Fractional Differential Calculus
Volume 15, Number 2 (2025), 323–339 doi:10.7153/fdc-2025-15-19

# LINKING AND EXISTENCE RESULT FOR THE FRACTIONAL $p$–LAPLACIAN PROBLEMS INVOLVING SINGULAR NONLINEARITY

MOHAMED LOUCHAICH

(*Communicated by R. Ashurov*)

*Abstract.* The purpose of the work is to investigate whether solutions exist for a certain class of fractional non-linear equations that are non-local and feature both singular and subcritical nonlinearities. The equation is given as follow

$$(\mathscr{P}_\lambda)\quad \begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u + \dfrac{\eta}{u^\delta} + \beta(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Here $\Omega$ is a bounded domain of $\mathbb{R}^N$ with $N \geqslant 2$ and Lipschitz boundary $\partial\Omega$, $\lambda, \eta > 0$ are two a real parameters, $(-\Delta_p)^s$ represents the fractional $p$-Laplacian operator with $s \in (0,1)$ and $p > 1$ satisfies $sp < N$, $q \in (p, p_s^\star)$. $\beta : \Omega \to \mathbb{R}$ is a bounded function, $\delta$ is a positive real number, satisfying $\delta \in (0,1)$. The study makes use of variational methods to prove that solutions exist. The author uses some abstract linking theorem based on the $\mathscr{Z}_2$-cohomological index to determine the critical points of a suitable functional that is related to the equation. The paper shows that the equation has at least one nontrivial solution for any positive value of the parameter $\lambda$.

## 1. Introduction

This work is concerned with the existence of weak solutions of the following non-local problem:

$$(\mathscr{P}_\lambda)\quad \begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u + \dfrac{\eta}{u^\delta} + \beta(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega$ is a smooth bounded subset of $\mathbb{R}^N$, $N > 1$, with Lipschitz boundary $\partial\Omega$, $\lambda, \eta > 0$ are two a real parameters, $\delta \in (0,1)$, $q \in (p, p_s^\star)$, $\beta : \Omega \to \mathbb{R}$ is a bounded function such that there exist $\beta_0, \beta_\infty > 0$ such that $\beta_0 \leqslant \beta(x) \leqslant \beta_\infty$ a.e. $x \in \Omega$. Here

*Mathematics subject classification* (2020): 35D30, 35J60, 35J75, 35R11, 46E35.
*Keywords and phrases*: Fractional $p$-Laplacian, fractional Sobolev space, variational methods, linking over cones, cohomological index.

$(-\Delta_p)^s$, $s \in (0,1)$ is the fractional $p$-Laplacian operator defined for every smooth function $\varphi \in C_0^\infty(\mathbb{R}^N)$ by

$$(-\Delta_p)^s \varphi(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dy, \quad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius $\varepsilon > 0$ at the center $x \in \mathbb{R}^N$. When $p = 2$, $(-\Delta)_p^s$ reduces to the fractional Laplacian operator $(-\Delta)^s$ which (up to normalization factors) may be defined as

$$(-\Delta)^s \varphi(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{\varphi(x+y) + \varphi(x-y) - 2\varphi(y))}{|y|^{N+2s}}\, dy,$$

for $x \in \mathbb{R}^N$ (see [6] and references therein for further details on the fractional Laplacian and on the fractional Sobolev space $H^s(\mathbb{R}^N)$). In this case, our problem becomes as follows

$$\begin{cases} (-\Delta)^s u = \lambda u + \dfrac{\eta}{u^\delta} + \beta(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}$$

We call that $\lambda \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p)^s$ in $\Omega$ if the problem

$$\begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

has a nontrivial weak solution. Define

$$\sigma_{s,p} = \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue}\}. \tag{1.3}$$

The eigenvalue problem (1.2) was first introduced by Lindgren and Lindqvist in [16] and considered by several authors afterwards, we cite for example [12, 15]. For all $\lambda \in \sigma_{s,p}$, the set of $\lambda$-eigenfunctions is called $\lambda$-eigenspace. Clearly, $\sigma_{s,p} \subset \mathbb{R}+$ and all eigenspaces are star-shaped sets, as both sides of (1.2) are $(p-1)$-homogeneous. Next, we recall some properties of $\sigma_{s,p}$ (see [16]).

- $\sigma_{s,p}$ is closed set,
- $\lambda_1 = \min \sigma_{s,p} > 0$ is simple, isolated, and has an associated eigenfunction $e_1$ that is positive in $\Omega$,
- for all $\lambda \in \sigma_{s,p}$ with $\lambda > \lambda_1$, any $\lambda$-eigenfunction $e_\lambda$ is sign-changing in $\Omega$,
- if $\Omega$ is a ball, then any positive (resp. negative) $\lambda_1$-eigenfunction is radially symmetric and radially decreasing (resp. increasing).

In literature, when $\eta = 0$ and $\lambda \in (0, \lambda_1)$, elliptic equations of type $(\mathscr{P}_\lambda)$ have been extensively studied by many authors; see for exemple [2–4, 10, 13, 18, 20, 21, 25] and references therein. When $\lambda \geqslant \lambda_1$, $\eta = 0$ and $p = 2$, Servadei discussed problem (1.1) via the Linking Theorem; see [22] for more details, see also [23, 24] in the case $q = 2_s^\star$. The classical proof is based on the fact that each eigenvalue $\lambda_n$, $n \in \mathbb{N}$ of the fractional Laplacian $(-\Delta)^s$ induces a suitable direct sum decomposition of the space $H_0^s(\Omega)$; see ([21], section 3). These arguments do not extend to the fractional p-Laplacian, which is a nonlinear operator and hence lacks linear eigenspaces. However, a linking argument over cones, rather than over linear subspaces, has been firstly developed in the local case, namely $s = 1$, by Fan and Li (see [7]) for $\lambda$ near to $\lambda_1$ and by Degiovanni and Lancelotti (see [5]) for any $\lambda > 0$.

Later, in [14], Iannizzotto et al. considered the following problem

$$\begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

By the help of Morse theory and the spectral properties of the operator $(-\Delta)_p^s$, they proved the existence of a nonzero solution for problem (1.4) for all $\lambda \in \mathbb{R}$. They treated, respectively, the cases where $f$ is $p$-superlinear, $p$-sublinear or asymptotically $p$-linear. Using the same tools, the authors in [13] extended the above results to

$$f(x,u) = \lambda h(x)|u|^{p-2}u + k(x)|u|^{r-2}u + g(x,u),$$

where $h$ and $k$ are two measurable functions belong to a class of singular weights (for more details see [13]).

Motivated by the papers mentioned above, we aim to investigate the existence of solutions for the problem $(\mathscr{P}_\lambda)$ for any $\lambda > 0$ using variational techniques and critical point theory. To present the main findings, we introduce some notation. Let us define the space

$$X = \left\{ u \in L^p(\mathbb{R}^N) \mid \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy < \infty \right\},$$

endowed with the norm

$$\|u\|_X = |u|_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy.$$

We also define the space

$$X_0 = \left\{ u \in X \text{ satisfying } u(x) = 0 \text{ a.e. } x \text{ in } \mathbb{R}^N \setminus \Omega \right\},$$

where $\Omega$ is a given domain. According to Theorem A.3 in [17], the space $X_0$ is a separable and reflexive Banach space, which can be equipped with the norm

$$\|u\| = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}}, \ \forall u \in X_0.$$

Given that $\Omega$ is a bounded smooth domain, it is well-known that the embedding $X_0 \hookrightarrow L^\nu(\Omega)$ holds continuously for $\nu \in [1, p_s^\star]$ and compactly for $\nu \in [1, p_s^\star)$, where $p_s^\star =$

$\frac{Np}{N-sp}$ (refer to [6], Theorems 6.5, 7.1). Moreover, there exists a positive constant $C_\nu$ such that the following inequality holds:

$$|u|_{L^\nu(\Omega)} \leqslant C_\nu \|u\|, \quad \forall u \in X_0. \tag{1.5}$$

DEFINITION 1.1. We say that $u_\lambda \in X_0$ is a weak solution of problem $(\mathscr{P}_\lambda)$ if $u_\lambda > 0$ and for any $\phi \in X_0$, we have

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(u)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\, dx\, dy$$
$$= \lambda \int_\Omega (u^+)^{p-2} u\phi\, dx + \int_\Omega \frac{\eta\phi}{(u^+)^\delta}\, dx + \int_\Omega \beta(x)(u^+)^{q-2} u\phi\, dx$$

where $u^+ = \max\{u, 0\}$.

To obtain weak solutions for problem $(\mathscr{P}_\lambda)$, we will employ variational techniques. Specifically, we will seek critical points of the Euler-Lagrange functional associated with problem $(\mathscr{P}_\lambda)$, which is represented by:

$$\mathscr{F}(u) = \mathscr{I}_\lambda(u) - \mathscr{J}_\eta(u), \tag{1.6}$$

where

$$\mathscr{I}_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy - \frac{\lambda}{p} \int_\Omega (u^+)^p\, dx,$$

and

$$\mathscr{J}_\eta(u) = \frac{\eta}{1-\delta} \int_\Omega (u^+)^{1-\delta}\, dx + \frac{1}{q} \int_\Omega \beta(x)(u^+)^q\, dx.$$

Our main result can be summarized as follows:

THEOREM 1.2. *Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $s \in (0,1)$ with $sp < N$. Assume that $q \in (p, p_s^\star)$. Then, for any $\lambda > 0$, there exists $\eta_0 > 0$ such that if $\eta \in (0, \eta_0)$, problem $(\mathscr{P}_\lambda)$ admits a nontrivial positive weak solution $u_\lambda \in X_0$ with $\mathscr{F}(u_\lambda) > 0$.*

The rest of this paper is organized as follows. In section 2, we recall some notations, definitions and some useful lemmas. Section 3 is devoted to study the approximated problem, in section 4, we prove our main result.

## 2. Preliminaries

We briefly recall the definition of $\mathbb{Z}_2$-cohomological index by Fadell and Rabinowitz [8]. For any closed, symmetric subset $E$ of a Banach space $X$, let $\overline{E} = E/\mathbb{Z}_2$ be the quotient space (in which $u$ and $-u$ are identified), and let $\varphi : \overline{E} \to \mathbb{R}P^\infty$ be the classifying map of $\overline{E}$, which induces a homomorphism $\varphi^\star : H^\star(\mathbb{R}P^\infty) \to H^\star(\overline{E})$ of the Alexander-Spanier cohomology rings with coefficients in $\mathbb{Z}_2$. We may identify

$H^{\star}(\mathbb{R}P^{\infty})$ with the polynomial ring $\mathbb{Z}_2[\omega]$. The cohomological index of $E$ is defined by

$$\begin{cases} i(E) = \sup\{n \in \mathbb{N} : \varphi^{\star}(\omega^n) \neq 0\} & \text{if } E \neq \emptyset, \\ 0 & \text{if } E = \emptyset. \end{cases}$$

Now we define the sequence $(\lambda_n)$. For any $u \in X_0$, define

$$S_0 = \{u \in X_0 : \int_{\Omega} |u|^p \, dx = 1\}.$$

We denote by $\mathscr{A}$ the family of all nonempty, closed, symmetric subsets of $S_0$ and for all $n \in \mathbb{N}$ we set

$$\mathscr{A}_n = \{M \in \mathscr{A} : i(M) \geqslant n\},$$

and

$$\lambda_n = \inf_{M \in \mathscr{A}_n} \sup_{u \in M} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy. \tag{2.1}$$

Then, $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \to +\infty$ is a sequence of eigenvalues of problem (1.2), (see ([15], Proposition 2.2). For each $\lambda_n$, we define the following cones

$$C_n^- = \Big\{u \in X_0 : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \leqslant \lambda_n \int_{\Omega} |u|^p \, dx\Big\}, \tag{2.2}$$

and

$$C_n^+ = \Big\{u \in X_0 : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \geqslant \lambda_{n+1} \int_{\Omega} |u|^p \, dx\Big\}. \tag{2.3}$$

Now, we recall some notions on linking sets and Alexander-Spanier cohomology, referring to [5].

DEFINITION 2.1. Let $D$, $S$, $A$, and $B$ be four subsets of a metric space $X$ with $S \subseteq D$ and $B \subseteq A$. We say that $(D,S)$ links $(A,B)$, if $S \cap A = B \cap D = \emptyset$ and, for every deformation $\eta : D \times [0,1] \to X \setminus B$ with $\eta(S \times [0,1]) \cap A = \emptyset$, we have that $\eta(D \times \{1\}) \cap A \neq \emptyset$.

To show the existence of critical points, we shall use the following result (see Theorem 2.2 [5].

LEMMA 2.2. *Let $X$ be a complete Finsler manifold of class $C^1$ and let $\mathscr{F} : X \to \mathbb{R}$ be a function of class $C^1$. Let $D$, $S$, $A$, and $B$ be four subsets of $X$, with $S \subseteq D$ and $B \subseteq A$, such that $(D,S)$ links $(A,B)$ and*

$$\sup_S \mathscr{F} < \inf_A \mathscr{F}, \quad \sup_D \mathscr{F} < \inf_B \mathscr{F}$$

*we agree that* $\sup\{\emptyset\} = -\infty$ *and* $\inf\{\emptyset\} = +\infty$. *Define*

$$c = \inf_{\eta \in \mathscr{N}} \sup \mathscr{F}(\eta(D \times \{1\})),$$

*where* $\mathscr{N}$ *is the set of deformation* $\eta : D \times [0,1] \to X \setminus B$ *with* $\eta(S \times [0,1]) \cap A = \emptyset$. *Then, we have*

$$\inf_A \mathscr{F} \leqslant c \leqslant \sup_D \mathscr{F}.$$

*Moreover, if* $\mathscr{F}$ *satisfies the Palais-Smale condition at level* $c$*, then* $c$ *is a critical value of* $\mathscr{F}$.

DEFINITION 2.3. Let $D$, $S$, $A$, and $B$ be four subsets of $X$, with $S \subseteq D$ and $B \subseteq A$, let $n$ be a nonnegative integer and let $\mathbb{K}$ be a field. We say that $(D,S)$ links $(A,B)$ cohomologically in dimension $n$ over $\mathbb{K}$ if $S \cap A = B \cap D = \emptyset$ and the restriction homomorphism $H^n(X \setminus B, X \setminus A; \mathbb{K}) \to H^n(D,S;\mathbb{K})$ is not identically zero.

LEMMA 2.4. ([11], Theorem 2.8) *Let* $X$ *be a real normed space and let* $C_-$ *and* $C_+$ *be two cones such that* $C_+$ *is closed in* $X$, $C_- \cap C_+ = \{0\}$ *and such that* $(X, C_- \setminus \{0\})$ *links* $C_+$ *cohomologically in dimension* $n$ *over* $\mathbb{K}$. *Let* $r_-$, $r_+ > 0$ *and let*

$$D_- = \{u \in C_- : \|u\| \leqslant r_-\}, \quad D_+ = \{u \in C_+ : \|u\| \leqslant r_+\}.$$

*Then, the following assertions hold*

(1) $(D_-, S_-)$ *links* $C_+$ *cohomologically in dimension* $n$ *over* $\mathbb{K}$,

(2) $(D_-, S_-)$ *links* $(D_+, S_+)$ *cohomologically in dimension* $n$ *over* $\mathbb{K}$.

*Moreover, let* $e \in X$ *with* $-e \notin C_-$, *and*

$$M = \{u + te : \ u \in C_-, \ t \geqslant 0, \ \|u + te\| \leqslant r_-\},$$

$$N = \{u + te : \ u \in C_-, \ t \geqslant 0, \ \|u + te\| = r_-\},$$

*and assume that* $r_- > r_+$. *Then, the following assertions hold*

(3) $(M, D_- \cup N)$ *links* $S_+$ *cohomologically in dimension* $n+1$ *over* $\mathbb{K}$,

(4) $D_- \cup N$ *links* $(D_+, S_+)$ *cohomologically in dimension* $n$ *over* $\mathbb{K}$.

*In particular, in each* $(1)$–$(4)$, *there is a geometry of the type described in Definition* 2.1.

COROLLARY 2.5. ([5], Corollary 2.9) *Let* $X$ *be a real normed space and let* $C_-$ *and* $C_+$ *be two symmetric cones in* $X$ *such that* $C_+$ *is closed in* $X$, $C_- \cap C_+ = \{0\}$ *and such that*

$$i(C_- \setminus \{0\}) = i(X \setminus C_+) < \infty.$$

*Then the assertions* $(1)$–$(4)$ *of Lemma* 2.4 *hold for* $n = i(C_- \setminus \{0\})$ *and* $\mathbb{K} = \mathbb{Z}_2$.

Going back to the definitions of $C_n^-$ and $C_n^+$, this is the transcription of Theorem 3.2 in [5] in our situation, yielding the following result.

LEMMA 2.6. *Let $n \geqslant 1$ be such that $\lambda_n < \lambda_{n+1}$, then we have*

$$i(C_n^- \setminus \{0\}) = i(X \setminus C_n^+) = n.$$

Finally, in order to use Lemma 2.2, the crucial tool is

LEMMA 2.7. ([5], Proposition 2.4) *If $(D,S)$ links $(A,B)$ cohomologically (in some dimension), then $(D,S)$ links $(A,B)$.*

## 3. Auxiliary problem

Classic variational methods cannot be applied to problem $(\mathscr{P}_\lambda)$ since its functional $\mathscr{F}$ is not $C^1$ in $X_0$ due to a singular term $u \mapsto \eta(u^+)^{-\delta}$. So, to obtain a nontrivial weak solution for problem $(\mathscr{P}_\lambda)$, we shall consider a modified problem $(\mathscr{P}_\varepsilon)$, $\varepsilon \in (0,1)$, which is given by

$$(\mathscr{P}_\varepsilon) \quad \begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u + \dfrac{\eta}{(u+\varepsilon)^\delta} + \beta(x)|u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

We associate an energy functional $\mathscr{F}_\varepsilon$ with problem $(\mathscr{P}_\varepsilon)$, defined as:

$$\mathscr{F}_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{\lambda}{p}\int_\Omega (u^+)^p\,dx \\ - \frac{\eta}{1-\delta}\int_\Omega ((u^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx - \frac{1}{q}\int_\Omega \beta(x)(u^+)^q\,dx.$$

It can be shown that $\mathscr{F}_\varepsilon$ is of $C^1$ in $X_0$. Furthermore, for any $u,\phi \in X_0$, the derivative of $\mathscr{F}_\varepsilon$ is given by:

$$(\mathscr{F}'_\varepsilon(u)\cdot\phi) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(u)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ -\lambda\int_\Omega (u^+)^{p-1}\phi\,dx - \int_\Omega \frac{\eta\phi}{(u+\varepsilon)^\delta}\,dx + \int_\Omega \beta(x)(u^+)^{q-1}\phi\,dx. \tag{3.2}$$

Therefore, by considering problem $(\mathscr{P}_\varepsilon)$ and the associated energy functional $\mathscr{F}_\varepsilon$, we can apply variational methods to find nontrivial weak solutions, which correspond to critical points of $\mathscr{F}_\varepsilon$.

### 3.1. Compactness structure

In this subsection, we discuss the compactness structure of the functional $\mathscr{F}_\varepsilon$ using the Palais-Smale condition. We recall that a functional $\mathscr{F}_\varepsilon$ satisfies the Palais-Smale condition at level $c_\varepsilon \in \mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}} \subset X_0$ satisfies

$$\mathscr{F}_\varepsilon(u_n) \to c_\varepsilon \text{ and } \mathscr{F}'_\varepsilon(u_n) \to 0 \text{ in } X_0^\star \text{ as } n \to +\infty,$$

admits a convergent subsequence in $X_0$.

LEMMA 3.1. ([4], Lemma 2.3) *The operator* $\mathscr{L}_s^p : X_0 \to X_0^\star$ *defined by*

$$(\mathscr{L}_s^p(u)\cdot v) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(u)|^{p-2}(u(x)-v(y))(v(x)-v(y))}{|x-y|^{N+sp}}\, dx\, dy,$$

*satisfies the* $(S+)$ *property. That is for every sequence* $(u_n)_{n\in\mathbb{N}}$ *such that if* $u_n$ *converges weakly to some* $u$ *in* $X_0$ *and satisfies*

$$\lim_{n\to+\infty}(\mathscr{L}_s^p(u_n)\cdot u_n - u) \to 0,$$

*then* $u_n$ *converges strongly to* $u$ *in* $X_0$.

LEMMA 3.2. *Assume that* $q\in(p,p_s^\star)$. *Then for any* $\lambda>0$, $\eta>0$, $\mathscr{F}_\varepsilon$ *satisfies the Palais-Smale condition at any level* $c_\varepsilon\in\mathbb{R}$.

*Proof.* Let $\eta>0$, $\lambda\in\mathbb{R}$, $c_\varepsilon\in\mathbb{R}$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $X_0$ be such that

$$\mathscr{F}_\varepsilon(u_n)\to c_\varepsilon \ \text{ and } \ \mathscr{F}'_\varepsilon(u_n)\to 0 \ \text{ in } \ X_0^\star, \tag{3.3}$$

as $n\to+\infty$. Firstly, let us show that $(u_n)_{n\in\mathbb{N}}$ is bounded in $X_0$. Indeed, due to the fact that

$$(u^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta} \leqslant (u^+)^{1-\delta}, \tag{3.4}$$

we obtain

$$\begin{aligned}
c_\varepsilon + o_n(1) &= \mathscr{F}_\varepsilon(u_n) - \frac{1}{p}(\mathscr{F}'_\varepsilon(u_n)\cdot u_n)\\
&= \left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega \beta(x)(u_n^+)^q\,dx - \frac{\eta}{1-\delta}\int_\Omega ((u_n^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx\\
&\quad + \frac{\eta}{p}\int_\Omega (u_n^+ + \varepsilon)^{-\delta} u_n\,dx\\
&\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega \beta(x)(u_n^+)^q\,dx - \frac{\eta}{1-\delta}\int_\Omega (u_n^+)^{1-\delta}\,dx\\
&\quad - \frac{\eta}{p}\int_\Omega (u_n^+ + \varepsilon)^{-\delta} u_n^+\,dx\\
&\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega \beta_0(u_n^+)^q\,dx - \eta\left(\frac{1}{1-\delta}+\frac{1}{p}\right)\int_\Omega (u_n^+)^{1-\delta}\,dx\\
&\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\beta_0|u_n^+|^q_{L^q(\Omega)} - \eta\left(\frac{1}{1-\delta}+\frac{1}{p}\right)|\Omega|^{\frac{q+\delta-1}{q}}|u_n^+|^{1-\delta}_{L^q(\Omega)},
\end{aligned}$$

as $n \to +\infty$. Consequently, $(u_n^+)_{n\in\mathbb{N}}$ is bounded sequence in $L^q(\Omega)$ and $q > p$ gives the boundedness of $(u_n^+)_{n\in\mathbb{N}}$ in $L^p(\Omega)$. As a consequence, we obtain

$$\begin{aligned}
c_\varepsilon + o(1) &= \mathscr{F}_\varepsilon(u_n) - \frac{1}{q}(\mathscr{F}_\varepsilon'(u_n)\cdot u_n) \\
&= \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p - \lambda\left(\frac{1}{p}-\frac{1}{q}\right)|u_n^+|^p_{L^p(\Omega)} - \frac{\eta}{1-\delta}\int_\Omega (u_n^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta}\,dx \\
&\quad + \frac{\eta}{q}\int_\Omega (u_n^+ + \varepsilon)^{-\delta} u_n\,dx \\
&\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p - \lambda\left(\frac{1}{p}-\frac{1}{q}\right)|u_n^+|^p_{L^p(\Omega)} - \eta\left(\frac{1}{1-\delta}+\frac{1}{p}\right)|\Omega|^{\frac{p+\delta-1}{p}}|u_n^+|^{1-\delta}_{L^p(\Omega)} \\
&\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p - \lambda C_1 - \eta C_2,
\end{aligned}$$

where $C_1, C_2 > 0$. This latter with the fact that $p < q$ imply the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $X_0$. Since $X_0$ is reflexive space, there exists a function $u \in X_0$ such that, up to a subsequence, (still denoted by $(u_n)_{n\in\mathbb{N}}$), $u_n \to u$ weakly in $X_0$, strongly in $L^\alpha(\Omega)$ for any $\alpha \in [1, p_s^\star)$, and almost everywhere in $\Omega$. Additionally, there exists $h \in L^1(\Omega)$, $h > 0$ such that

$$|u_n| \leqslant h, \quad \text{a.e. in } \Omega.$$

From the inequality

$$\frac{u_n - u}{(u_n^+ + \varepsilon)^\delta} \leqslant \varepsilon^{-\delta}|u + h|,$$

we can apply the dominated convergence theorem to obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{u_n - u}{(u_n^+ + \varepsilon)^\delta}\,dx = 0. \tag{3.5}$$

Furthermore, utilizing Hölder's inequality, we get

$$\begin{aligned}
\left|\int_\Omega (u_n^+)^{p-1}(u_n - u)\,dx\right| &\leqslant \left(\int_\Omega |u_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u_n - u|^p\,dx\right)^{\frac{1}{p}} \\
&\leqslant |u_n|^{p-1}_{L^p(\Omega)}|u_n - u|_{L^p(\Omega)},
\end{aligned}$$

$$\begin{aligned}
\left|\int_\Omega \beta(x)(u_n^+)^{q-1}(u_n - u)dx\right| &\leqslant \beta_\infty\left(\int_\Omega |u_n|^q\,dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |u_n - u|^q\,dx\right)^{\frac{1}{q}} \\
&\leqslant \beta_\infty|u_n|^{q-1}_{L^q(\Omega)}|u_n - u|_{L^q(\Omega)}.
\end{aligned}$$

Therefore, we obtain

$$\lim_{n\to+\infty}\int_\Omega (u_n^+)^{p-1}(u_n - u)\,dx = 0,$$

$$\lim_{n\to+\infty}\int_\Omega \beta(x)(u_n^+)^{q-1}(u_n - u)\,dx = 0.$$

This latters with equations (3.3), and (3.5), we get

$$
\begin{aligned}
o_n(1) &= (\mathscr{F}'_\varepsilon(u)\cdot u_n - u)\\
&= (\mathscr{L}^p_s(u_n)\cdot u_n - u) - \lambda\int_\Omega (u_n^+)^{p-1}(u_n-u)\,dx - \eta\int_\Omega \frac{u_n-u}{(u_n^+ + \varepsilon)^\delta}\,dx\\
&\quad - \int_\Omega \beta(x)(u_n^+)^{q-1}(u_n-u)\,dx\\
&= (\mathscr{L}^p_s(u_n)\cdot u_n - u) + o_n(1).
\end{aligned}
$$

Since the operator $\mathscr{L}^p_s$ satisfy the $(S+)$ propriety, it yields that $u_n \to u$ strongly in $X_0$. The proof of Lemma 3.2 is complete. □

### 3.2. Case $\lambda \in (0,\lambda_1)$: Mountain Pass type solution

In this subsection, we show that problem $(\mathscr{P}_\varepsilon)$ admits a nontrivial weak solution for any $\lambda \in (0,\lambda_1)$ by using the Mountain Pass Theorem of Ambrosetti-Rabinowitz in [1]. Firstly, we start by proving the necessary geometric features of the functional $\mathscr{F}_\varepsilon$.

LEMMA 3.3. *Assume that $q > p$ and $\lambda \in (0,\lambda_1)$. Then, there exists $\mu_0 > 0$, $\rho_\varepsilon, \gamma_\varepsilon > 0$ such that for any $\mu \in (0,\mu_0)$, $u \in X_0 \cap B_{\rho_\varepsilon}$, it results that $\mathscr{F}_\varepsilon(u) = \gamma_\varepsilon > 0$.*

*Proof.* Let $u \in X_0$. By the use of inequality (3.4) we get

$$
\begin{aligned}
\frac{1}{1-\delta}\int_\Omega (u^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta}\,dx &\leqslant \frac{1}{1-\delta}\int_\Omega (u^+)^{1-\delta}\,dx\\
&\leqslant \frac{1}{1-\delta}|\Omega|^{\frac{q+\delta-1}{q}}|u|^{1-\delta}_{L^q(\Omega)}\\
&\leqslant \frac{1}{1-\delta}|\Omega|^{\frac{q+\delta-1}{q}}C_q^{1-\delta}\|u\|^{1-\delta}\\
&= c_{\delta,q}\|u\|^{1-\delta},
\end{aligned}
\tag{3.6}
$$

where $c_{\delta,q} = \frac{C_q^{1-\delta}}{1-\delta}|\Omega|^{\frac{q+\delta-1}{q}}$. From this, we obtain

$$
\begin{aligned}
\mathscr{F}_\varepsilon(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}|u^+|^p_{L^p(\Omega)} - \frac{\eta}{1-\delta}\int_\Omega ((u^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx\\
&\quad - \frac{1}{q}\int_\Omega \beta(x)(u^+)^q\,dx\\
&\geqslant \frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u\|^p - \eta c_{\delta,q}\|u\|^{1-\delta} - \frac{C_q^q\beta_\infty}{q}\|u\|^q\\
&= \|u\|^p\Big(\frac{1}{p}\Big(1-\frac{\lambda}{\lambda_1}\Big) - \frac{C_q^q\beta_\infty}{q}\|u\|^{q-p}\Big) - \eta c_{\delta,q}\|u\|^{1-\delta}.
\end{aligned}
$$

Define

$$h_\lambda(t) = t^p\Big(\frac{1}{p}\Big(1-\frac{\lambda}{\lambda_1}\Big) - \frac{C_q^q\beta_\infty}{q}t^{q-p}\Big), \ \ t \geqslant 0.$$

Since $\lambda < \lambda_1$ and $p < q$. Then, there exists $\rho_\varepsilon \in (0,1)$ such that

$$h_\lambda(\rho_\varepsilon) = \max_{t>0} h_\lambda(t) > 0.$$

Now, taking $\|u\| = \rho_\varepsilon$ and $\eta_0 = \frac{1}{2}c_{\delta,q}^{-1}\rho_\varepsilon^{\delta-1}h_\lambda(\rho_\varepsilon)$. Then, if $\eta < \eta_0$, we obtain

$$\mathscr{F}_\varepsilon(u) \geqslant \frac{h_\lambda(\rho_\varepsilon)}{2} = \gamma_\varepsilon > 0.$$

Hence, Lemma 3.3 is proved. □

LEMMA 3.4. *There exists $e \in X_0$ such that $\|e\| > \rho_\lambda$ and $\mathscr{F}_\varepsilon(e) < \gamma_\varepsilon$, where $\rho_\varepsilon$ and $\gamma_\varepsilon$ are given in Lemma 3.3.*

*Proof.* Let $\varphi \in X_0$, $\varphi \geqslant 0$ be such that $\|\varphi\| = 1$ and let $\zeta > 0$. Then, we have

$$\begin{aligned}
\mathscr{F}_\varepsilon(\zeta\varphi) &= \frac{\zeta^p}{p}\|\varphi\|^p - \frac{\lambda\zeta^p}{p}|\varphi|^p_{L^p(\Omega)} - \frac{\eta}{1-\delta}\int_\Omega ((\zeta\varphi+\varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx \\
&\quad - \frac{\zeta^q}{q}\int_\Omega \beta(x)\varphi^q\,dx \\
&\leqslant \frac{\zeta^p}{p}(1-\lambda|\varphi|^p_{L^p(\Omega)}) + \frac{\eta}{1-\delta}\Big(\frac{\zeta^{1-\gamma}}{2}\int_\Omega \varphi^{1-\delta}\,dx + \varepsilon^{1-\delta}|\Omega|\Big) \\
&\quad - \frac{\zeta^q}{q}\beta_0\int_\Omega \varphi^q\,dx.
\end{aligned}$$

Since $q > p > 1-\delta$, passing to the limit as $\zeta \to +\infty$, we obtain $\mathscr{F}_\varepsilon(\zeta\varphi) \to -\infty$, so that the assertion follows taking $e = \zeta\varphi$, with $\zeta$ sufficiently large. According to Lemma 3.2, Lemma 3.3 and Lemma 3.4, the functional $\mathscr{F}_\varepsilon$ satisfies all the assumptions of the Mountain Pass Theorem. Then, there exists $u_\varepsilon \in X_0$ a critical point of the functional $\mathscr{F}_\varepsilon$ such that

$$\mathscr{F}_\varepsilon(u_\varepsilon) \geqslant \gamma_\varepsilon > 0 = \mathscr{F}_\varepsilon(0),$$

so that $u_\varepsilon \neq 0$.

### 3.3. Case $\lambda \geqslant \lambda_1$: Linking type solution

Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of eigenvalues given in (2.1). Since this sequence is divergent, there exists $n \geqslant 1$ such that $\lambda_n \leqslant \lambda < \lambda_{n+1}$. Defining $C_n^-$ and $C_n^+$ as in (2.2) and (2.3). It is easy to see that $C_n^-$ and $C_n^+$ are two symmetric closed cones in $X_0$ with $C_n^- \cap C_n^+ = \{0\}$. Moreover, by Lemma 2.6, it holds that

$$i(C_n^- \setminus \{0\}) = i(X \setminus C_n^+) = n.$$

Let $u \in C_n^+$. By using assertion (3.6), we have

$$\begin{aligned}
\mathscr{F}_\varepsilon(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{p}|u^+|^p_{L^p(\Omega)} - \frac{\eta}{1-\delta}\int_\Omega ((u^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx \\
&\quad - \frac{1}{q}\int_\Omega \beta(x)(u^+)^q\,dx \\
&\geqslant \frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_{n+1}}\Big)\|u\|^p - \eta c_{\delta,q}\|u\|^{1-\delta} - \frac{C_q^q \beta_\infty}{q}\|u\|^q \\
&= \|u\|^p\Big(\frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_{n+1}}\Big) - \frac{C_q^q\beta_\infty}{q}\|u\|^{q-p}\Big) - \eta c_{\delta,q}\|u\|^{1-\delta}.
\end{aligned}$$

Here $c_{\delta,q} = \frac{1}{1-\delta}|\Omega|^{\frac{q+\delta-1}{q}}C_q^{1-\delta}$. Let $h_\lambda(t) = t^p\Big(\frac{1}{p}(1-\frac{\lambda}{\lambda_{n+1}}) - \frac{1}{q}C_q^q\beta_\infty t^{q-p}\Big)$, $t \geqslant 0$. Since $\lambda < \lambda_{n+1}$ and $p < q$, then, there exists $r_\varepsilon^+ \in (0,1)$ such that

$$h_\lambda(r_\varepsilon^+) = \max_{t>0} h_\lambda(t) > 0.$$

Now, taking $\|u\| = r_\varepsilon^+$, $\eta_0 = \frac{1}{2}c_{\delta,q}^{-1}(r_\varepsilon^+)^{\delta-1}h_\lambda(r_\varepsilon^+)$. Then, if $\eta < \eta_0$, we obtain

$$\mathscr{F}_\varepsilon(u) \geqslant \frac{h_\lambda(r_\varepsilon^+)}{2} = \gamma > 0.$$

On the other hand, let $u \in C_n^-$, $e \in X_0 \setminus C_n^-$ and $t > 0$. Since $\lambda \in [\lambda_n, \lambda_{n+1})$, then, using (3.4), we obtain

$$\begin{aligned}
\mathscr{F}_\varepsilon(u+te) &= \frac{1}{p}\|u+te\|^p - \frac{\lambda}{p}|u+te|^p_{L^p(\Omega)} - \frac{1}{q}\int_\Omega \beta(x)[(u+te)^+]^q\,dx \\
&\quad - \frac{\eta}{1-\delta}\int_\Omega \big(((u+te)^+ + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta}\big)\,dx \\
&\leqslant \frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_n}\Big)\|u+te\|^p - \frac{1}{q}\int_\Omega \beta(x)[(u+te)^+]^q\,dx \\
&\quad + \frac{\eta}{1-\delta}\int_\Omega [(u+te)^+]^{1-\delta}\,dx + \frac{\eta}{1-\delta}\varepsilon^{1-\delta}|\Omega| \\
&\leqslant -t^q\frac{\beta_0}{q}\int_\Omega \Big[\Big(\frac{u}{t} + e\Big)^+\Big]^q\,dx + \frac{\eta t^{1-\delta}}{1-\delta}\int_\Omega \Big(\Big(\frac{u}{t} + e\Big)^+\Big)^{1-\delta}\,dx \\
&\quad + \frac{\eta}{1-\delta}\varepsilon^{1-\delta}|\Omega|.
\end{aligned}$$

This latter gives that $\mathscr{F}_\varepsilon(u+te) \to -\infty$ as $t \to +\infty$. Hence, there exists $r_\varepsilon^- > r_\varepsilon^+$ such that

$$w \in C_n^- + [\mathbb{R}^+ e], \text{ and } \|w\| \geqslant r_\varepsilon^- \Rightarrow \mathscr{J}_\varepsilon(w) < 0.$$

Defining $D_-$, $S+$, $M$ and $N$ as in Lemma 2.4. From Corollary 2.5, we have that $(M, D_- \cup N)$ links $S+$ cohomologically in dimension $n+1$ over $\mathbb{Z}_2$. In particular, $(M, D_- \cup N)$ links $S+$ by Lemma 2.7. In addition, $\mathscr{F}_\varepsilon$ is bounded on $M$, $\mathscr{F}_\varepsilon(u) \leqslant 0$

for every $u \in D_- \cup N$ and $\mathscr{F}_\varepsilon(u) \geqslant \alpha_\varepsilon > 0$ for every $u \in S+$. By Lemma 3.2, $\mathscr{F}_\varepsilon$ satisfies the Palais-Smale condition at any level $c_\varepsilon \in \mathbb{R}$. Finally, by applying Lemma 2.2 with $S = D_- \cup N$, $D = M$, $A = S+$ and $B = \emptyset$, $\mathscr{F}_\varepsilon$ has a critical value $c_\varepsilon \geqslant \alpha_\varepsilon$, so that there exists a critical point $u_\varepsilon$ with $\mathscr{F}_\varepsilon(u_\varepsilon) = c_\varepsilon > 0$. It follows that $u_\varepsilon$ is a nontrivial weak solution of problem $(\mathscr{P}_\varepsilon)$.

Now, to verify the positivity of the solution $u_\varepsilon$, we replace the test function $\phi$ in the equation (3.2) by $u_\varepsilon^- = \max\{-u_\varepsilon, 0\}$ and using the elementary inequality

$$(a-b)(a^- - b^-) \leqslant -(a^- - b^-)^2,$$

we obtain $\|u_\varepsilon^-\| = 0$ implying that $u_\varepsilon$ is a nonnegative function. By applying the maximum principle (Proposition 2.17, [19]), we conclude that $u_\varepsilon$ is a positive solution for problem $(\mathscr{P}_\varepsilon)$. This complete the proof. □

## 4. Proof of our main result

In order to finish the proof our main result, we will show that problem $(\mathscr{P}_\lambda)$ admits a nontrivial weak solution $u \in X_0$ as a limit of solutions of problem $(\mathscr{P}_\varepsilon)$.

Let $\lambda > 0$, $\eta \in (0, \eta_0)$. Let $(u_\varepsilon)$, $\varepsilon \in (0,1)$, be a family of positive solutions of problem $(\mathscr{P}_\varepsilon)$. Then, using the fact that $(u_\varepsilon + \varepsilon)^{1-\delta} - \varepsilon^{1-\delta} \leqslant u_\varepsilon^{1-\delta}$, we obtain

$$\begin{aligned}
c_\varepsilon + o_\varepsilon(1) &= \mathscr{F}_\varepsilon(u_\varepsilon) - \frac{1}{p}(\mathscr{F}'_\varepsilon(u_\varepsilon)\cdot u_\varepsilon) \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega \beta(x) u_\varepsilon^q\,dx - \frac{\eta}{1-\delta}\int_\Omega ((u_\varepsilon+\varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx \\
&\quad + \frac{\eta}{p}\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta} u_\varepsilon\,dx \\
&\geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega \beta(x) u_\varepsilon^q\,dx - \frac{\eta}{1-\delta}\int_\Omega u_\varepsilon^{1-\delta}\,dx - \frac{\eta}{p}\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta} u_\varepsilon\,dx \\
&\geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\int_\Omega \beta(x) u_\varepsilon^q\,dx - \eta\left(\frac{1}{1-\delta} + \frac{1}{p}\right)\int_\Omega u_\varepsilon^{1-\delta}\,dx \\
&\geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\beta_0 |u_\varepsilon|^q_{L^q(\Omega)} - \eta\left(\frac{1}{1-\delta} + \frac{1}{p}\right)|\Omega|^{\frac{q+\delta-1}{q}} |u_\varepsilon|^{1-\delta}_{L^q(\Omega)}.
\end{aligned}$$

Therefore, $(u_\varepsilon)$ is bounded in $L^q(\Omega)$. Since $q > p$, then $(u_\varepsilon)$ is bounded also in $L^p(\Omega)$ and as a consequence, we obtain

$$\begin{aligned}
c_\varepsilon + o_\varepsilon(1) &= \mathscr{F}_\varepsilon(u_\varepsilon) - \frac{1}{q}(\mathscr{F}'_\lambda(u_\varepsilon)\cdot u_\varepsilon) \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\|u_\varepsilon\|^p - \lambda\left(\frac{1}{p} - \frac{1}{q}\right)|u_\varepsilon|^p_{L^p(\Omega)} \\
&\quad - \frac{\eta}{1-\delta}\int_\Omega ((u_\varepsilon+\varepsilon)^{1-\delta} - \varepsilon^{1-\delta})\,dx + \frac{\eta}{q}\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta} u_\varepsilon\,dx
\end{aligned}$$

$$\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\|u_\varepsilon\|^p-\lambda\left(\frac{1}{p}-\frac{1}{q}\right)|u_\varepsilon|^p_{L^p(\Omega)}-\eta\left(\frac{1}{1-\delta}+\frac{1}{q}\right)|\Omega|^{\frac{p+\delta-1}{p}}|u_\varepsilon|^{1-\delta}_{L^p(\Omega)}$$
$$\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\|u_\varepsilon\|^p-\lambda C_1-\eta C_2,$$

for some $C_1,C_2>0$. This latter with the fact that $p<q$ imply the boundedness of $(u_\varepsilon)$ in $X_0$. Since $X_0$ is a reflexive space, there exists $u_\lambda\in X_0$ such that, up to a subsequence, (still denoted by $(u_\varepsilon)$), $u_\varepsilon\to u_\lambda$ weakly in $X_0$, strongly in $L^\nu(\Omega)$, $\nu\in[1,p_s^\star)$, and a.e.in $\Omega$, as $\varepsilon\to 0^+$. Since

$$0\leqslant \frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\delta}\leqslant u_\varepsilon^{1-\delta} \text{ a.e. in } \Omega. \tag{4.1}$$

It follows that

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\delta}\,dx &\leqslant \int_\Omega u_\varepsilon^{1-\delta}\,dx\\ &\leqslant \int_\Omega |u_\varepsilon-u_\lambda|^{1-\delta}\,dx+\int_\Omega u_\lambda^{1-\delta}\,dx\\ &\leqslant |\Omega|^{\frac{p+\delta-1}{p}}|u_\varepsilon-u_\lambda|^{1-\delta}_{L^p(\Omega)}+\int_\Omega u_\lambda^{1-\delta}\,dx\\ &\leqslant \int_\Omega u_\lambda^{1-\delta}\,dx+o_\varepsilon(1).\end{aligned}$$

That is

$$\lim_{\varepsilon\to0^+}\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\delta}\,dx\leqslant \int_\Omega u_\lambda^{1-\delta}\,dx. \tag{4.2}$$

Furthermore, since $\frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\delta}$ converges to $u_\lambda^{1-\delta}$ a.e. in $\Omega$, it follows by the Fatou's lemma that

$$\liminf_{\varepsilon\to0^+}\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\delta}\,dx\geqslant \int_\Omega u_\lambda^{1-\delta}\,dx. \tag{4.3}$$

So, using (4.2) and (4.3), we deduce that

$$\lim_{\varepsilon\to0^+}\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^\delta}\,dx= \int_\Omega u_\lambda^{1-\delta}\,dx. \tag{4.4}$$

On the other hand, from (3.1), we have

$$\begin{cases}(-\Delta_p)^s u_\varepsilon=\lambda u_\varepsilon^{p-1}+\dfrac{\eta}{(u_\varepsilon+\varepsilon)^\delta}+\beta(x)u_\varepsilon^{q-1}\geqslant\dfrac{\eta}{(u_\varepsilon+\varepsilon)^\delta}\geqslant\dfrac{\eta}{2^\delta} & \text{if } u_\varepsilon\leqslant 1,\\ (-\Delta_p)^s u_\varepsilon=\lambda u_\varepsilon^{p-1}+\dfrac{\eta}{(u_\varepsilon+\varepsilon)^\delta}+\beta_0u_\varepsilon^{q-1}\geqslant\lambda u_\varepsilon^{p-1}+\beta(x)u_\varepsilon^{q-1}\geqslant\lambda+\beta_0 & \text{if } u_\varepsilon\geqslant 1.\end{cases}$$

Therefore, we get

$$(-\Delta_p)^s u_\varepsilon\geqslant c_0:=\min\left\{\frac{\eta}{2^\delta},\lambda+\beta_0\right\}.$$

Now, by the use of the maximum principle (see [19], Proposition 2.17), there exist a constant $m_0 > 0$ that is independent of $\varepsilon$ and $\Omega_0 \subset \Omega$ such that

$$u_\varepsilon(x) \geqslant m_0 > 0, \quad \text{a.e. } x \in \Omega_0. \tag{4.5}$$

Now, consider $\phi \in C_0^\infty(\Omega)$ such that $supp(\phi) = \Omega_0 \subset \Omega$. Then, from (4.5) we obtain

$$0 \leqslant \frac{\phi}{(u_\varepsilon + \varepsilon)^\delta} \leqslant \frac{\phi}{m_0^\delta}, \quad \text{a.e. in } \Omega.$$

Therefore, by the dominated convergence theorem we deduce that

$$\lim_{\varepsilon \to 0^+} \int_\Omega \frac{\phi}{(u_\varepsilon + \varepsilon)^\delta}\, dx = \int_\Omega \frac{\phi}{u_\lambda^\delta}\, dx. \tag{4.6}$$

Since $\partial\Omega$ is continuous, the space $C_0^\infty(\Omega)$ is dense in $X_0$ (see [9], Theorem 6). Thus, by a standard density argument, equation (4.6) holds true for any $\phi \in X_0$. By combining (4.4) and (4.6) with the test function $\phi = u_\lambda$ we get

$$\lim_{\varepsilon \to 0^+} \int_\Omega \frac{u_\varepsilon - u_\lambda}{(u_\varepsilon + \varepsilon)^\delta}\, dx = 0. \tag{4.7}$$

On the other hand, by the use of Hölder's inequality, we obtain

$$\begin{aligned} \left| \int_\Omega u_\varepsilon^{p-1}(u_\varepsilon - u_\lambda) dx \right| &\leqslant \left( \int_\Omega |u_\varepsilon|^p\, dx \right)^{\frac{p-1}{p}} \left( \int_\Omega |u_\varepsilon - u_\lambda|^p\, dx \right)^{\frac{1}{p}} \\ &\leqslant |u_\varepsilon|_{L^p(\Omega)}^{p-1} |u_\varepsilon - u_\lambda|_{L^p(\Omega)}, \end{aligned}$$

$$\begin{aligned} \left| \int_\Omega \beta(x) u_\varepsilon^{q-1}(u_\varepsilon - u_\lambda) dx \right| &\leqslant \beta_\infty \left( \int_\Omega |u_\varepsilon|^q\, dx \right)^{\frac{q-1}{q}} \left( \int_\Omega |u_\varepsilon - u_\lambda|^q\, dx \right)^{\frac{1}{q}} \\ &\leqslant \beta_\infty |u_\varepsilon|_{L^q(\Omega)}^{q-1} |u_\varepsilon - u_\lambda|_{L^q(\Omega)}, \end{aligned}$$

Consequently, we get

$$\lim_{\varepsilon \to 0^+} \int_\Omega u_\varepsilon^{p-1}(u_\varepsilon - u_\lambda) dx = 0 \tag{4.8}$$

$$\lim_{\varepsilon \to 0^+} \int_\Omega \beta(x) u_\varepsilon^{q-1}(u_\varepsilon - u_\lambda) dx = 0. \tag{4.9}$$

Combining equations (4.8)–(4.9) with (4.7) we get

$$\begin{aligned} o_\varepsilon(1) &= (\mathscr{F}_\varepsilon'(u_\varepsilon) \cdot u_\varepsilon - u_\lambda) \\ &= (\mathscr{L}_s^p(u_\varepsilon).u_\varepsilon - u_\lambda) - \lambda \int_\Omega u_\varepsilon^{p-1}(u_\varepsilon - u_\lambda)\, dx - \eta \int_\Omega \frac{u_\varepsilon - u_\lambda}{(u_\varepsilon + \varepsilon)^\delta}\, dx \\ &\quad - \int_\Omega \beta(x) u_\varepsilon^{q-1}(u_\varepsilon - u_\lambda)\, dx \\ &= (\mathscr{L}_s^p(u_\varepsilon) \cdot u_\varepsilon - u_\lambda) + o_\varepsilon(1). \end{aligned}$$

Since $\mathscr{L}_s^p$ satisfies the $(S+)$ property, we obtain $u_\varepsilon \to u_\lambda$ strongly in $X_0$ as $\varepsilon \to 0^+$. Thus, we get a nontrivial positive weak solution for problem $(\mathscr{P}_\lambda)$ for any $\lambda > 0$. The proof of Theorem 1.2 is complete.

## REFERENCES


[1] A. AMBROSETTI, P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] V. AMBROSIO, *Nontrivial solutions for a fractional p-Laplacian problem via Rabier Theorem*, Complex Variables and Elliptic Equations 2017, **62** (6): 838–847.

[3] G. M. BISCI, V. RADULESCU, R. SERVADEI, *Variational Methods for Nonlocal Fractional Problems*, Cambridge University Press, 2016.

[4] A. DAOUAS, M. LOUCHAICH, *On fractional p-Laplacian type equations with general nonlinearities*, Turk. J. Math. (2021) **45**: 2477–2491, doi:10.3906/mat-2010-100.

[5] M. DEGIOVANNI, S. LANCELOTTI, *Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007) 907–919.

[6] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI, *Hitchhiker's guide to the fractional Sobolev spaces*, B. Sci. Math., in press, available on line at http://arxiv.org/abs/1104.4345.

[7] X. FAN, Z. LI, *Linking and existence results for perturbations of the p-Laplacian*, Nonlinear Anal. **42** (2000) 1413–1420.

[8] E. R. FADELL, P. H. RABINOWITZ, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. **45** (1978) 139–174.

[9] A. FISCELLA, R. SERVADEI AND E. VALDINOCI, *Density properties for fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **40** (2015), no. 1, 235–253.

[10] Y. FU, P. PUCCI, *Multiplicity existence for sublinear fractional Laplacian problems*, Applicable Analysis 2017, **96** (9): 1497–1508.

[11] M. FRIGON, *On a new notion of linking and application to elliptic problems at resonance*, J. Differential Equations **153** (1999), no. 1, 96–120.

[12] G. FRANZINA, G. PALATUCCI, *Fractional p-eigenvalues*, Riv. Mat. Univ. Parma **5** (2014).

[13] K. HO, K. PERERA, I. SIM, M. SQUASSINA, *A note on fractional p-Laplacian problems with singular weights*, Journal of Fixed Point Theory and Applications 2017, **19** (1): 157–173.

[14] A. IANNIZZOTTO, S. LIU, K. PERERA, M. SQUASSINA, *Existence results for fractional p-Laplacian problems via Morse theory*, Advances in Calculus of Variations 2016, **9** (2): 101–125.

[15] A. IANNIZZOTTO, M. SQUASSINA, *Weyl-type laws for fractional p-eigenvalue problems*, Asymptotic Anal. **88** (2014), 233–245.

[16] E. LINDGREN, P. LINDQVIST, *Fractional eigenvalues*, Calc. Var. Partial Differential Equations **49** (2014), 795–826.

[17] P. PUCCI, M. XIANG, B. ZHANG, *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*, Advances in Nonlinear Analysis. (2016), 5.

[18] H. QIU, M. XIANG, *Existence of solutions for fractional p-Laplacian problems via Leray-Schauder's nonlinear alternative*, Boundary Value Problems 2016, 1–8.

[19] L. SILVESTRE, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), 67–112.

[20] R. SERVADEI, E. VALDINOCI, *Mountain Pass solutions for non-local elliptic operators*, Journal of Mathematical Analysis and Applications 2012, **389** (2): 887–898.

[21] R. SERVADEI, E. VALDINOCI, *Variational methods for non-local operators of elliptic type*, Discrete and Continuous Dynamical Systems 2013, **33** (5): 2105–2137.

[22] R. SERVADEI, *Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity*, Contemp. Math. 2013, **595**: 317–40.

[23] R. SERVADEI, *The Yamabe equation in a non-local setting*, Advances in Nonlinear Analysis, vol. 2, no. 3, 2013, pp. 235–270, https://doi.org/10.1515/anona-2013-0008.

[24] R. SERVADEI, *A critical fractional Laplace equation in the resonant case*, Topol. Methods Nonlinear Anal. **43**, 251–267 (2014).

[25] B. L. ZHANQ, G. M. BISCI, R. SERVADEI, *Superlinear nonlocal fractional problems with infinitely many solutions*, Nonlinearity 2015, **28**: 2247–2264.

(Received January 4, 2025)

*Mohamed Louchaich*
*High school of sciences and technology*
*Sousse University*
*4011 Hammam Sousse, Tunisia*
*e-mail:* mohamedlouchaiech@gmail.com