2 \leqslant s \leqslant \infty$.

LEMMA 2.2. [14] *Under assumption* (L), *the space* X^α *is compactly embedded in* $L^s(\mathbb{R})$ *for any* $s \in [2, \infty[$. *Moreover, for all* $s \in [2, \infty]$, *there exists a constant* $\eta_s > 0$ *such that*

$$\|u\|_{L^s} \leqslant \eta_s \|u\|, \ \ \forall u \in X^\alpha. \tag{2.8}$$

To prove our results, we will employ the spherical fibering method as introduced by Pohozaev in [18, 19]. For the sake of completeness, we will recall this method here. Consider a real Banach space X with a norm $\|u\|_X$ that is differentiable for $u \neq 0$. Let I be a functional on X of class $C^1(X \setminus \{0\})$. We can associate I with a functional $\widetilde{I}$ defined on $\mathbb{R} \times X$ as follows:

$$\widetilde{I}(t, v) = I(tv), \ \ \forall (t, v) \in \mathbb{R} \times X.$$

Let S denote the unit sphere in X. The following result holds:

LEMMA 2.3. [18, Theorem 1.2.1] *Let* X *be a real Banach space with a norm differentiable on* $X \setminus \{0\}$, *and let* $(t, v) \in (\mathbb{R} \setminus \{0\}) \times S$ *be a conditionally critical point of the functional* $\widetilde{I}$ *considered on* $\mathbb{R} \times S$. *Then the vector* $u = tv$ *is a critical point of the functional* I, *that is,* $I'(u) = 0$. *In other words, any critical point* (t, v) *of* $\widetilde{I}$ *restricted on* $(\mathbb{R} \setminus \{0\}) \times S$ *generates the free nontrivial critical point* u *of* I *and vice-versa, that is, the system* $I'(u) = 0$, $u \neq 0$ *is equivalent to*

$$\begin{cases} \widetilde{I}'_t(t, v) = 0 \\ \widetilde{I}'_v(t, v) = 0 \end{cases}$$

for $\|v\| = 1$. *In the following, we will call the first scalar system of the previous system the "bifurcation system".*

3. Proof of Theorem 1.1

We will proceed by successive lemmas.

LEMMA 3.1. *Let* $K : \mathbb{R}^N \to \mathbb{R}$ *be a function and* $\beta > 0$ *be a constant. We have the following properties:*

a) Equivalence of Homogeneity and Evenness: K *is homogeneous of degree* β *if and only if* K *is even and positively homogeneous of degree* β.

b) Bounds on Positively Homogeneous Functions: If K *is positively homogeneous of degree* β, *then there exist constants* $m_K, M_K \in \mathbb{R}$ *such that*

$$m_K |x|^\beta \leqslant K(x) \leqslant M_K |x|^\beta, \ \ \forall x \in \mathbb{R}^N.$$

c) Derivative of Positively Homogeneous Functions: If K is differentiable and positively homogeneous of degree β, then ∇K is positively homogeneous of degree $\beta-1$. Furthermore, for all $x\in\mathbb{R}^N$, the following identity holds:

$$\nabla K(x)\cdot x=\beta K(x).$$

Proof. a) It suffices to observe that

$$K(-x)=K((-1)x)=|-1|^{\beta}K(x)=K(x).$$

b) For $x\in\mathbb{R}^N\setminus\{0\}$, we have

$$K(x)=K\left(|x|\frac{x}{|x|}\right)=|x|^{\beta}K\left(\frac{x}{|x|}\right).$$

Let $m_K=\min\{K(x)/\,|x|=1\}$ and $M_K=\max\{K(x)/\,|x|=1\}$, we have

$$m_K|x|^{\beta}\leqslant K(x)\leqslant M_K|x|^{\beta},\ \ \forall x\in\mathbb{R}^N.$$

c) For $s>0$, we have for any $y\in\mathbb{R}^N$

$$\begin{aligned}\nabla K(sx)\cdot y&=\lim_{t\to0^+}\frac{K(sx+ty)-K(sx)}{t}\\&=\lim_{t\to0^+}s^{\beta-1}\frac{K(x+\frac{t}{s}y)-K(x)}{\frac{t}{s}}\\&=s^{\beta-1}\nabla K(x)\cdot y.\end{aligned}$$

Since y is arbitrary, then $\nabla K(sx)=s^{\beta-1}\nabla K(x)$. For the equality, differentiate the system

$$K(sx)=s^{\beta}K(x)$$

with respect to s to get

$$\nabla K(sx)\cdot x=\beta s^{\beta-1}K(x).$$

Now, substitute $s=1$ to get the desired result. □

LEMMA 3.2. *Assume that (W_1)–(W_4) are satisfied. Then, we have*
a) If $u_n\to u$ in $L^2(\mathbb{R})$, then $a\nabla G(u_n)\to a\nabla G(u)$ in $L^2(\mathbb{R})$.
b) If $u_n\to u$ in $L^{\mu}(\mathbb{R})$, then $a\nabla H(u_n)\to a\nabla H(u)$ in $L^{\frac{\mu}{\mu-1}}(\mathbb{R})$.

Proof. a) If $u_n\to u$ in $L^2(\mathbb{R})$. We claim that $a\nabla G(u_n)\to a\nabla G(u)$ in $L^2(\mathbb{R})$. Arguing indirectly that there exist a subsequence (u_{n_k}) and a constant $\varepsilon_0>0$ such that

$$\int_{\mathbb{R}}a^2(t)\left|\nabla G(u_{n_k}(t))-\nabla G(u(t))\right|^2dt\geqslant\varepsilon_0,\ \ \forall k\in\mathbb{N}. \tag{3.1}$$

Up to a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty}\left\|u_{n_k}-u\right\|_{L^2}<\infty$ and $u_{n_k}\to u$ almost everywhere on $\mathbb{R}$. Let $v(t)=\sum_{k=1}^{\infty}\left|u_{n_k}(t)-u(t)\right|$, then $v\in L^2(\mathbb{R})$ and we have

$$\begin{aligned}
&a^2(t)\left|\nabla G(u_{n_k}(t))-\nabla G(u(t))\right|^2\\
&\leqslant a^2(t)M_{|\nabla G|}^2\left[\left|u_{n_k}(t)\right|^{\nu-1}+|u(t)|^{\nu-1}\right]^2\\
&\leqslant 2a^2(t)M_{|\nabla G|}^2\left[\left|u_{n_k}(t)\right|^{2(\nu-1)}+|u(t)|^{2(\nu-1)}\right]\\
&\leqslant 2a^2(t)M_{|\nabla G|}^2\left[\left(\left|u_{n_k}(t)-u(t)\right|+|u(t)|\right)^{2(\nu-1)}+|u(t)|^{2(\nu-1)}\right]\\
&\leqslant c_1a^2(t)\left[v^{2(\nu-1)}(t)+|u(t)|^{2(\nu-1)}\right]=w(t),
\end{aligned}$$

where c_1 is a positive constant. By (W_3), $w\in L^1(\mathbb{R})$, hence by the dominated convergence theorem, one gets

$$\int_{\mathbb{R}}a^2(t)\left|\nabla G(u_{n_k}(t))-\nabla G(u(t))\right|^2dt\to 0\ \text{ as }\ k\to\infty,$$

which contradicts (3.1). Hence $a\nabla G(u_n)\to a\nabla G(u)$ in $L^2(\mathbb{R})$.

b) Let $u_n\to u$ in $L^{\mu}(\mathbb{R})$. We claim that $b\nabla H(u_n)\to b\nabla H(u)$ in $L^{\frac{\mu}{\mu-1}}(\mathbb{R})$. Arguing indirectly that there exist a subsequence (u_{n_k}) and a constant $\varepsilon_0>0$ such that

$$\int_{\mathbb{R}}b^{\frac{\mu}{\mu-1}}(t)\left|\nabla H(u_{n_k}(t))-\nabla H(u(t))\right|^{\frac{\mu}{\mu-1}}dt\geqslant\varepsilon_0,\ \ \forall k\in\mathbb{N}. \tag{3.2}$$

Taking a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty}\left\|u_{n_k}-u\right\|_{L^{\mu}}<\infty$ and $u_{n_k}\to u$ almost everywhere on $\mathbb{R}$. Let $v(x)=\sum_1^{\infty}\left|u_{n_k}(t)-u(t)\right|$, then $v\in L^{\mu}(\mathbb{R})$ and we have

$$\begin{aligned}
&b^{\frac{\mu}{\mu-1}}(t)\left|\nabla H(u_{n_k}(t))-\nabla H(u(t))\right|^{\frac{\mu}{\mu-1}}\\
&\leqslant b^{\frac{\mu}{\mu-1}}(t)M_{|\nabla H|}^{\frac{\mu}{\mu-1}}\left[\left|u_{n_k}(t)\right|^{\mu-1}+|u(t)|^{\mu-1}\right]^{\frac{\mu}{\mu-1}}\\
&\leqslant 2^{\frac{1}{\mu-1}}b^{\frac{\mu}{\mu-1}}(t)M_{|\nabla H|}^{\frac{\mu}{\mu-1}}\left[\left|u_{n_k}(t)\right|^{\mu}+|u(t)|^{\mu}\right]\\
&\leqslant 2^{\frac{1}{\mu-1}}b^{\frac{\mu}{\mu-1}}(t)M_{|\nabla H|}^{\frac{\mu}{\mu-1}}\left[\left(\left|u_{n_k}(t)-u(t)\right|+|u(t)|\right)^{\mu}+|u(t)|^{\mu}\right]\\
&\leqslant c_2\left[v^{\mu}+|u(t)|^{\mu}\right]=w(t),
\end{aligned}$$

where c_2 is a positive constant. Since $w\in L^1(\mathbb{R})$, then by the dominated convergence theorem, one gets

$$\int_{\mathbb{R}}b^{\frac{\mu}{\mu-1}}(t)\left|\nabla H(u_{n_k}(t))-\nabla H(u(t))\right|^{\frac{\mu}{\mu-1}}dt\to 0\ \text{ as }\ k\to\infty,$$

which contradicts (3.2). Hence the claim is verified. □

LEMMA 3.3. *Assume that* (W_1)–(W_3) *are satisfied. Then the functionals*

$$I_1:u\mapsto\int_{\mathbb{R}}a(t)G(u(t))dt$$

and

$$I_2 : u \mapsto \int_{\mathbb{R}} b(t)H(u(t))dt$$

are continuously differentiable respectively on $L^2(\mathbb{R})$ *and* $L^\mu(\mathbb{R})$*, and we have*

$$I_1'(u)v = \int_{\mathbb{R}} a(t)\nabla G(u(t)) \cdot v(t)dt, \ \ \forall u, v \in L^2(\mathbb{R})$$

and

$$I_2'(u)v = \int_{\mathbb{R}} b(t)\nabla H(u(t)) \cdot v(t)dt, \ \ \forall u, v \in L^\mu(\mathbb{R}).$$

Proof. a) For $u, v \in L^2(\mathbb{R})$, by the Mean Value Theorem and Hölder's inequality, we have

$$\begin{aligned}
&\left| I_1(u+v) - I_1(u) - \int_{\mathbb{R}} a(x)\nabla G(u(t)) \cdot v(t)dt \right| \\
&= \left| \int_{\mathbb{R}} a(t)\big[G(u(t)+v(t)) - G(u(t)) - \nabla G(u(t)) \cdot v(t)\big]dt \right| \\
&= \left| \int_{\mathbb{R}} a(t)\big[\nabla G(u(t)+\theta(t)v(t)) - \nabla G(u(t))\big] \cdot v(t)dt \right| \\
&\leqslant \Big(\int_{\mathbb{R}} a^2(t)\left|\nabla G(u(t)+\theta(t)v(t)) - \nabla G(u(t))\right|^2 dt \Big)^{\frac{1}{2}} \|v\|_{L^2}
\end{aligned}$$

where $\theta(t) \in]0,1[$. By Lemma 3.2, the functional defined on $L^2(\mathbb{R})$ by

$$v \mapsto \int_{\mathbb{R}} a^2(t)\left|\nabla G(u(t)+v(t)) - \nabla G(u(t))\right|^2 dt$$

goes to zero as $v \to 0$. Hence

$$I_1(u+v) - I_1(u) - \int_{\mathbb{R}} a(t)\nabla G(u(t)) \cdot v(t)dt = o(\|v\|_{L^2}).$$

So I_1 is differentiable on u and $I_1'(u)v = \int_{\mathbb{R}} a(t)\nabla G(u(t)) \cdot v(t)dt$ for all $v \in L^2(\mathbb{R})$. Let us prove that I_1' is continuous. Let $u_n \to u$ in $L^2(\mathbb{R})$, Lemma 3.2 and Hölder's inequality imply

$$\begin{aligned}
\left\|I_1'(u_n) - I_1'(u)\right\| &= \sup_{\|v\|_{L^2}=1} \left|I_1'(u_n)v - I_1'(u)v\right| \\
&= \sup_{\|v\|_{L^2}=1} \left| \int_{\mathbb{R}} a(t)\big[\nabla G(u_n(t)) - \nabla G(u(t))\big] \cdot v(t)dt \right| \\
&\leqslant \Big(\int_{\mathbb{R}} a^2(t)\left|\nabla G(u_n(t)) - \nabla G(u(t))\right|^2 dt \Big)^{\frac{1}{2}} \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Hence I_1 is continuously differentiable on $L^2(\mathbb{R})$.

b) For $u,v\in L^{\mu}(\mathbb{R})$, by the Mean Value Theorem and Hölder's inequality, we have

$$\begin{aligned}&\left|I_2(u+v)-I_2(u)-\int_{\mathbb{R}} b(t)\nabla H(u(t))\cdot v(t)dt\right|\\&=\left|\int_{\mathbb{R}} b(t)\big[H(u(t)+v(t))-H(u(t))-\nabla H(u(t))\cdot v(t)\big]dt\right|\\&=\left|\int_{\mathbb{R}} b(t)\big[\nabla H(u(t)+\theta(t)v(t))-\nabla H(u(t))\big]\cdot v(t)dt\right|\\&\leqslant M_b\Big(\int_{\mathbb{R}}|\nabla H(u(t)+\theta(t)v(t))-\nabla H(u(t))|^{\frac{\mu}{\mu-1}}dt\Big)^{\frac{\mu-1}{\mu}}\|v\|_{L^{\mu}},\end{aligned}$$

where $\theta(t)\in]0,1[$. By Lemma 3.2, the functional defined on $L^{\mu}(\mathbb{R})$ by

$$v\mapsto\int_{\mathbb{R}}|\nabla H(u(t)+v(t))-\nabla H(u(t))|^{\frac{\mu}{\mu-1}}dt$$

goes to zero as $v\to 0$. Hence

$$I_2(u+v)-I_2(u)-\int_{\mathbb{R}} b(t)\nabla H(u(t))\cdot v(t)dt=o(\|v\|_{L^{\mu}}).$$

So I_2 is differentiable on $L^{\mu}(\mathbb{R})$ and as above, we prove that $I_2\in C^1(L^{\mu}(\mathbb{R}))$. The proof of Lemma 3.3 is completed. □

REMARK 3.1. Using Lemmas 2.2, 3.3, it is easy to see that I_1 and I_2 are continuously differentiable on X^{α}, where X^{α} is defined in Section 2.

Associated to system $(\mathcal{FHS})$, is the energy functional $J_0:X^{\alpha}\to\mathbb{R}$ defined by

$$\begin{aligned}J_0(u)&=\frac{1}{2}\int_{\mathbb{R}}\big[|{}_{-\infty}D_t^{\alpha}u(t)|^2+L(t)u(t)\cdot u(t)\big]dt-\int_{\mathbb{R}}W(t,u(t))dt\\&=\frac{1}{2}\|u\|^2+\int_{\mathbb{R}}a(t)G(u(t))dt-\int_{\mathbb{R}}b(t)H(u(t))dt.\end{aligned}$$

From Remark 3.1, it is clear that J_0 is continuously differentiable on X^{α} with derivative

$$\begin{aligned}J_0'(u)v=&\int_{\mathbb{R}}\Big[{}_{-\infty}D_t^{\alpha}u(t)\cdot{}_{-\infty}D_t^{\alpha}v(t)+L(t)u(t)\cdot v(t)\Big]dt\\&+\int_{\mathbb{R}}\big[a(t)\nabla G(u(t))-b(t)\nabla H(u(t))\big]\cdot v(t)dt\end{aligned}$$

for all $u,v\in X^{\alpha}$. Moreover, the critical points of J_0 on X^{α} correspond to the solutions of system $(\mathcal{FHS})$.

According to the spherical fibering method, we look for critical points $u\in X^{\alpha}$ of the functional J_0 of the type

$$u=sv\ \text{ where }\ s\in\mathbb{R},\ \ v\in X^{\alpha},\ \ \|v\|=1.$$

Therefore the functional J_0 can be extended to the space $\mathbb{R}\times X^{\alpha}$ by setting

$$\begin{aligned}\widetilde{J_0}(s,v)=J_0(sv)&=\frac{s^2}{2}\|v\|^2+\int_{\mathbb{R}}a(t)G(sv(t))dt-\int_{\mathbb{R}}b(t)H(sv(t))dt\\&=\frac{s^2}{2}\|v\|^2+|s|^{\nu}\int_{\mathbb{R}}a(t)G(v(t))dt-|s|^{\mu}\int_{\mathbb{R}}b(t)H(v(t))dt\end{aligned}$$

for $(s,v)\in\mathbb{R}\times X^{\alpha}$. Thus, the restriction of $\widetilde{J_0}$ on $\mathbb{R}\times S$, with $S=\{v\in X^{\alpha}/\ \|v\|=1\}$ becomes

$$\widetilde{J_0}(s,v)=\frac{s^2}{2}+|s|^{\nu}\int_{\mathbb{R}}a(t)G(v(t))dt-|s|^{\mu}\int_{\mathbb{R}}b(t)H(v(t))dt$$

for $(s,v)\in\mathbb{R}\times S$. Hence, if $s\neq0$, the bifurcation system $\frac{\partial\widetilde{J_0}}{\partial s}(s,v)=0$ takes the form

$$s+\nu|s|^{\nu-2}s\int_{\mathbb{R}}a(t)G(v(t))dt-\mu|s|^{\mu-2}s\int_{\mathbb{R}}b(t)H(v(t))dt=0$$

which is equivalent to

$$1+\nu|s|^{\nu-2}\int_{\mathbb{R}}a(t)G(v(t))dt-\mu|s|^{\mu-2}\int_{\mathbb{R}}b(t)H(v(t))dt=0. \tag{3.3}$$

LEMMA 3.4. *For any $v\in\widetilde{X}=L^2(\mathbb{R})\cap L^{\mu}(\mathbb{R})$, the function*

$$\varphi_v(s)=1+\nu|s|^{\nu-2}\int_{\mathbb{R}}a(t)G(v(t))dt-\mu|s|^{\mu-2}\int_{\mathbb{R}}b(t)H(v(t))dt$$

possesses exactly two zeros $\pm s(v)$. Moreover, the functional $v\mapsto s(v)$ is continuously differentiable on $\widetilde{X}$.

Proof. Since $1<\nu\leqslant\max\{2,\nu\}<\mu$, then it is clear that

$$\lim_{|s|\to\infty}\varphi_v(s)=-\infty.$$

Moreover

$$\lim_{s\to0^+}\varphi_v(s)=\begin{cases}+\infty, & \text{if } 1<\nu<2\\ 1+\nu\int_{\mathbb{R}}a(t)G(v(t))dt, & \text{if } \nu=2\\ 1, & \text{if } \nu>2.\end{cases}$$

Since φ_v is continuous, we deduce by the Mean Value Theorem that φ_v has at least two zeros. It remains to prove that φ_v has exactly two zeros. Indeed, for $s\neq0$, we have

$$\varphi_v'(s)=\nu(\nu-2)|s|^{\nu-4}s\int_{\mathbb{R}}a(t)G(v(t))dt-\mu(\mu-2)|s|^{\mu-4}s\int_{\mathbb{R}}b(t)H(v(t))dt.$$

We discuss two cases.

a) *First case*: $1 < \nu \leqslant 2$, φ_v' does not admit zeros. Hence φ_v has exactly two zeros: $\pm s(v)$.

b) *Second case*: $\nu > 2$. In this case, φ_v' possesses two zeros $\pm \overline{s}(v)$ with

$$\overline{s}(v) = \Big(\frac{\nu(\nu-2)\int_{\mathbb{R}} a(t)G(v(t))dt}{\mu(\mu-2)\int_{\mathbb{R}} b(t)H(v(t))dt}\Big)^{\frac{1}{\mu-\nu}}.$$

Using the variation table, we see directly that φ_v admits exactly two zeros $\pm s(v)$ with $s(v) > \overline{s}(v)$.

Now, we shall prove that the functional $v \mapsto s(v)$ obtained above is continuously differentiable on $\widetilde{X}$. Let $v_0 \in \widetilde{X}$, we consider the functional $\Phi : \widetilde{X} \times I \to \mathbb{R}$ defined by $\Phi(v,s) = \varphi_v(s)$, where $I = \mathbb{R}_+^*$ if $1 < \nu \leqslant 2$ and $]\overline{s}(v), +\infty[$ if $\nu > 2$. Lemma 3.3 implies that Φ is continuously differentiable on $\widetilde{X} \times I$. Moreover, we have $\Phi(v_0, s(v_0)) = 0$ and $\frac{\partial \Phi}{\partial s}(v_0, s(v_0)) \neq 0$. By the Implicit Function Theorem, there exist an open neighborhood $V \subset \widetilde{X}$ of v_0 and a unique $\theta : V \to \mathbb{R}$ which is continuously differentiable such that

$$\Phi(v, \theta(v)) = 0, \ \ \forall v \in V.$$

By the uniqueness of s and θ, we deduce that $s = \theta$ on V, so s is continuously differentiable on V and in particular at v_0. Since v_0 is arbitrary, then s is continuously differentiable on $\widetilde{X}$.

Next, consider the functional $\widehat{J_0}$ defined on $\widetilde{X}$ by

$$\widehat{J_0}(v) = \frac{1}{2}s^2(v) + |s(v)|^\nu \int_{\mathbb{R}} a(t)G(v(t))dt - |s(v)|^\mu \int_{\mathbb{R}} b(t)H(v(t))dt$$

for $v \in \widetilde{X}$. We deduce from Lemmas 3.3, 3.4 that $\widehat{J_0}(v) = \widetilde{J_0}(s(v), v)$ on S. From system $\varphi_v(s(v)) = 0$, we deduce that for all $v \in S$

$$\widehat{J_0}(v) = \left(\frac{1}{2} - \frac{1}{\mu}\right) s^2(v) + \nu \left(\frac{1}{\nu} - \frac{1}{\mu}\right) |s(v)|^\nu \int_{\mathbb{R}} a(t)G(v(t))dt.$$

Since $1 < \nu \leqslant \max\{2, \nu\} < \mu$, $b \geqslant 0$ and $G \geqslant 0$, then $\widehat{J_0}$ is bounded from below on S as the sum of two non-negative terms. Let $(v_n) \subset S$ be such that $\widehat{J_0}(v_n) \to \inf_{v \in S} \widehat{J_0}(v)$. Since (v_n) is bounded, then up to a subsequence, we can assume that $v_n \rightharpoonup \overline{v}$ weakly in X^α. By Lemma 2.2, we can assume after going to a subsequence, that $v_n \to \overline{v}$ in both $L^2(\mathbb{R})$ and $L^\mu(\mathbb{R})$. By Lemmas 3.3, 3.4, $\widehat{J_0}$ is continuous on $\widetilde{X}$, then $\widehat{J_0}(v_n) \to \widehat{J_0}(\overline{v})$. Thus $\widehat{J_0}$ attains its minimum on S at a point $\overline{v}$ with $\|\overline{v}\| \leqslant 1$. It remains to prove that $\overline{v} \in S$. Indeed, by using system $\varphi_v(s(v)) = 0$, we get for all $v \in S$ and $\xi \in [0,1]$

$$\begin{aligned}&\frac{d}{d\xi}(\widehat{J_0}(\xi v)) \\ &= \frac{d}{d\xi}\Big[\frac{1}{2}s^2(\xi v) + |s(\xi v)|^\nu \int_{\mathbb{R}} a(t)G(\xi v(t))dt - |s(\xi v)|^\mu \int_{\mathbb{R}} b(t)H(\xi v(t))dt\Big]\end{aligned}$$

$$
\begin{aligned}
&= s(\xi v)\frac{d}{d\xi}(s(\xi v)) + \nu\,|s(\xi v)|^{\nu-2} s(\xi v)\frac{d}{d\xi}(s(\xi v))\int_{\mathbb{R}} a(t)G(\xi v(t))dt \\
&\quad - \mu\,|s(\xi v)|^{\mu-2} s(\xi v)\frac{d}{d\xi}(s(\xi v))\int_{\mathbb{R}} b(t)H(\xi v(t))dt \\
&\quad + |s(\xi v)|^{\nu}\int_{\mathbb{R}} a(t)\nabla G(\xi v(t))\cdot v(t)dt \\
&\quad - |s(\xi v)|^{\mu}\int_{\mathbb{R}} b(t)\nabla H(\xi v(t))\cdot v(t)dt \\
&= s(\xi v)\frac{d}{d\xi}(s(\xi v))\Big[1 + \nu\,|s(\xi v)|^{\nu-2}\int_{\mathbb{R}} a(t)G(\xi v(t))dt \\
&\quad - \mu\,|s(\xi v)|^{\mu-2}\int_{\mathbb{R}} b(t)H(\xi v(t))dt\Big] \\
&\quad + \nu\frac{|s(\xi v)|^{\nu}}{\xi}\int_{\mathbb{R}} a(t)G(\xi v(t))dt - \mu\frac{|s(\xi v)|^{\mu}}{\xi}\int_{\mathbb{R}} b(t)H(\xi v(t))dt \\
&= \frac{|s(\xi v)|^2}{\xi}\Big[\nu\,|s(\xi v)|^{\nu-2}\int_{\mathbb{R}} a(t)G(\xi v(t))dt - \mu\,|s(\xi v)|^{\mu-2}\int_{\mathbb{R}} b(t)H(\xi v(t))dt\Big] \\
&= -\frac{|s(\xi v)|^2}{\xi} < 0.
\end{aligned}
$$

Thus, $\widehat{J_0}(\xi v)$ decreases with respect to $\xi \in [0,1]$ and reaches its minimum at $\xi = 1$. This implies that $\widehat{J_0}$ attains its minimum on S at $\overline{v} \in S$. According to the spherical fibering method, we obtain that $\pm s(\overline{v})\overline{v}$ are two solutions of problem $(\mathcal{FHS})$. $\square$

Given that $\widehat{J_0}$ is an even function, bounded from below, weakly continuous, and of class C^1 on S, the Lusternik-Schnirelmann theory (as discussed in [12]) ensures that $\widehat{J_0}$ has a sequence of conditionally critical points $(v_n)_{n\in\mathbb{N}} \subset S$ such that $\widehat{J_0}(v_n) \to +\infty$ as $n \to \infty$. By applying Lemma 2.3, we deduce that when $h = 0$, the system $(\mathcal{FHS})$ possesses a sequence of distinct solutions $(\pm u_n)_{n\in\mathbb{N}}$, where $u_n = s(v_n)v_n$ and $J_0(v_n) \to +\infty$ as $n \to \infty$.

4. Proof of Theorem 1.2

In the nonsymmetric case $h \neq 0$, the energy functional $J_h : X^\alpha \to \mathbb{R}$ is defined by

$$
\begin{aligned}
J_h(u) &= \frac{1}{2}\int_{\mathbb{R}}\big[|{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t)\cdot u(t)\big]dt - \int_{\mathbb{R}} W(t,u(t))dt - \int_{\mathbb{R}} h(t)\cdot u(t)dt \\
&= \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}} a(t)G(u(t))dt - \int_{\mathbb{R}} b(t)H(u(t))dt - \int_{\mathbb{R}} h(t)\cdot u(t)dt \\
&= J_0(u) - \int_{\mathbb{R}} h(t)\cdot u(t)dt
\end{aligned}
$$

for $u \in X^\alpha$. According to the spherical fibering method, we look for critical points $u \in X^\alpha$ of the functional J_h of the type

$$u = sv \ \text{ where } \ s \in \mathbb{R}, \ \ v \in X^\alpha, \ \ \|v\| = 1.$$

So, the energy functional J_h extended to $\mathbb{R}\times X^{\alpha}$ becomes:

$$\begin{aligned}\widetilde{J}_h(s,v) &= \widetilde{J}_0(s,v) - s\int_{\mathbb{R}} h(t)\cdot v(t)dt \\ &= \frac{s^2}{2}\|v\|^2 + |s|^{\nu}\int_{\mathbb{R}} a(t)G(v(t))dt \\ &\quad - |s|^{\mu}\int_{\mathbb{R}} b(t)H(v(t))dt - s\int_{\mathbb{R}} h(t)\cdot v(t)dt\end{aligned}$$

for $(s,v)\in\mathbb{R}\times X^{\alpha}$, and its restriction to $\mathbb{R}\times S$ is

$$\widetilde{J}_h(s,v) = \frac{s^2}{2} + |s|^{\nu}\int_{\mathbb{R}} a(t)G(v(t))dt - |s|^{\mu}\int_{\mathbb{R}} b(t)H(v(t))dt - s\int_{\mathbb{R}} h(t)\cdot v(t)dt$$

for $(s,v)\in\mathbb{R}\times S$. The bifurcation system $\frac{\partial \widetilde{J}_0}{\partial s}(s,v)=0$ involves

$$s+\nu|s|^{\nu-2}s\int_{\mathbb{R}} a(t)G(v(t))dt - \mu|s|^{\mu-2}s\int_{\mathbb{R}} b(t)H(v(t))dt = \int_{\mathbb{R}} h(t)\cdot v(t)dt. \quad (4.1)$$

Let $\psi_v:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$\psi_v(s) = s+\nu|s|^{\nu-2}s\int_{\mathbb{R}} a(t)G(v(t))dt - \mu|s|^{\mu-2}s\int_{\mathbb{R}} b(t)H(v(t))dt.$$

LEMMA 4.1. *For every $v\in S$, the function ψ_v is odd, and it has a local minimum m_v and a local maximum M_v such that $M_v=-m_v$.*

Proof. By derivation, we have

$$\psi_v'(s) = 1+\nu(\nu-1)|s|^{\nu-3}s\int_{\mathbb{R}} a(t)G(v(t))dt - \mu(\mu-1)|s|^{\mu-3}s\int_{\mathbb{R}} b(t)H(v(t))dt.$$

Since $1<\nu<\mu$, then we have $\lim_{s\to\infty}\psi_v'(s)=-\infty$ and

$$\lim_{s\to 0^+}\psi_v'(s) = \begin{cases} +\infty, & \text{if } 1<\nu<2 \\ 1+\nu(\nu-1)\int_{\mathbb{R}} a(t)G(v(t))dt, & \text{if } \nu=2 \\ 1, & \text{if } \nu>2. \end{cases}$$

It results that the function ψ_v' possesses at least two zeros. On the other hand, we have

$$\begin{aligned}\psi_v''(s) &= \nu(\nu-1)(\nu-2)|s|^{\nu-4}s\int_{\mathbb{R}} a(t)G(v(t))dt \\ &\quad -\mu(\mu-1)(\mu-2)|s|^{\mu-4}s\int_{\mathbb{R}} b(t)H(v(t))dt.\end{aligned}$$

If $1<\nu\leqslant 2$, we have $\psi_v''(s)<0$ if $s>0$. Moreover if $\nu>2$, it is clear that

$$\psi_v''(s)=0 \to \bar{s}(v) = \left(\frac{\nu(\nu-1)(\nu-2)\int_{\mathbb{R}} a(t)G(v(t))dt}{\mu(\mu-1)(\mu-2)\int_{\mathbb{R}} b(t)H(v(t))dt}\right)^{\frac{1}{\mu-\nu}}.$$

In both cases, ψ_v' has precisely two zeros at $\pm s(v)$. Similarly to Lemma 3.4, we can demonstrate that $s(v)$ is continuously differentiable on $\widetilde{X}=L^2(\mathbb{R})\cap L^{\mu}(\mathbb{R})$. Since $\lim s\to\infty\psi_v'(s)=-\infty$ and ψ_v is an odd function, ψ_v must have a local minimum m_v and a local maximum M_v, where $M_v=-m_v$. □

LEMMA 4.2. *If* $\|h\|_{L^{\mu'}} < (\mu-2)\Big[\mu(\mu-1)^{\mu-1}\eta_\mu^{2(\mu-1)}M_bM_H\Big]^{-\frac{1}{\mu-2}}$*, where* μ' *is the conjugate exponent of* μ*, then system*

$$\psi_v(s) = \int_{\mathbb{R}} h(t)\cdot v(t)dt$$

has three distinct solutions.

Proof. Set

$$\overline{\psi}_v(s) = s - \mu\,|s|^{\mu-2}\,s\int_{\mathbb{R}} b(t)H(v(t))dt.$$

We have $\psi_v \geqslant \overline{\psi}_v$ and direct calculations show that, denoting by $\overline{M}_v$ the local maximum of $\overline{\psi}_v$, it is

$$\begin{aligned}\overline{M}_v &= \overline{\psi}_v\Big(\Big[\mu(\mu-1)\int_{\mathbb{R}} b(t)H(v(t))dt\Big]^{-\frac{1}{\mu-2}}\Big)\\ &= (\mu-2)\Big[\mu(\mu-1)^{\mu-1}\int_{\mathbb{R}} b(t)H(v(t))dt\Big]^{-\frac{1}{\mu-2}}.\end{aligned} \tag{4.2}$$

Note that Hölder's inequality implies

$$\begin{aligned}&\left|\int_{\mathbb{R}} h(t)\cdot v(t)dt\right|\Big(\int_{\mathbb{R}} b(t)H(v(t))dt\Big)^{\frac{1}{\mu-2}}\\ &\leqslant \|h\|_{L^{\mu'}}\,\|v\|_{L^\mu}\,\big(M_bM_H\,\|v\|_{L^\mu}^{\mu}\,\big)^{\frac{1}{\mu-2}}\\ &\leqslant \|h\|_{L^{\mu'}}\,\big(M_bM_H\eta_\mu^{2(\mu-1)}\big)^{\frac{1}{\mu-2}}.\end{aligned}$$

If we take $\|h\|_{L^{\mu'}} < (\mu-2)\Big[\mu(\mu-1)^{\mu-1}\eta_\mu^{2(\mu-1)}M_bM_H\Big]^{-\frac{1}{\mu-2}}$, then Hölder's inequality implies

$$\sup_{v\in S}\left[\left|\int_{\mathbb{R}} h(t)\cdot v(t)dt\right|\Big(\int_{\mathbb{R}} b(t)H(v(t))dt\Big)^{\frac{1}{\mu-2}}\right] < (\mu-2)\big[\mu(\mu-1)^{(\mu-1)}\big]^{\frac{1}{\mu-2}}. \tag{4.3}$$

Since $M_v \geqslant \overline{M}_v$, then (4.2), (4.3) imply

$$\left|\int_{\mathbb{R}} h(t)\cdot v(t)dt\right| < \overline{M}_v \leqslant M_v$$

which implies that system (4.1) has three distinct solutions. □

Given the bifurcation system (4.1) with three distinct solutions $s_i(v),\ i=1,2,3,$ we consider the three induced functionals

$$\begin{aligned}\widehat{J}_{h,i}(v) = \frac{1}{2}\,|s_i(v)|^2 + |s_i(v)|^\nu\int_{\mathbb{R}} a(t)G(v(t))dt - |s_i(v)|^\mu\int_{\mathbb{R}} b(t)H(v(t))dt\\ - s_i(v)\int_{\mathbb{R}} h(t)\cdot v(t)dt\end{aligned}$$

which are defined and distinct on $B\setminus\{0\}$, where $B=\{v\in X^{\alpha}/\,\|v\|\leqslant 1\}$. Using Hölder's inequality and the properties of the bifurcation system (4.1), we obtain

$$\begin{aligned}\widehat{J}_{h,i}(v) &= \widehat{J}_{0,i}(v)-s_i\int_{\mathbb{R}} h(t)\cdot v(t)dt\\ &\geqslant \left(\frac{1}{2}-\frac{1}{\mu}\right)s_i^2(v)+\left(1-\frac{\nu}{\mu}\right)|s_i(v)|^{\nu}\int_{\mathbb{R}} a(t)G(v(t))dt\\ &\quad+\left(\frac{1}{\mu}-1\right)\eta_{\mu}\,\|h\|_{L^{\mu'}}\,|s_i(v)|.\end{aligned}$$

Since $\max\{2,\nu\}<\mu$, it follows that $\widehat{J}_{h,i}$ is bounded from below. By applying Lemmas 3.3 and 3.4, we know that $\widehat{J}_{h,i}$ is continuously differentiable on the space $\widetilde{X}=L^2(\mathbb{R})\cap L^{\mu}(\mathbb{R})$. Combining this with Lemma 2.2, we conclude that $\widehat{J}_{h,i}$ is weakly continuous on the space X^{α}. Therefore, $\widehat{J}_{h,i}$ attains its minimum on the set S at some point $\overline{v}_i\in B$ where $s_i(\overline{v}_i)\neq 0$. What remains to be shown is that $\overline{v}_i\in S$. Indeed, by utilizing equation (4.1), we can demonstrate for any $v\in S$ and $\xi\in[0,1]$ that

$$\begin{aligned}\frac{d}{d\xi}(\widehat{J}_{h,i}(\xi v)) &= \frac{d}{d\xi}\Big[\frac{1}{2}s_i^2(\xi v)+|s_i(\xi v)|^{\nu}\int_{\mathbb{R}} a(t)G(\xi v(t))dt\\ &\quad-|s_i(\xi v)|^{\mu}\int_{\mathbb{R}} b(t)H(\xi v(t))dt-s_i(\xi v)\int_{\mathbb{R}} h(t)\cdot\xi v(t)dt\Big]\\ &= s_i(\xi v)\frac{d}{d\xi}(s_i(\xi v))+\nu\,|s_i(\xi v)|^{\nu-2}s_i(\xi v)\frac{d}{d\xi}(s_i(\xi v))\int_{\mathbb{R}} a(t)G(\xi v(t))dt\\ &\quad-\mu\,|s_i(\xi v)|^{\mu-2}s_i(\xi v)\frac{d}{d\xi}(s_i(\xi v))\int_{\mathbb{R}} b(t)H(\xi v(t))dt\\ &\quad+|s_i(\xi v)|^{\nu}\int_{\mathbb{R}} a(t)\nabla G(\xi v(t))\cdot v(t)dt\\ &\quad-|s_i(\xi v)|^{\mu}\int_{\mathbb{R}} b(t)\nabla H(\xi v(t))\cdot v(t)dt\\ &\quad-s_i(\xi v)\int_{\mathbb{R}} h(t)\cdot v(t)dt-\frac{d}{d\xi}(s_i(\xi v))\int_{\mathbb{R}} h(t)\cdot\xi v(t)dt\\ &= \frac{d}{d\xi}(s_i(\xi v))\Big[s_i(\xi v)+\nu\,|s_i(\xi v)|^{\nu-2}s_i(\xi v)\int_{\mathbb{R}} a(t)G(\xi v(t))dt\\ &\quad-\mu\,|s_i(\xi v)|^{\mu-2}s_i(\xi v)\int_{\mathbb{R}} b(t)H(\xi v(t))dt-\int_{\mathbb{R}} h(t)\cdot\xi v(t)dt\Big]\\ &\quad+\frac{|s_i(\xi v)|^{\nu}}{\xi}\int_{\mathbb{R}} a(t)G(\xi v(t))dt-\frac{|s_i(\xi v)|^{\mu}}{\xi}\int_{\mathbb{R}} b(t)H(\xi v(t))dt\\ &\quad-s_i(\xi v)\int_{\mathbb{R}} h(t)\cdot v(t)dt\\ &= \nu\frac{|s_i(\xi v)|^{\nu}}{\xi}\int_{\mathbb{R}} a(t)G(\xi v(t))dt\\ &\quad-\mu\frac{|s_i(\xi v)|^{\mu}}{\xi}\int_{\mathbb{R}} b(t)H(\xi v(t))dt-s_i(\xi v)\int_{\mathbb{R}} h(t)\cdot v(t)dt\end{aligned}$$

$$= \frac{s_i(\xi v)}{\xi}\Big[v\,|s_i(\xi v)|^{v-2}s_i(\xi v)\int_{\mathbb{R}} a(t)G(\xi v(t))dt$$
$$-\mu\,|s_i(\xi v)|^{\mu-2}s_i(\xi v)\int_{\mathbb{R}} b(t)H(\xi v(t))dt - \int_{\mathbb{R}} h(t)\cdot\xi v(t)dt\Big]$$
$$= -\frac{|s_i(\xi v)|^2}{\xi} < 0.$$

As ξ varies over $[0,1]$, $\widehat{J}_{h,i}(\xi v)$ decreases, reaching its minimum when $\xi = 1$. This minimum occurs at $\overline{v}_i \in S$, indicating that $\widehat{J}_{h,i}$ achieves its minimum on S at $\overline{v}_i$. According to the spherical fibering method, the solutions $\pm s_i(\overline{v}_i)\overline{v}$ represent three solutions of problem $(\mathcal{FHS})$.

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