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A DILUTION TEST FOR THE CONVERGENCE OF SUBSERIES OF A MONOTONE SERIES

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Abstract. Cauchy's condensation test allows to determine the convergence of a monotone series by looking at a weighted subseries that only involves terms of the original series indexed by the powers of two. It is natural to ask whether the converse is also true: Is it possible to determine the convergence of an arbitrary subseries of a monotone series by looking at a suitably weighted version of the original series? In this note we show that the answer is affirmative and introduce a new convergence test particularly designed for this purpose.

1. Cauchy–Schlömilch condensation test

Consider a series which is monotone in the sense that its terms satisfy $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$. Cauchy's condensation test (e.g. [3, Theorem 2.3]) states that a monotone series $\sum_{n\geqslant 1} a_n$ converges if and only if

$$\sum_{k\geqslant 0} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 16a_{16} + \cdots$$

converges, thereby allowing to determine the convergence of a monotone series by only looking at its terms indexed by the powers of two. Schlömilch's extension (e.g. [3, Theorem 2.4]) allows to replace the powers of two by a more general subsequence $s(1) < s(2) < s(3) < \cdots$ of the positive integers, assuming that the forward differences

$$\Delta s(k) = s(k+1) - s(k) \tag{1.1}$$

do not grow too fast. We will present a short proof of Schlömilch's classical result, to highlight its structural similarity to our new test in Section 2.

THEOREM 1.1. (Schlömilch) *For any monotone series $\sum_{n\geqslant 1} a_n$ and subsequence of the integers such that for some $c > 0$,*

$$\frac{\Delta s(k+1)}{\Delta s(k)} \leqslant c \quad \text{for all } k \geqslant 1, \tag{1.2}$$

the series $\sum_{n\geqslant 1} a_n$ converges if and only if $\sum_{k\geqslant 1} a_{s(k)}\, \Delta s(k)$ converges.

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Proof. Because the terms of $\sum_{n\geqslant 1} a_n$ are nonincreasing, we see that

$$a_{s(k+1)}\Delta s(k) \leqslant a_{s(k)} + \cdots + a_{s(k+1)-1} \leqslant a_{s(k)}\Delta s(k) \tag{1.3}$$

for all $k \geqslant 1$. Inequalities (1.3) combined with assumption (1.2) imply that

$$c^{-1}a_{s(k+1)}\Delta s(k+1) \leqslant a_{s(k)} + \cdots + a_{s(k+1)-1} \leqslant a_{s(k)}\Delta s(k).$$

By summing the above display over k we now find that

$$c^{-1}\sum_{k\geqslant 2} a_{s(k)}\Delta s(k) \leqslant \sum_{n\geqslant s(1)} a_n \leqslant \sum_{k\geqslant 1} a_{s(k)}\Delta s(k),$$

so that all three series above either converge or diverge together. □

2. Dilution test

The Cauchy–Schlömilch condensation test is designed for determining the convergence of a monotone series by looking at a weighted subseries of the original series. We will now reverse this line of thought and prove a converse to these results, which allows to determine the convergence of a subseries of a monotone series using a weighted version of the original series. Suitable weights can be defined in terms of the forward differences (1.1) and the counting function

$$S(n) = \#\{k : s(k) \leqslant n\}$$

of a sequence $s(1) < s(2) < \cdots$. It is interesting to note that the growth condition of the forward differences in Theorem 1.1 is not needed below. The weights $\Delta s(S(n))$ in Theorem 2.1 measure the distance between points of the subsequence nearest to n; see Figure 1.

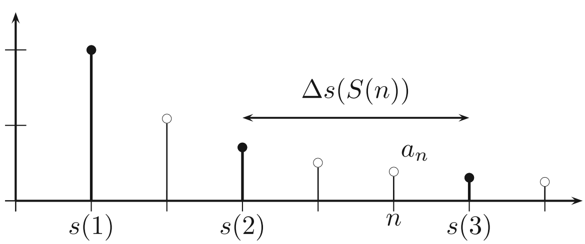

Figure 1: *The weight* $\Delta s(S(n))$.

THEOREM 2.1. *For any monotone series* $\sum_{n\geqslant 1} a_n$ *and any infinite subsequence of the integers, the subseries* $\sum_{k\geqslant 1} a_{s(k)}$ *converges if and only if*

$$\sum_{n\geqslant s(1)} \frac{a_n}{\Delta s(S(n))} < \infty. \tag{2.1}$$

Proof. The monotonicity of the series implies the validity of (1.3). After dividing the terms in (1.3) by $\Delta s(k)$, we find that

$$a_{s(k+1)} \leqslant \frac{a_{s(k)}}{\Delta s(k)} + \cdots + \frac{a_{s(k+1)-1}}{\Delta s(k)} \leqslant a_{s(k)}. \tag{2.2}$$

Because $S(n)$ indexes the last member of the sequence $s(1) < s(2) < \cdots$ not exceeding n, it follows that $S(n) = k$ for all n such that $s(k) \leqslant n \leqslant s(k+1) - 1$. This is why (2.2) may be rephrased as

$$a_{s(k+1)} \leqslant \sum_{n=s(k)}^{s(k+1)-1} \frac{a_n}{\Delta s(S(n))} \leqslant a_{s(k)}.$$

After summing the above display over k, we find that

$$\sum_{k\geqslant 2} a_{s(k)} \leqslant \sum_{n\geqslant s(1)} \frac{a_n}{\Delta s(S(n))} \leqslant \sum_{k\geqslant 1} a_{s(k)},$$

which shows that all series above either converge or diverge together. □

3. Thinning out a divergent series into a convergent one

Given a divergent monotone series $\sum_{n\geqslant 1} a_n$, one may ask whether it can be made convergent by deleting some of its terms. If $\lim_{n\to\infty} a_n > 0$, this is obviously not possible, while if $\lim_{n\to\infty} a_n = 0$ this can always be done by selecting terms of the series along a sparse enough subsequence. Indeed, in this case the series may even be thinned out to sum to an arbitrary positive real number (Banerjee and Lahiri [1]).

For a divergent monotone series $\sum_{n\geqslant 1} a_n$ such that $a_n \to 0$, a more specific question is to quantify a sufficient degree or sparsity required for the thinning subsequence. The following corollary of Theorem 2.1 presents a sufficient condition.

THEOREM 3.1. *Consider a monotone divergent series $\sum_{n\geqslant 1} a_n$ such that $\sum_{n\geqslant 1} a_n^p$ converges for some $p > 1$. A sufficient condition for the convergence of the subseries $\sum_{k\geqslant 1} a_{s(k)}$ is that*

$$\sum_{k\geqslant 1} s(k)^{-1/p} < \infty. \tag{3.1}$$

Proof. Observe that the series (2.1) in Theorem 2.1 can be written as $\sum_{k\geqslant 1} A_k$, where

$$A_k = \frac{1}{\Delta s(k)} \sum_{n=s(k)}^{s(k+1)-1} a_n$$

is the average of the terms $a_{s(k)}, \ldots, a_{s(k+1)-1}$. Because all these terms are less than or equal to the terms $a_1, \ldots, a_{s(k)}$, we see that A_k is bounded from above by the average

$$B_k = \frac{1}{s(k)} \sum_{n=1}^{s(k)} a_n.$$

Jensen's inequality implies that

$$B_k^p \leqslant \frac{1}{s(k)} \sum_{n=1}^{s(k)} a_n^p \leqslant \frac{1}{s(k)} \sum_{n=1}^{\infty} a_n^p,$$

so that

$$\sum_{n \geqslant s(1)} \frac{a_n}{\Delta s(S(n))} = \sum_{k \geqslant 1} A_k \leqslant \sum_{k \geqslant 1} B_k \leqslant \left(\sum_{n=1}^{\infty} a_n^p \right)^{1/p} \sum_{k \geqslant 1} s(k)^{-1/p}.$$

The now claim follows as a consequence of Theorem 2.1. □

4. Sparse subseries of the harmonic series

This section illustrates how Theorem 3.1 can be applied to find convergent subseries of the harmonic series, the archetype of a divergent series (see [3, Section 3] for a lively discussion). Kempner [5] has shown that, rather surprisingly, we obtain a convergent series by deleting from the harmonic series all terms whose decimal representation contains the digit '9' (see [6] and references therein for refinements). The following corollary of Theorem 3.1 shows that the harmonic series converges over any polynomially sparse subsequence. A subsequence of the integers is called *polynomially sparse* if its density among the first n positive integers decreases fast enough as n grows, according to

$$S(n)/n \leqslant cn^{-\alpha} \tag{4.1}$$

for some $c > 0$ and $\alpha \in (0,1)$. A simple counting argument (e.g. Behforooz [2]) may be used to verify that Kempner's no-'9' fulfills (4.1) with $c = 10$ and $\alpha = 1 - \frac{\log 9}{\log 10}$.

THEOREM 4.1. *The harmonic series $\sum_{n \geqslant 1} \frac{1}{n}$ converges over any polynomially sparse subsequence.*

Proof. Fix an integer $k \geqslant 1$, and let c and α be such that (4.1) holds for all n. The definition of the counting function implies that $s(k) \geqslant n+1$ for all integers n such that $S(n) < k$, and in particular for all integers n such that $n < (k/c)^{\beta}$, where $\beta = 1/(1-\alpha)$. By letting n be the largest integer strictly less than $(k/c)^{\beta}$, we see that $s(k) \geqslant (k/c)^{\beta}$. Therefore, condition (3.1) of Theorem 3.1 is valid for any $p \in (1, \beta)$. The claim now follows by Theorem 3.1, because $\sum_{n \geqslant 1} n^{-p}$ converges for $p > 1$. □

Note that in general, the harmonic series may diverge over a subsequence which is sparse (in the sense that $S(n)/n \to 0$) but not polynomially sparse. A classical example is the harmonic series over the prime numbers, which diverges (e.g. [4, Sec. 2.6]), whereas the density of the prime numbers decreases to zero according to $S(n)/n \sim (\log n)^{-1}$ (e.g. [4, Sec. 1.8]).

5. Concluding remark

Many nonmonotone series $\sum_{n\geqslant 1} a_n$ encountered in applications admit a monotone majorant $\sum_{n\geqslant 1} b_n$. In this case, the dilution test can be applied to subseries of the majorant series; if $\sum_{k\geqslant 1} b_{s(k)}$ converges, then so does the corresponding subseries $\sum_{k\geqslant 1} a_{s(k)}$ of the original nonmonotone series.

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REFERENCES

[1] C. R. BANERJEE AND B. K. LAHIRI, *On subseries of divergent series*, Amer. Math. Monthly, 71:767–768, 1964.

[2] G. H. BEHFOROOZ, *Thinning out the harmonic series*, Math. Mag., 68(4):289–293, 1995.

[3] D. D. BONAR AND M. KHOURY, JR., *Real Infinite Series*, Mathematical Association of America, Washington DC, 2006.

[4] G. H. HARDY AND E. M. WRIGHT, *An Introduction to the Theory of Numbers*, Oxford University Press, sixth edition, 2008.

[5] A. J. KEMPER, *A curious convergent series*, Amer. Math. Monthly, 21(2):48–50, 1914.

[6] T. SCHMELZER AND R. BAILLIE, *Summing a curious, slowly convergent series*, Amer. Math. Monthly, 115(6):525–540, 2008.

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