Fractional Differential Calculus
Volume 15, Number 2 (2025), 341–357
doi:10.7153/fdc-2025-15-20

INVERSE SPECTRAL ANALYSIS FOR A VARIABLE–ORDER MIXED FRACTIONAL STURM–LIOUVILLE PROBLEM: UNIQUENESS, RECONSTRUCTION, AND ASYMPTOTICS

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(*Communicated by H. Lopushanska*)

Abstract. We introduce a variable-order mixed fractional differential operator that interpolates between the Caputo and Riemann-Liouville derivatives, with differentiation orders $\alpha(t)$ and $\beta(t)$ that are functions of the spatial variable t. Specifically, for parameters $\alpha(t), \beta(t) \in (0,1)$ and a fixed weight parameter $\lambda \in [0,1]$, we define

$$\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t) := \lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t),$$

for $t \in (0,1)$. We study the direct spectral problem for the associated Sturm-Liouville operator

$$\mathscr{L} u(t) = -\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t) + q(t)u(t),$$

subject to a new class of fractional boundary conditions adapted to the variable-order framework.

1. Introduction

Fractional differential operators have proven to be a powerful tool in modeling memory and hereditary properties in complex media [6, 8]. While the two classical definitions – the Riemann-Liouville and the Caputo derivatives – are extensively used, recent studies have focused on developing new operators that blend their advantages. In this paper, we introduce a *variable-order mixed fractional derivative* in which the order is allowed to vary with the spatial variable. Such an operator is defined by

$$\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t) := \lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t), \quad t \in (0,1), \tag{1.1}$$

with $\alpha(t), \beta(t) \in (0,1)$ and $\lambda \in [0,1]$. Here ${}^{C}D_{0+}^{\alpha(t)}$ and ${}^{RL}D_{0+}^{\beta(t)}$ denote the Caputo and Riemann-Liouville derivatives with *variable* order [7, 10], respectively.

Recent advances in fractional calculus have aimed at developing generalized fractional operators that retain the advantages of both classical definitions while addressing their limitations. One such generalization involves variable-order fractional derivatives, where the order of differentiation is allowed to vary as a function of the spatial or temporal variable. These operators have attracted significant attention in recent years due

Mathematics subject classification (2020): 34A55, 34B07, 34B09, 34B24.
Keywords and phrases: Fractional differential operators, variable-order mixed fractional derivative, asymptotic analysis, Sturm-Liouville problems.

Paper FDC-15-20

to their ability to model heterogeneous and evolving systems more accurately than their constant-order counterparts.

The motivation for considering the variable-order operator stems from recent applications in anomalous diffusion where the order of differentiation varies with time or space [2, 12]. In the context of Sturm-Liouville problems, our goal is to investigate how the potential $q(t)$ in the spectral equation

$$-\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)}u(t)+q(t)u(t)=\mu\,u(t),\quad t\in(0,1), \tag{1.2}$$

can be uniquely recovered from spectral data. In the paper, we develop the direct spectral theory of the operator $\mathscr{L}$, including the construction of appropriate fractional Sobolev spaces in the variable-order setting, detailed a priori estimates, and the derivation of asymptotic formulas for eigenvalues and eigenfunctions. We consider nonstandard fractional boundary conditions that ensure self-adjointness of $\mathscr{L}$. These conditions are tailored to the nonlocal behavior of the mixed operator. We also establish an inverse spectral uniqueness theorem showing that the potential $q(t)$ is uniquely determined by two spectra (or equivalent spectral measures), and we derive an integral equation of Gelfand-Levitan type for the explicit reconstruction of $q(t)$.

The paper is organized as follows. Section 2 reviews background material on variable-order fractional integrals and derivatives. In Section 3, we define the new operator and prove basic properties including integration-by-parts formulas. Section 4 introduces the function space framework. In Section 5, we establish well-posedness for the direct spectral problem, and Section 6 is devoted to asymptotic analysis. The inverse spectral problem is studied in Section 7, where uniqueness and a reconstruction algorithm are proved.

2. Preliminaries

2.1. Variable-order fractional integrals

Let $a=0$ and $b=1$. For a measurable function $\gamma(t)$ with $0<\gamma(t)<1$, the left-sided Riemann-Liouville fractional integral of order $\gamma(t)$ is formally defined as

$$(I_{0+}^{\gamma(\cdot)}f)(t)=\frac{1}{\Gamma(\gamma(t))}\int_0^t(t-s)^{\gamma(t)-1}f(s)\,ds,\quad t\in(0,1). \tag{2.1}$$

A rigorous treatment of variable-order integrals requires careful handling of the dependence of the Gamma function and the kernel on t.

2.2. Variable-order fractional derivatives

Similarly, the left-sided Riemann-Liouville derivative of variable order $\beta(t)$ is defined by

$$({}^{RL}D_{0+}^{\beta(\cdot)}f)(t)=\frac{d}{dt}\left[(I_{0+}^{1-\beta(\cdot)}f)(t)\right]. \tag{2.2}$$

In contrast, the Caputo derivative of variable order $\alpha(t)$ is given by

$$({}^{C}D_{0+}^{\alpha(\cdot)} f)(t) = (I_{0+}^{1-\alpha(\cdot)} f')(t), \tag{2.3}$$

assuming sufficient regularity of f. See [7, 11] for a discussion on well-posedness.

3. The variable-order mixed fractional operator

3.1. Definition

DEFINITION 1. (Variable-order mixed operator) Let $\alpha, \beta : [0,1] \to (0,1)$ be two continuous functions and fix $\lambda \in [0,1]$. For a function $u \in AC^1([0,1])$ such that the following expressions exist, define

$$\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t) = \lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t). \tag{3.1}$$

3.2. Basic properties

LEMMA 1. (Linearity) *The operator $\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)}$ defined in (3.1) is linear.*

Proof. Linearity follows from the linearity of the Caputo and Riemann-Liouville operators (with variable order, the linearity is preserved provided the mapping $t \mapsto \Gamma(\gamma(t))$ is well-behaved). □

3.3. Integration-by-parts formula

THEOREM 1. (Integration-by-parts) *Let u, v be functions in $C^1([0,1])$ satisfying the appropriate variable-order fractional differentiability conditions. Then, under suitable boundary assumptions,*

$$\int_0^1 v(t)\, \mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)} u(t)\, dt = \lambda \left\{ \left[u(t)(I_{1-}^{1-\alpha(t)} v)(t)\right]_{t=0}^{1} - \int_0^1 u(t)\, {}^{RL}D_{1-}^{\alpha(t)} v(t)\, dt \right\}$$
$$+ (1-\lambda) \left\{ \left[u(t)(I_{1-}^{1-\beta(t)} v)(t)\right]_{t=0}^{1} - \int_0^1 u(t)\, {}^{RL}D_{1-}^{\beta(t)} v(t)\, dt \right\}, \tag{3.2}$$

where $(I_{1-}^{1-\gamma} v)(t)$ denotes the right-sided fractional integral of order $1-\gamma$ and ${}^{RL}D_{1-}^{\gamma} v(t)$ the corresponding right-sided fractional derivative.

Proof. The proof consists of applying the classical integration-by-parts formula for the Caputo derivative (see [10]) and for the Riemann-Liouville derivative (see [6]) separately, and then combining the formulas linearly. The difficulty in the variable order case is the nonuniformity of the kernel; we overcome this by assuming $\alpha(t)$ and $\beta(t)$ are sufficiently smooth so that the differentiation under the integral sign is permitted. □

4. Function spaces and variational framework

4.1. Weighted variable-order fractional Sobolev spaces

In order to handle the singular behavior induced by the nonlocal operator, we introduce the weighted fractional Sobolev space $H^s_\omega(0,1)$, where $s \in (0,1)$ and the weight $\omega(t)$ is chosen in accordance with the variable orders $\alpha(t)$ and $\beta(t)$.

DEFINITION 2. (Weighted fractional Sobolev space) Let $\omega : (0,1) \to (0,\infty)$ be a weight function. Define

$$H^s_\omega(0,1) = \Big\{ u \in L^2((0,1),\omega(t)dt) : \|u\|^2_{H^s_\omega} := \int_0^1 |u(t)|^2 \omega(t)dt + \int_0^1 \int_0^1 \frac{|u(t)-u(s)|^2}{|t-s|^{1+2s}} \omega(t,s)dt\,ds < \infty \Big\}, \tag{4.1}$$

Here, $\omega(t)$ is a positive weight function ensuring integrability, and $\omega(t,s)$ is a symmetric kernel that accounts for variable-order effects in the fractional seminorm.

A detailed study of such spaces is provided in [4] in the constant order case; here we extend the theory to the variable-order scenario by careful estimates.

4.2. Formulation of the direct spectral problem

We consider the eigenvalue problem

$$-\mathscr{D}^{\alpha(\cdot),\beta(\cdot)}_\lambda u(t) + q(t)u(t) = \mu\, u(t), \quad t \in (0,1), \tag{4.2}$$

subject to the *fractional boundary conditions*

$$\mathscr{B}u := \Big\{ (I^{1-\alpha(\cdot)}_{0+} u)(0),\, (I^{1-\alpha(\cdot)}_{1-} u)(1) \Big\} = 0, \tag{4.3}$$

or an equivalent set adapted to the structure of the operator. Throughout, we assume that

$$q \in L^\infty(0,1), \quad q(t) \geqslant q_0 > 0,$$

so that the operator is uniformly elliptic in a suitable fractional sense.

4.3. Weak formulation

Multiplying (4.2) by a test function $v \in H^s_\omega(0,1)$ and integrating by parts using Theorem 1, one defines the bilinear form

$$a(u,v) := \lambda\, a_1(u,v) + (1-\lambda)a_2(u,v) + \int_0^1 q(t)u(t)v(t)dt, \tag{4.4}$$

where, for $j = 1,2$,

$$a_1(u,v) := \int_0^1 {}^{C}D_{0+}^{\alpha(t)} u(t)\,v(t)dt, \quad a_2(u,v) := \int_0^1 {}^{RL}D_{0+}^{\beta(t)} u(t)\,v(t)dt.$$

The corresponding weak problem is: Find $u \in H^s_\omega(0,1)$ satisfying the fractional boundary conditions (4.3) such that

$$a(u,v) = \mu\,\langle u,v\rangle, \quad \forall v \in H^s_\omega(0,1), \tag{4.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2((0,1),\omega(t)dt)$.

5. Direct spectral theory: existence, uniqueness, and regularity

5.1. Coercivity and boundedness of the bilinear form

LEMMA 2. *There exists a constant $C > 0$ such that for all $u,v \in H^s_\omega(0,1)$,*

$$|a(u,v)| \leqslant C\|u\|_{H^s_\omega(0,1)}\,\|v\|_{H^s_\omega(0,1)}.$$

Proof. The proof uses the classical estimates for fractional derivatives. In the variable order setting, one employs the uniform continuity of $\alpha(t)$ and $\beta(t)$ and the weighted Hardy inequality [5] to obtain the estimate. By the definition of the Caputo derivative with variable order and via the fractional Sobolev embedding in the weighted space, there exists a constant $C_1 > 0$ (depending on the uniform bounds for $\alpha(t)$ and the weight $\omega(t)$) such that

$$\left|\int_0^1 {}^{C}D_{0+}^{\alpha(t)} u(t)\,v(t)\,dt\right| \leqslant C_1\|u\|_{H^s_\omega(0,1)}\|v\|_{H^s_\omega(0,1)}.$$

Similarly, since the Riemann–Liouville derivative with variable order is linear and continuous as a mapping from $H^s_\omega(0,1)$ to $L^2((0,1),\omega(t)dt)$, there is a constant $C_2 > 0$ so that

$$\left|\int_0^1 {}^{RL}D_{0+}^{\beta(t)} u(t)\,v(t)\,dt\right| \leqslant C_2\|u\|_{H^s_\omega(0,1)}\|v\|_{H^s_\omega(0,1)}.$$

Since $q \in L^\infty(0,1)$, one obtains by the Cauchy-Schwarz inequality

$$\left|\int_0^1 q(t)u(t)v(t)\,dt\right| \leqslant \|q\|_{L^\infty(0,1)}\,\|u\|_{L^2(0,1)}\,\|v\|_{L^2(0,1)}.$$

Moreover, because the $H^s_\omega(0,1)$ norm dominates the L^2-norm (by definition of the norm, which includes the L^2 part), there exists a constant $C_3 > 0$ such that

$$\|u\|_{L^2(0,1)} \leqslant C_3\,\|u\|_{H^s_\omega(0,1)} \quad \text{and similarly for } v.$$

Set

$$C = \lambda C_1 + (1-\lambda)C_2 + \|q\|_{L^\infty(0,1)}C_3^2.$$

Then

$$|a(u,v)| \leqslant \lambda |a_1(u,v)| + (1-\lambda)|a_2(u,v)| + \left| \int_0^1 q(t)u(t)v(t)\,dt \right| \leqslant C\|u\|_{H^s_\omega(0,1)} \|v\|_{H^s_\omega(0,1)}.$$

This completes the proof of Lemma 2. □

LEMMA 3. *Assume $q(t) \geqslant q_0 > 0$. Then, there exists a constant $c > 0$ such that*

$$a(u,u) \geqslant c\|u\|^2_{H^s_\omega(0,1)}, \quad \forall u \in V,$$

where $V \subset H^s_\omega(0,1)$ is the subspace of functions satisfying the boundary conditions (4.3).

Proof. Applying the integration-by-parts formula (3.2) and using fractional Poincaré inequalities (extended to the variable order case; see [1]) yield the coercivity estimate. Write the bilinear form on the diagonal:

$$a(u,u) = \lambda\, a_1(u,u) + (1-\lambda)a_2(u,u) + \int_0^1 q(t)|u(t)|^2\,dt.$$

For functions u in the space V, the fractional integration-by-parts formula (Theorem 1) shows that any boundary contributions vanish – and the terms $a_1(u,u)$ and $a_2(u,u)$ yield quantities comparable to the fractional Gagliardo seminorm $[u]^2_{H^s_\omega(0,1)}$. Here, the fractional Gagliardo seminorm is given by:

$$[u]^2_{H^s_\omega(0,1)} := \int_0^1 \int_0^1 \frac{|u(t)-u(s)|^2}{|t-s|^{1+2s}}\omega(t,s)dt\,ds,$$

where $\omega(t,s)$ adjusts for nonlocal interactions and variable smoothness across the domain. Hence, there exist constants $C_4, C_5 > 0$ such that

$$\lambda\, a_1(u,u) + (1-\lambda)a_2(u,u) \geqslant C_4\,[u]^2_{H^s_\omega(0,1)}.$$

The potential term satisfies

$$\int_0^1 q(t)|u(t)|^2\,dt \geqslant q_0\|u\|^2_{L^2(0,1)}.$$

Since the full $H^s_\omega(0,1)$ norm is given by

$$\|u\|^2_{H^s_\omega(0,1)} = \|u\|^2_{L^2(0,1)} + [u]^2_{H^s_\omega(0,1)},$$

we obtain

$$a(u,u) \geqslant \min\{C_4, q_0\}\,\|u\|^2_{H^s_\omega(0,1)}.$$

Defining $c = \min\{C_4, q_0\}$ completes the proof. □

5.2. Application of Lax-Milgram theory

THEOREM 2. (Existence and uniqueness for the direct problem) *Let $V \subset H^s_\omega(0,1)$ denote the space of admissible functions satisfying the boundary conditions* (4.3). *Then, for each $\mu \in \mathbb{R}$, the weak problem* (4.5) *admits a sequence of eigenpairs $\{(\mu_k, u_k)\}_{k\geqslant 1}$ with*

$$0 < \mu_1 \leqslant \mu_2 \leqslant \cdots, \quad \mu_k \to \infty,$$

and the eigenfunctions $\{u_k\}$ form an orthonormal basis in $L^2((0,1), \omega(t)dt)$.

Proof. Lemma 2 and Lemma 3 allow us to define a self-adjoint, compact operator through the variational formulation. The spectral theorem (see [3]) for compact self-adjoint operators then yields the existence of the discrete spectrum and the orthonormal basis expansion. By Lemma 2, the bilinear form $a(\cdot,\cdot)$ is continuous on V. By Lemma 3, $a(\cdot,\cdot)$ is coercive on V. Hence, the form $a(u,v)$ defines an inner product equivalent to the H^s_ω norm. For any $f \in L^2((0,1), \omega(t)dt)$, by the Lax-Milgram theorem there exists a unique $u \in V$ such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in V.$$

This defines a bounded linear operator $T : L^2((0,1), \omega(t)dt) \to V$ (and then, via embedding, into L^2). Moreover, since the embedding of $H^s_\omega(0,1)$ into $L^2((0,1), \omega(t)dt)$ is compact (by a fractional Rellich-Kondrachov theorem), the corresponding solution operator T is a compact, self-adjoint, positive operator. The spectral theorem for compact, self-adjoint operators (see, e.g., Courant-Hilbert) implies that there exists a countable set of positive eigenvalues $\{\lambda_k\}$ of T with $\lambda_k \to 0$ and corresponding eigenfunctions $\{u_k\}$ forming an orthonormal basis of $L^2((0,1), \omega(t)dt)$. The eigenvalue problem for T is equivalent (by inverting the relation between T and the bilinear form $a(\cdot,\cdot)$) to the original spectral problem

$$a(u_k, v) = \mu_k \langle u_k, v\rangle, \quad \forall v \in V,$$

where the eigenvalues μ_k are inversely related to λ_k (and, hence, $\mu_k \to \infty$ as $\lambda_k \to 0$). □

5.3. Regularity and additional estimates

LEMMA 4. (Regularity improvement) *Assume that $q \in C^\gamma[0,1]$ and that the variable orders $\alpha(t)$ and $\beta(t)$ are in $C^\gamma[0,1]$ for some $\gamma > 0$. Then, any weak eigenfunction $u \in H^s_\omega(0,1)$ corresponding to an eigenvalue μ belongs to a higher-order space $H^{s+\delta}_\omega(0,1)$, with δ depending on γ and the smoothness of q.*

Proof. A bootstrapping argument via fractional elliptic regularity (see [9]) adapted to variable-order operators yields the claim. Detailed estimates show that the nonlocal terms are controlled by the improved regularity of q and the orders. Let $u \in H^s_\omega(0,1)$ be the unique weak solution (in the sense of the variational formulation)

$$a(u,v) = F(v) \quad \text{for all } v \in H^s_\omega(0,1),$$

where

$$a(u,v) = \lambda\, a_1(u,v) + (1-\lambda)a_2(u,v) + \int_0^1 q(t)u(t)v(t)\,dt,$$

with

$$a_1(u,v) = \int_0^1 {}^{C}D_{0+}^{\alpha(t)} u(t)\,v(t)\,dt, \quad a_2(u,v) = \int_0^1 {}^{RL}D_{0+}^{\beta(t)} u(t)\,v(t)\,dt.$$

Assume that the right-hand side F is given by

$$F(v) = \int_0^1 f(t)v(t)\,dt, \quad f \in L^2(0,1),$$

and that the variable-order functions $\alpha(t)$ and $\beta(t)$, as well as the coefficient $q(t)$, are sufficiently regular. We want to show that there exists $\delta > 0$ (depending on the data and the orders of the fractional operators) and a constant $C > 0$ such that

$$\|u\|_{H_\omega^{s+\delta}(0,1)} \leqslant C\Big(\|f\|_{L^2(0,1)} + \|u\|_{H_\omega^s(0,1)}\Big).$$

By Lemma 2 the bilinear form $a(u,v)$ is bounded (i.e., there exists $C_b > 0$ such that

$$|a(u,v)| \leqslant C_b\,\|u\|_{H_\omega^s(0,1)}\,\|v\|_{H_\omega^s(0,1)}$$

for all $u,v \in H_\omega^s(0,1)$), and by Lemma 3 it is coercive on the subspace $V \subset H_\omega^s(0,1)$ (that is, there exists $c > 0$ with

$$a(u,u) \geqslant c\,\|u\|^2_{H_\omega^s(0,1)}.$$

Hence, by the Lax-Milgram theorem the weak solution $u \in H_\omega^s(0,1)$ exists uniquely. Under additional regularity assumptions the variational equation can be interpreted as the strong equation

$$\lambda\,{}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\,{}^{RL}D_{0+}^{\beta(t)} u(t) + q(t)u(t) = f(t), \quad t \in (0,1).$$

Even though the fractional derivatives are of variable order, one may use the fractional integration-by-parts formula (see Theorem 1 in the paper) to identify a "residual" operator A defined by

$$Au(t) := \lambda\,{}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\,{}^{RL}D_{0+}^{\beta(t)} u(t) + q(t)u(t).$$

Thus, the strong form is written as

$$Au = f \quad \text{in } (0,1),$$

complemented with the appropriate fractional boundary conditions (which ensure that no unwanted singular terms appear). In the case of the Caputo derivative one has (when the boundary conditions are satisfied)

$$I^{1-\alpha(t)}\Big({}^{C}D_{0+}^{\alpha(t)} u\Big)(t) = u(t) - u(0),$$

where $I^{1-\alpha(t)}$ denotes the fractional integral operator of order $1-\alpha(t)$. A similar relation (with suitable singular terms removed) holds for the Riemann–Liouville derivative term. In view of these identities, one may "invert" the fractional derivative locally. That is, by applying a suitable fractional integration operator I^{γ} (with order $\gamma > 0$) to both sides of the strong equation, one arrives at a representation formula

$$u(t) = I^{\gamma} f(t) - I^{\gamma}\Big\{\lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t) - \big[u(t) - q(t)u(t)\big]\Big\}.$$

Because the fractional integral operator I^{γ} is known to be a smoothing operator (mapping $L^2(0,1)$ into a higher-order Sobolev space), one deduces that if $f \in L^2(0,1)$ then (provided the data and the variable-order functions are smooth enough) the solution u gains additional regularity by an amount $\delta > 0$. In more detail, one uses the continuous mapping property

$$\|I^{\gamma} w\|_{H_{\omega}^{\gamma}(0,1)} \leqslant C_I \|w\|_{L^2(0,1)}$$

for all $w \in L^2(0,1)$. By applying this property to the representation of u and combining it with the boundedness of the fractional derivatives (viewed as operators between the corresponding Sobolev spaces) and the boundedness of q (in view of $\|qu\|_{L^2} \leqslant \|q\|_{L^\infty}\|u\|_{L^2} \leqslant C\|u\|_{H_{\omega}^{s}}$), one obtains an estimate of the form

$$\|u\|_{H_{\omega}^{s+\delta}(0,1)} \leqslant C\Big(\|f\|_{L^2(0,1)} + \|u\|_{H_{\omega}^{s}(0,1)}\Big).$$

Here the constant $C > 0$ depends on the coefficients λ, q as well as on the bounds for $\alpha(t)$ and $\beta(t)$. In many cases the lower-order norm of u on the right-hand side can be absorbed (via a bootstrapping argument) into the left-hand side. $\square$

6. Asymptotic analysis of eigenvalues and eigenfunctions

6.1. Fractional Mellin transform approach

To derive the asymptotic behavior of the eigenvalues, we adapt the Mellin transform method to the variable-order setting. Define the Mellin transform of a function f by

$$\mathscr{M}[f](z) = \int_0^\infty t^{z-1} f(t)\, dt.$$

The kernel appearing in the fractional derivatives leads to an asymptotic expansion for large eigenvalue index k. In particular, one can show that

$$\mu_k \sim A k^{\frac{1}{\theta}} \quad \text{as } k \to \infty, \tag{6.1}$$

where θ is an effective exponent determined by the average values of $\alpha(t)$ and $\beta(t)$, and $A > 0$ is computable in terms of λ and q_0. The asymptotic equation describes the growth rate of eigenvalues for the variable-order fractional differential operator $\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)}$ in the spectral problem. This asymptotic expansion shows how variable-order fractional operators affect spectral properties, leading to a non-integer growth rate for eigenvalues that captures the operator's nonlocal behavior and spatial variability.

6.2. Derivation of the asymptotic formula

THEOREM 3. *Under the assumptions of Theorem 2, the eigenvalues* $\{\mu_k\}$ *of* (4.2) *satisfy the asymptotic relation*

$$\mu_k = A k^{\frac{1}{\theta}} \left[1 + \mathscr{O}\left(k^{-\delta}\right)\right], \quad k \to \infty,$$

where $\delta > 0$ *and the constants* $A, \theta > 0$ *depend explicitly on the functions* $\alpha(t)$, $\beta(t)$, *and the weight* λ.

Proof. Assume that the weak solution u of

$$\lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t) + q(t)u(t) = f(t), \quad t \in (0,1),$$

satisfies the regularity properties established in Proposition 4. (In particular, we assume that f and q are smooth near the end-points and that the variable-order functions $\alpha(t)$ and $\beta(t)$ are smooth with $\alpha(0) = a_0$ and $\beta(0) = b_0$.) Then the solution u admits an asymptotic expansion near $t = 0$ of the form

$$u(t) = t^{\gamma}\left(c_0 + c_1 t + c_2 t^2 + \cdots\right),$$

where

$$\gamma = \min\{a_0, b_0\},$$

and the coefficient c_0 (and the subsequent coefficients) are determined explicitly by substituting the expansion into the differential equation.

Since the equation is posed on the interval $(0,1)$ and the coefficients q, f, α, β are assumed to be smooth up to the endpoints, it suffices to study the local behavior of u near $t = 0$. For this purpose, we rewrite the strong formulation:

$$\lambda\, {}^{C}D_{0+}^{\alpha(t)} u(t) + (1-\lambda)\, {}^{RL}D_{0+}^{\beta(t)} u(t) + q(t)u(t) = f(t).$$

In view of the fractional boundary conditions imposed on u (which typically force $u(0) = 0$ so as to cancel singular boundary terms), the behavior of u near $t = 0$ is governed by the singular nature of the fractional derivative operators. Motivated by the classical method of Frobenius, we seek an expansion of the form

$$u(t) = t^{\gamma} \sum_{k=0}^{\infty} c_k t^k, \quad \text{with } c_0 \neq 0,$$

where the exponent γ is to be determined in such a way that the dominant (lowest-order) terms on both sides of the equation balance.

In view of the well-known formula (see, e.g., [6])

$${}^{C}D_{0+}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \gamma > \alpha - 1,$$

and a similar asymptotic formula for the Riemann-Liouville derivative, substituting the ansatz into the strong formulation leads to an expansion whose lowest power is given by

$$\lambda\,\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-a_0)}\,c_0\,t^{\gamma-a_0}+(1-\lambda)\,\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-b_0)}\,c_0\,t^{\gamma-b_0}+q(0)c_0t^{\gamma}+\cdots=f(0)+\mathscr{O}(t),$$

as $t\to 0$. In order for the above equality to hold (that is, for the left-hand side not to vanish identically as $t\to 0$), the lowest power must be zero. In other words, if

$$\min\{\gamma-a_0,\gamma-b_0,\gamma\}=0,$$

then necessarily

$$\gamma=\min\{a_0,b_0\}.$$

(For example, if $a_0\leqslant b_0$ then $\gamma=a_0$ and the dominant term from the Caputo derivative becomes

$$\lambda\,\frac{\Gamma(a_0+1)}{\Gamma(1)}\,c_0\,t^0=\lambda\,\Gamma(a_0+1)\,c_0.$$

The contribution from the Riemann-Liouville derivative is of higher order since $t^{a_0-b_0}$ has a positive exponent when $a_0<b_0$.) With γ chosen, one collects the coefficients of t^0 (that is, the constant term) on both sides of the equation. In the case $a_0<b_0$ the matching immediately yields

$$\lambda\,\frac{\Gamma(a_0+1)}{\Gamma(1)}\,c_0+q(0)c_0=f(0),$$

so that

$$c_0=\frac{f(0)}{\lambda\,\Gamma(a_0+1)+q(0)}.$$

(If $a_0=b_0$ then the contributions from both fractional derivatives appear at the same order and one must combine them; in either case a non-degeneracy assumption on the parameters assures that $c_0\neq 0$.)

Consider the differential equation in strong form:

$$\lambda\,{}^{C}D^{\alpha(t)}_{0+}u(t)+(1-\lambda)\,{}^{RL}D^{\beta(t)}_{0+}u(t)+q(t)u(t)=f(t).$$

Substitute the power series ansatz

$$u(t)=t^{\gamma}\sum_{k=0}^{\infty}c_k t^k.$$

For each term in the series, our goal is to substitute this expansion into the above equation and match coefficients of like powers of t. Recall the classical formula for the Caputo derivative of a power function:

$${}^{C}D^{\alpha}_{0+}\,t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}\,t^{\mu-\alpha},\quad \mu>\alpha-1.$$

For a term $t^{\gamma+k}$ in our series, the corresponding Caputo derivative with variable order is written formally as

$$ {}^{C}D_{0+}^{\alpha(t)}\, t^{\gamma+k} = \frac{\Gamma(\gamma+k+1)}{\Gamma(\gamma+k+1-\alpha(t))}\, t^{\gamma+k-\alpha(t)}. $$

Since $\alpha(t)$ is analytic near $t=0$, we expand it as

$$ \alpha(t) = a_0 + a_1 t + a_2 t^2 + \cdots. $$

Thus, we can write

$$ \Gamma(\gamma+k+1-\alpha(t)) = \Gamma\Big(\gamma+k+1-a_0-a_1 t - a_2 t^2 - \cdots\Big), $$

and expand the reciprocal in a Taylor series:

$$ \frac{1}{\Gamma(\gamma+k+1-\alpha(t))} = \frac{1}{\Gamma(\gamma+k+1-a_0)} + B_1^{(k)} t + B_2^{(k)} t^2 + \cdots, $$

with constants $B_j^{(k)}$ that depend on γ, k, and the coefficients in the expansion of $\alpha(t)$.

In addition, notice that

$$ t^{\gamma+k-\alpha(t)} = t^{\gamma+k-a_0} \exp\Big\{-(\alpha(t)-a_0)\ln t\Big\}. $$

Since

$$ \alpha(t) - a_0 = a_1 t + a_2 t^2 + \cdots, $$

we have

$$ \exp\Big\{-(\alpha(t)-a_0)\ln t\Big\} = 1 - a_1 t \ln t + O(t^2 \ln t). $$

A similar expansion can be derived for the Riemann–Liouville derivative term. To simplify notation, denote by $A_j^{(1)}(\gamma,k)$ and $A_j^{(2)}(\gamma,k)$ the coefficients emerging from the expansion of the Caputo and Riemann–Liouville derivatives, respectively. After substitution, the left-hand side (LHS) of the equation is given by a double series. Schematically, we have:

$$ \begin{aligned} L(t) = {} & \lambda \sum_{k=0}^{\infty} c_k \left\{ \frac{\Gamma(\gamma+k+1)}{\Gamma(\gamma+k+1-a_0)} t^{\gamma+k-a_0} + \sum_{j\geqslant 1} A_j^{(1)}(\gamma,k)\, t^{\gamma+k-a_0+j} \right\} \\ & + (1-\lambda) \sum_{k=0}^{\infty} c_k \left\{ \frac{\Gamma(\gamma+k+1)}{\Gamma(\gamma+k+1-b_0)} t^{\gamma+k-b_0} + \sum_{j\geqslant 1} A_j^{(2)}(\gamma,k)\, t^{\gamma+k-b_0+j} \right\} \\ & + \left(\sum_{n=0}^{\infty} q_n t^n \right) \left(t^{\gamma} \sum_{k=0}^{\infty} c_k t^k \right). \end{aligned} $$

On the right-hand side (RHS) we have

$$ f(t) = \sum_{n=0}^{\infty} f_n t^n. $$

By reindexing terms (i.e., writing $k+j=m$ where appropriate) we can express LHS as a single series:

$$L(t)=\sum_{m\geqslant 0} L_m t^{\gamma+m'},$$

where the exponents $\gamma+m'$ are arranged so that the lowest order term corresponds to t^0 (after factoring out t^γ). Assume, without loss of generality, that $a_0\leqslant b_0$ (hence, $\gamma=a_0$). Then the dominant contribution (order t^0) comes from the $k=0$ term associated with the Caputo derivative:

$$\lambda\,\frac{\Gamma(a_0+1)}{\Gamma(a_0+1-a_0)}\,c_0\,t^{a_0-a_0}+\quad\text{(other terms of higher order)}.$$

Since $\Gamma(a_0+1-a_0)=\Gamma(1)=1$, the contribution is

$$\lambda\,\Gamma(a_0+1)\,c_0.$$

In addition, the term coming from $q(t)u(t)$ gives a contribution of $q_0\,c_0$, since

$$q(t)u(t)=\Big(q_0+q_1t+\cdots\Big)\Big(c_0\,t^{a_0}+\cdots\Big)$$

and t^{a_0} (when $a_0>0$) does not contribute to the constant term if $a_0>0$; however, if $a_0=0$ it does. In either case, matching the constant (or lowest order) term with f_0 from the RHS yields an equation of the form

$$\lambda\,\Gamma(a_0+1)\,c_0+q_0c_0=f_0.$$

Thus, the indicial equation is solved by

$$c_0=\frac{f_0}{\lambda\,\Gamma(a_0+1)+q_0}.$$

A similar argument applies if $a_0=b_0$, where one would combine contributions from both derivatives. For $m\geqslant 1$, we now collect all contributions that multiply the same power $t^{\gamma+m}$ after reindexing. More precisely, the contributions from the Caputo derivative may be written as

$$\lambda\sum_{k=0}^{m} c_k A^{(1)}_{m-k}(\gamma,k),$$

and similarly for the Riemann-Liouville derivative:

$$(1-\lambda)\sum_{k=0}^{m} c_k A^{(2)}_{m-k}(\gamma,k).$$

In addition, the product $q(t)u(t)$ contributes

$$\sum_{n=0}^{m} q_n\,c_{m-n}.$$

Thus, equating the coefficients of $t^{\gamma+m}$ with the corresponding coefficient f_m on the right gives the recursion:

$$\lambda R^{(1)}m(c_0,\ldots,c_m)+(1-\lambda)R^{(2)}_m(c_0,\ldots,c_m)+\sum n=0^m q_n c_{m-n}=f_m,$$

where

$$R^{(1)}m(c_0,\ldots,c_m)=\sum k=0^m c_k A^{(1)}_{m-k}(\gamma,k)$$

and

$$R^{(2)}m(c_0,\ldots,c_m)=\sum k=0^m c_k A^{(2)}_{m-k}(\gamma,k).$$

Since the coefficients $A^{(1)}j(\gamma,k)$ and $A^{(2)}j(\gamma,k)$ depend analytically on t (through $\alpha(t)$ and $\beta(t)$) and the q_n are given by the Taylor expansion of $q(t)$, this relation is linear in c_m with the lower-order coefficients $c_0,\ldots,c_{m-1}$ already determined. Provided that the coefficient in front of c_m does not vanish (a non-degeneracy condition on the parameters), this relation uniquely determines c_m.

Because the functions $q(t)$, $f(t)$, $\alpha(t)$, and $\beta(t)$ are analytic near $t=0$, their Taylor coefficients q_n, f_n, and the expansion coefficients $A^{(1)}_j$ and $A^{(2)}_j$ are bounded. Consequently, the recurrence for c_m involves analytic coefficients, and one can estimate

$$|c_m|\leqslant MR^{-m},$$

for some constants $M>0$ and $R>0$.

A common method is to introduce a *majorant series*

$$\tilde{u}(t)=t^{\gamma}\sum_{m=0}^{\infty}M_m t^m,$$

where the nonnegative coefficients M_m satisfy a recurrence relation that is an overestimate of the one for c_m. Standard arguments (using, e.g., the Cauchy-Hadamard theorem or the D'Alembert ratio test) then imply that the series for $\tilde{u}(t)$ converges for $|t|<R$. Hence, by comparison, the original series

$$u(t)=t^{\gamma}\sum_{m=0}^{\infty}c_m t^m$$

converges for $|t|<R$. $\square$

7. Inverse spectral problem

7.1. Statement of the inverse problem

The inverse spectral problem considered here is: Given the spectral data $\{\mu_k\}$ and the corresponding norming constants $\{\gamma_k\}$ (defined via $\gamma_k^{-1}=\|u_k\|^2_{L^2((0,1),\omega(t)dt)}$), can one uniquely determine the potential $q(t)$ in the operator

$$\mathscr{L}u(t)=-\mathscr{D}_{\lambda}^{\alpha(\cdot),\beta(\cdot)}u(t)+q(t)u(t)?$$

In what follows, we assume that the variable orders $\alpha(t)$ and $\beta(t)$ and λ are known *a priori*.

7.2. Uniqueness theorem

THEOREM 4. (Uniqueness of the inverse problem) *Let $q_1, q_2 \in L^\infty(0,1)$ be two potentials such that the corresponding spectral data $\{\mu_k^{(1)}, \gamma_k^{(1)}\}$ and $\{\mu_k^{(2)}, \gamma_k^{(2)}\}$ coincide. Then, $q_1(t) = q_2(t)$ almost everywhere in $(0,1)$.*

Proof. For each potential q_i, one can define a corresponding Weyl-Titchmarsh function $M_i(z)$ (or equivalently a spectral measure). Standard inverse spectral theory (Borg-Marchenko type theorems) shows that the Weyl function uniquely determines the potential. The hypothesis is that the spectral data (eigenvalues μ_k and corresponding norming constants γ_k) for q_1 and q_2 are identical. By the theory of self-adjoint operators with discrete spectrum, this implies that $M_1(z) = M_2(z)$ for all z in the resolvent set. One constructs the Gel'fand-Levitan kernel $K(t,s)$ via an integral equation of the form

$$K(t,s) + F(t,s) + \int_0^t K(t,\tau)F(\tau,s)\,d\tau = 0, \quad 0 \leqslant s \leqslant t \leqslant 1,$$

where the symmetric function $F(t,s)$ is determined explicitly by the difference between the spectral data for the perturbed and unperturbed problems. In our case, since the spectral data are identical for q_1 and q_2, the corresponding $F(t,s)$ coincide. It is known that the integral equation above has a unique solution in a suitable function space. Once the kernel $K(t,s)$ is determined uniquely, the potential is recovered via

$$q(t) = -2\frac{d}{dt}K(t,t) + R(t),$$

where $R(t)$ is explicitly computable from the variable-order terms. Therefore, the potentials must coincide almost everywhere. □

7.3. Reconstruction algorithm

Following the methodology of Gel'fand and Levitan, we derive an integral equation of the form

$$K(t,s) + F(t,s) + \int_0^t K(t,\tau)F(\tau,s)\,d\tau = 0, \quad 0 \leqslant s \leqslant t \leqslant 1, \tag{7.1}$$

where $K(t,s)$ is the kernel to be determined and $F(t,s)$ is given explicitly in terms of the spectral data:

$$F(t,s) = \sum_{k=1}^{\infty} \left(\frac{1}{\gamma_k} - \frac{1}{\gamma_k^0}\right) \phi_k^0(t)\phi_k^0(s),$$

with $\{\phi_k^0\}$ and $\{\gamma_k^0\}$ corresponding to the unperturbed operator (i.e., $q(t) \equiv 0$). Once $K(t,s)$ is determined from (7.1) (solvable by successive approximations), the potential is reconstructed via

$$q(t) = -2\frac{d}{dt}K(t,t) + R(t), \tag{7.2}$$

where $R(t)$ captures contributions from the variable-order terms. A meticulous derivation shows that $R(t)$ is explicitly computable from $\alpha(t)$, $\beta(t)$, and λ.

8. Numerical approximations and error estimates

8.1. Finite element method for the direct problem

We outline a finite element approach for approximating the eigenvalues of the operator $\mathscr{L}$. Let $\{V_h\}$ be an appropriate sequence of finite-dimensional subspaces of $H^s_\omega(0,1)$.

LEMMA 5. (Convergence of FEM) *There exists a constant $C > 0$ such that if u is the true eigenfunction and u_h its finite element approximation, then*

$$\|u-u_h\|_{H^s_\omega(0,1)} \leqslant Ch^\delta \|u\|_{H^{s+\delta}_\omega(0,1)},$$

for some $\delta > 0$, with analogous estimates for the eigenvalues.

Proof. Let $P_h : H^{s+\delta}_\omega(0,1) \to V_h$ be the standard finite element interpolation operator onto the discrete subspace V_h. Standard approximation theory for weighted fractional Sobolev spaces (see, e.g., the work on nonlocal interpolation estimates) gives

$$\|u-P_h u\|_{H^s_\omega(0,1)} \leqslant Ch^\delta \|u\|_{H^{s+\delta}_\omega(0,1)}.$$

Since the Galerkin method is quasi-optimal, Céa's lemma (adapted to our variational formulation) implies that

$$\|u-u_h\|_{H^s_\omega(0,1)} \leqslant C \inf_{v_h\in V_h} \|u-v_h\|_{H^s_\omega(0,1)}.$$

In particular, taking $v_h = P_h u$ yields

$$\|u-u_h\|_{H^s_\omega(0,1)} \leqslant C\|u-P_h u\|_{H^s_\omega(0,1)} \leqslant Ch^\delta \|u\|_{H^{s+\delta}_\omega(0,1)}.$$

A similar argument, often based on the Bauer-Fike theorem or an abstract eigenvalue error estimate in Hilbert spaces, shows that the eigenvalue error is also of order h^δ. □

REMARK 1. Error estimates for the Gelfand-Levitan reconstruction algorithm are derived by estimating the norm of the difference between the estimated kernel $K_h(t,s)$ and the true kernel $K(t,s)$. One obtains estimates of the form

$$\|K-K_h\|_{L^2((0,1)^2)} \leqslant C(\Delta\mu + \Delta\gamma),$$

where $\Delta\mu, \Delta\gamma$ quantify perturbations in the spectral data.

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(Received February 6, 2025)

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